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Chapter 6

The inertial class with Levi-subgroup of the type (2,2)

6.1 Preliminary elimination of atypical representations

We denote by I the partition $(2,2)$. Let σ_1 and σ_2 be two cuspidal representations of $\mathrm{GL}_2(F)$. We denote by P, M and U the standard parabolic subgroup, the standard Levi-subgroup and the unipotent radical of P corresponding to the partition I. We denote by $\bar{P}$ and $\bar{U}$ the opposite parabolic subgroup of P with respect to the Levi-subgroup M and unipotent radical of $\bar{P}$. In this chapter we are interested in the classification of typical representations for the inertial class $[M, \sigma_1 \boxtimes \sigma_2]$. We denote by π the canonical quotient map

$$\pi : P(\mathcal{O}_F) \to M(\mathcal{O}_F)$$

For any positive integer r we denote by $P(r)$ the inverse image of $P(\mathcal{O}_F/\mathfrak{P}_F^r)$ under the mod-$\mathfrak{P}_F^r$ reduction of $\mathrm{GL}_4(\mathcal{O}_F)$. Let τ_1 and τ_2 be $\mathrm{GL}_2(\mathcal{O}_F)$-typical representations occurring in σ_1 and σ_2 respectively. From the lemma 2.2.4 we get that the representation

$$\mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

has a complement say Γ in

$$\mathrm{res}_{\mathrm{GL}_4(\mathcal{O}_F)}\, i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$$

such that $\mathrm{GL}_4(\mathcal{O}_F)$-irreducible sub-representations of Γ are atypical.

Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining simple types (J_1, λ_1) and (J_2, λ_2) contained in σ_1 and σ_2 respectively. We may and do assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined by lattice chains $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1(0) = \mathcal{L}_2(0) = \mathcal{O}_F \oplus \mathcal{O}_F$. We deduce that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are contained in $M_2(\mathcal{O}_F)$.

The representation λ_i restricted to $U^{[n_i/2]+1}(\mathfrak{A}_i)$ is isomorphic to a direct sum $\oplus^{p^{k_i}} \psi_{\beta_i}$ (for the definition of ψ_β we refer to [BK93][1.1.6]) and hence λ_i is trivial on $U^{n_i+1}(\mathfrak{A}_i)$ for $i \in \{1, 2\}$. If $e(\mathfrak{A}_i) = 2$ then

$$K_2([(n_i+1)/2]+1) \subset U^{n_i+1}(\mathfrak{A}_i) \subset K_2([(n_i+1)/2]).$$

Notation 6.1. *We denote by* N_{λ_i} *the positive integer* $[(n_i+1)/2]$ *if* $e_i = 2$ *and* n_i *if* $e_i = 1$*. Let* N *be the positive integer* $\max\{N_{\lambda_1}, N_{\lambda_2}\}$*.*

The representation $\tau_1 \boxtimes \tau_2$ contains $M \cap K_4(N+1)$ in its kernel. We extend the representation $\tau_1 \boxtimes \tau_2$ of $M(\mathcal{O}_F / \mathfrak{P}_F{}^r)$ to a representation of $P(r)$ for $r \geq N+1$ via the inflation map

$$\pi_r : P(r) \to P(\mathcal{O}_F / \mathfrak{P}_F{}^r) \to M(\mathcal{O}_F / \mathfrak{P}_F{}^r).$$

Note that $P(r) \cap U$ and $P(r) \cap \bar{U}$ are contained in the kernel of this extension. The groups $P(r)$ for $r \geq N+1$ and $\tau = \tau_1 \boxtimes \tau_2$ satisfy the hypothesis for the lemma 2.2.5 hence we obtain

$$\mathrm{ind}_{P \cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) = \bigcup_{r \geq N+1} \mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2).$$

Hence to classify typical representations we need to examine the typical representations occurring as irreducible sub-representations of

$$\mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

for all $r \geq N+1$. As we did in the previous chapter, it is convenient to work with a smaller subgroup $P^0(r)$. Let r be a positive integer greater than N then $P^0(r) = K_4(r)\pi^{-1}(J_1 \times J_2)$ (We consider $J_1 \times J_2$ as a subgroup of $M(\mathcal{O}_F)$). The group $P^0(r)$ satisfies Iwahori decomposition with respect to P and M. Observe that $P^0(r) \cap U = P(r) \cap U$ and $P^0(r) \cap \bar{U} = P(r) \cap \bar{U}$ and since $r \geq N+1$ we get that $P^0(r) \cap M = J_1^0 \times J_2^0$. Now we apply lemma 2.2.6 for $J_1 = P(r)$ and $J_2 = P^0(r)$ and $\lambda = \lambda_1 \boxtimes \lambda_2$ and obtain an isomorphism

$$\mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P^0(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

We will need the decomposition of the representation

$$\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id})$$

for the proof of the theorem 6.1.2. We will prove the theorem by induction on r and the decomposition of the above representation is crucial. Let r be a positive integer greater than or equal to $N+1$. Let $K_I(r)$ be the group

$$K_4(r)\pi^{-1}(K_2(N+1) \times K_2(N+1)).$$

We note that $K_I(r)P^0(r+1) = P^0(r)$. It follows from Mackey decomposition that

$$\mathrm{res}_{K_I(r)}\, \mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(r+1)}^{K_I(r)}(\mathrm{id}).$$

Lemma 6.1.1. *The group $K_I(r)$ is a normal subgroup of $P_I^0(r)$ and $K_I(r+1)$ is a normal subgroup of $K_I(r)$.*

Proof. By definition of the groups $K_I(r)$ we have $K_I(r) \cap U = P^0(r) \cap U$ and $K_I(r) \cap \bar{U} = P^0(r) \cap \bar{U}$. To show the normality of $K_I(r)$ in $P^0(r)$ it is enough to verify that $P^0(r) \cap M$ normalizes the group $K_I(r)$. $P^0(r) \cap M$ normalizes the group $K_I(r) \cap U = U(\mathcal{O}_F)$ and $K_I(r) \cap \bar{U} = \bar{U}(\mathfrak{P}_F^r)$. The group $K_I(r) \cap M$ is a normal subgroup of $P^0(r) \cap M$ and hence $P^0(r) \cap M$ normalizes $K_I(r) \cap M$. This shows the first part of the lemma.

Since $K_I(r) \cap P = K_I(r+1) \cap P$, we have to check that $K_I(r) \cap \bar{U}$ normalizes the group $K_I(r+1)$. We note that $\bar{U}$ is abelian hence we have to check that the conjugations $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $K_I(r+1)$ for all $u^- \in K_I(r) \cap \bar{U}_I$, $j \in K_I(r+1) \cap M_I = K_I(r) \cap M_I$ and $u^- \in K_I(r+1) \cap U_I = U_I(\mathcal{O}_F)$. Let us begin with the element $u^- j (u^-)^{-1}$. We have $u^- j (u^-)^- = j\{j^{-1} u^- j (u^-)^{-1}\}$. Let

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1 \end{pmatrix}$$

be the block diagonal form of j and u^-; $J_1 \in K_2(N+1), J_2 \in K_2(N+1)$ and $U^- \in \varpi_F^r M_{2\times 2}(\mathcal{O}_F)$. The element $j^{-1} u^- j (u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_2 & 0 \\ J_2^{-1} U^- J_1 - U^- & 1_2 \end{pmatrix}$$

We note that the matrix $J_1^{-1} U^- J_1 - U^-$ belongs to $\varpi_F^{r+1} M_{2\times 2}(\mathcal{O}_F)$. This shows that $j^{-1} u^- j (u^-)^{-1} \in K_I(r+1) \cap \bar{U}_I$. Hence the element $u^- j (u^-)^-$ which can be rewritten as $j\{j^{-1} u^- j (u^-)^{-1}\}$ belongs to $K_I(r+1)$.

We now consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in its block matrix form as

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

where $U^+ \in M_{2\times 2}(\mathcal{O}_F)$. Now the conjugation $u^- u^+ (u^-)^{-1}$ in its block matrix from is as follows

$$\begin{pmatrix} 1_2 - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1_2 \end{pmatrix}.$$

Since $U^- U^+ U^- \in \varpi^{2r} M_{2\times 2}(\mathcal{O}_F)$ and $2r \geq r+1$, we conclude that $u^- u^+ (u^-)^{-1}$ belongs to $K_I(r+1)$. This ends the proof of this lemma. $\square$

From the above lemma and Iwahori decomposition for the group $K_I(r)$ we get that the inclusion $K_I(r) \cap \bar{U}$ in $K_I(r)$ induces an isomorphism of the quotient $K_I(r)/K_I(r+1)$ with

$$(K_I(r) \cap \bar{U})/(K_I(r+1) \cap \bar{U}). \tag{6.1}$$

The representation $\mathrm{ind}_{K_I(r+1)}^{K_I(r)}(\mathrm{id})$ splits as a direct sum of characters of $K_I(r)$ which are trivial on $K_I(r+1)$. We denote these characters by η_k for $1 \leq k \leq t$. The group $P^0(r)$ acts on these characters and let $Z(\eta_k)$ be the $P^0(r)$-stabilizer of the character η_k. We note that the trivial character id occurs with multiplicity one. From Clifford theory we get that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) = \mathrm{id} \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(r)}(U_{\eta_{n_k}}) \tag{6.2}$$

where η_{n_k} is a representative for the $P^0(r)$-orbit and $U_{\eta_{n_k}}$ is an irreducible representation of $Z(\eta_{n_k})$. Note that $Z(\eta_{n_k}) = (Z(\eta_{n_k}) \cap M)K_I(r)$.

The next step is to bound the group $Z(\eta_{n_k}) \cap M$ for some $\eta_{n_k} \neq \mathrm{id}$. Let u^- be an element of the group $K_I(r) \cap \bar{U}_I$. We represent the element u^- in its block form as

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

The map $u^- \mapsto \varpi_F^{-r} U^-$ gives us an isomorphism of $K_I(r) \cap \bar{U}_I$ with $M_2(\mathcal{O}_F)$. Further the map $u^- \mapsto \overline{\varpi_F^{-r} U^-}$ (we denote by $\overline{U}$ the class of $U \in M_2(\mathcal{O}_F)$ in the quotient $M_2(\mathcal{O}_F)/\varpi_F M_2(\mathcal{O}_F)$) gives an isomorphism of the quotient $(K_I(r) \cap \bar{U}_I)/(K_I(r+1) \cap \bar{U}_I)$ with $M_{2\times 2}(k_F)$. This is $M(\mathcal{O}_F)$-equivariant. We also have an $M(\mathcal{O}_F)$-equivariant isomorphism between the character group of $M_{2\times 2}(k_F)$ and $M_{2\times 2}(k_F)$ (see 3.0.14). We finally obtain an $M(\mathcal{O}_F)$-equivariant isomorphism

$$\widehat{K_I(r)/K_I(r+1)} \simeq M_{2\times 2}(k_F). \tag{6.3}$$

Note that the group $M(\mathcal{O}_F)$ acts through its quotient $M(k_F)$.

In order to calculate $Z(\eta_{n_k}) \cap M$ for some $\eta_{n_k} \neq \mathrm{id}$, we can as well calculate $Z_{J_1 \times J_2}(m)$ for some non-zero matrix m in $M_2(k_F)$. It will be useful to first recall the $Z_{M(k_F)}(m)$ for $m \neq 0$. We have the following possibilities

1. If m is a full rank matrix then $Z_{M(k_F)}(m) = \{(g, mgm^{-1}) \,|g \in \mathrm{GL}_2(k_F)\}$.
2. If m is not a full rank matrix then $Z_{M(k_F)}(m) = \{(g_1, g_2) \,|g_2 m = m g_1 \}$. Since m is non-zero and has a kernel, we can see that g_1 fixes the kernel hence the first projection of $Z_{M(k_F)}(m)$ is contained in a proper parabolic subgroup of $\mathrm{GL}_2(k_F)$. The conclusion in this case is symmetric for the second projection as well.

We denote by λ_s the representation $\lambda_1 \boxtimes \lambda_2$ of $J_1 \times J_2$. The representation λ_s occurs with multiplicity one in $\mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s)$. We denote by $U_r^0(\lambda_s)$ the complement of λ_s in $\mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s)$. Let $U_r(\lambda_s)$ be the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U_r^0(\lambda_s)).$$

Theorem 6.1.2. *Let $\#k_F > 3$. The $\mathrm{GL}_4(\mathcal{O}_F)$-irreducible sub-representations of $U_r(\lambda_s)$ are atypical for all $r \geq N+1$.*

Proof. We prove this theorem by induction on the integer $r \geq N+1$. The theorem is vacuously true for $r = N+1$ since $U_r(\lambda_s) = 0$. We suppose that the theorem is true for some positive integer r we will prove the same holds for $r+1$. We first note that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(N+1)}(\lambda_s) \simeq \mathrm{ind}_{P^0(r)}^{P^0(N+1)}\{\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) \otimes \lambda_s\}.$$

Now using the decomposition 6.2 we get that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(N+1)}(\lambda_s) \simeq \mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s) \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(N+1)}(\lambda_s \otimes U_{\eta_{n_k}}).$$

From the definition of $U_r^0(\lambda_s)$ we get that

$$U_{r+1}^0(\lambda_s) \simeq U_r^0(\lambda_s) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(N+1)}(\lambda_s \otimes U_{\eta_{n_k}}). \tag{6.4}$$

Now applying the induction functor to the maximal compact subgroup $\mathrm{GL}_4(\mathcal{O}_F)$ we have

$$U_{r+1}(\lambda_s) \simeq U_r(\lambda_s) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}}). \tag{6.5}$$

Let $\eta_{n_k} \neq \mathrm{id}$. We will show that the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}}) \tag{6.6}$$

are atypical for the component $[M, \sigma_1 \boxtimes \sigma_2]$. We choose to treat this case by case depending on the different classes of supercuspidal representations σ_1 and σ_2.

Case 1: We will first consider the case where both σ_1 and σ_2 are twists of level-zero cuspidal representations. From our assumptions $\lambda_i \simeq \chi_i \otimes \lambda_i'$ where χ_i is a character of $F^\times$ and λ_i' is the inflation of a cuspidal representation of $\mathrm{GL}_2(k_F)$ for $i \in \{1, 2\}$. Let m be the non-zero matrix associated to the non-trivial character η_{n_k}. We observe that $Z(\eta_{n_k}) \cap M$ is equal to $Z_{\mathrm{GL}_2(\mathcal{O}_F)\times\mathrm{GL}_2(\mathcal{O}_F)}(m)$. Hence mod $\mathfrak{P}_F$ reduction of $Z(\eta_{n_k}) \cap M$ satisfies one of the properties listed in (6.3). We know from the results of chapter three (see 3.0.16, 3.0.17) that any irreducible sub-representation of

$$\mathrm{res}_{Z(\eta_{n_k})}(\chi_1\lambda_1' \boxtimes \chi_2\lambda_2')$$

occur in

$$\mathrm{res}_{Z(\eta_{n_k})}(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'')$$

where $\lambda_1''\boxtimes\lambda_2''$ is the inflation of a non-cuspidal representation of $M(k_F)$. Hence we deduce that the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

occurs in the representation

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_2(\mathcal{O}_F)}\{(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'') \otimes U_{\eta_{n_k}}\}$$

which occurs in the representation

$$\mathrm{ind}_{P\cap\mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'').$$

Since $\lambda_1'' \boxtimes \lambda_2''$ is the inflation of a non-cuspidal representation of $M(k_F)$, the above representation occurs in

$$\mathrm{res}_{\mathrm{GL}_4(\mathcal{O}_F)}\, i_{P'}^{\mathrm{GL}_4(F)}(\pi)$$

where P' is a parabolic subgroup contained properly in P. This shows that the irreducible sub-representations of (6.6) are atypical.

Case 2: Let (J_1, λ_1) be defined by a simple strata $[\mathfrak{A}, n_1, 0, \beta_1]$ such that $[E_1 := F[\beta_1] : F] > 1$ and (J_2, λ_2) be such that $J_2 = \mathrm{GL}_2(\mathcal{O}_F)$ and λ_2 is $\chi \otimes \lambda_2'$ where χ is a character of $\mathrm{GL}_2(F)$ and λ_2' is the inflation of a cuspidal representation of $\mathrm{GL}_2(k_F)$. Let m be the matrix in $M_{2\times 2}(k_F)$ associated to the non-trivial character η_{n_k} in the isomorphism (6.3). We first consider the easier case when m is a non-zero matrix with rank one. We note that the second projection of $Z_{M(k_F)}(m)$ is contained in a proper parabolic subgroup say B of $\mathrm{GL}_2(k_F)$. Hence by lemma 3.0.16 we get that for any irreducible sub-representation ξ of $\mathrm{res}_B\,\lambda_2'$ there exists a $\mathrm{GL}_2(k_F)$-irreducible non-cuspidal representation λ_3 such that ξ occurs as a sub-representation of $\mathrm{res}_B\,\lambda_3$ and $\lambda_2' \not\simeq \lambda_3$. This shows that any irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

is contained in

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \chi\lambda_3') \otimes U_{\eta_{n_k}}).$$

for some irreducible representation λ_3' of $\mathrm{GL}_2(\mathcal{O}_F)$ obtained by inflating an irreducible representation λ_3 of $\mathrm{GL}_2(k_F)$. Hence for every irreducible sub-representation say γ of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

there exists λ_3' (obtained by inflating an irreducible non-cuspidal representation of $\mathrm{GL}_2(k_F)$) such that γ occurs as sub-representation of

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \chi\lambda_3').$$

The above representation is contained in

$$\mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\sigma_1 \boxtimes \sigma_2')$$

where σ_2' is an irreducible smooth representation of $\mathrm{GL}_2(F)$ which contains the type $(\mathrm{GL}_2(\mathcal{O}_F), \chi\lambda_3')$. Since λ_3' is the inflation of a non-cuspidal representation of $\mathrm{GL}_2(k_F)$, σ_2' is not a cuspidal representation hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical.

This paragraph concerns the case where m is a matrix of full rank. We begin with the observation that $Z(\eta_{n_k}) \cap M = Z_{J_1\times J_2}(\eta_{n_k})$. Let $[E_1 = F[\beta_1] : F] > 1$. The group J_1 contains a normal subgroup J_1^1 such that $J_1 = \mathcal{O}_{E_1}^{\times} J_1^1$ and $J_1/J_1^1 \simeq \mathcal{O}_{E_1}^{\times}/U^1(E_1)$ and J_1^1 is a pro-p subgroup (see [BK93, Chapter 3, 3.1.14]). Now we have to understand $Z_{\mathcal{O}_{E_1}^{\times} J_1^1 \times \mathrm{GL}_2(\mathcal{O}_F)}(m)$. The action of the group $(\mathcal{O}_{E_1}^{\times} J_1^1) \times \mathrm{GL}_2(\mathcal{O}_F)$ factors through its quotient

$$(U^0(E_1)J_1^1K_2(1)/K_2(1)) \times \mathrm{GL}_2(k_F).$$

We now have two different possibilities: $e_1 = 1$ and $e_1 = 2$.

If $e_1 = 1$ then we have $U^0(E_1)J_1^1K_2(1)/K_2(1) = U^0(E_1)/U^{e_1}(E_1) = k_{E_1}^{\times}$ where k_{E_1} is a quadratic extension of k_F. Now it is clear that

$$Z_{U^0(E_1)/U^{e_1}(E_1)\times \mathrm{GL}_2(k_F)}(m) \cap (\{\mathrm{id}\} \times U) = \{\mathrm{id}\}$$

where U is the unipotent radical of any Borel subgroup of $\mathrm{GL}_2(k_F)$.

If $e_1 = 2$ the group $U^0(E_1)J_1^1K_2(1)/K_2(1)$ is $k_F^{\times}X$ where X is a p-group. Now the second projection of $Z_{k_F^{\times}X\times \mathrm{GL}_2(k_F)}(m)$ is contained in the product of the center $k_F^{\times}$ and a p-group. Hence we conclude that the mod $\mathfrak{P}_F$-reduction of the image of the second projection of $Z_{J_1\times J_2}(m)$ is contained in a Borel subgroup say B of $\mathrm{GL}_2(k_F)$. If $\bar{B}$ is the opposite Borel subgroup then its its unipotent radical U satisfies the property that

$$Z_{k_F^{\times}X\times \mathrm{GL}_2(k_F)}(m) \cap (\{\mathrm{id}\} \times U) = \{\mathrm{id}\}.$$

Let H be a subgroup of $\mathrm{GL}_2(k_F)$ such that $H \cap U = \{\mathrm{id}\}$ and σ be a cuspidal representation of $\mathrm{GL}_2(k_F)$. For any irreducible sub-representation ξ of $\mathrm{res}_H(\sigma)$ we can find an irreducible non-cuspidal representation σ' of $\mathrm{GL}_2(k_F)$ such that $\mathrm{Hom}_H(\xi, \sigma') \neq 0$ and $\sigma' \not\simeq \sigma$. This is because Mackey decomposition shows that

$$\mathrm{Hom}_U(\mathrm{ind}_H^{\mathrm{GL}_2(k_F)}(\xi), \mathrm{id}) \neq 0$$

and hence $\mathrm{ind}_H^{\mathrm{GL}_2(k_F)}(\xi)$ cannot be a sum of cuspidal representations. This shows that for any irreducible sub-representation γ of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

there exists an irreducible representation λ_3 of $\mathrm{GL}_2(\mathcal{O}_F)$ obtained by inflating a non-cuspidal representation of $\mathrm{GL}_2(k_F)$ such that γ occurs in

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}}).$$

The above representation occurs in the representation

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}})$$

and we have

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}}) \subset \mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\sigma_1 \boxtimes \sigma_2')$$

the representation σ_2' is an irreducible smooth representation containing the type $(\mathrm{GL}_2(\mathcal{O}_F), \chi\lambda_3)$. hence γ is not typical representation. This shows that irreducible sub-representations of (6.6) are atypical.

Case 3: We now consider the case where both (J_1, λ_1) and (J_2, λ_2) are defined by simple strata $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ respectively such that $[E_i = F[\beta_i] : F] > 1$ where $i \in \{1, 2\}$. We have to look at the stabilizer $Z(\eta_{n_k}) \cap M = Z_{J_1 \times J_2}(m)$. We consider the possibilities $(e_1, e_2) = (1, 1)$; $(e_1, e_2) = (2, 1)$ and $(e_1, e_2) = (2, 2)$, the other case is similar.

Case 3.1: Here $e_1 = 1$ and $e_2 = 1$, $J_1^1 \times J_2^1$ is contained in the group $K_2(1) \times K_2(1)$. The group $J_1 \times J_2$ acts through its mod-$\mathfrak{P}_F$ reduction hence it acts through its quotient $k_{E_1}^\times \times k_{E_2}^\times$. We are reduced to bound the group $Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$. Let m be a matrix of rank one and $(a, b) \in Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$. Now $am = mb$ and let v be a vector in k_F^2 which is contained in the kernel of m. Now $mb(v) = 0$ implies that $b(v)$ is in the kernel of m. This shows that b has eigen-values in k_F hence we must have $b \in k_F^\times$, similarly $a \in k_F^\times$ and $am = mb$ implies that $a = b$ (since at least one of the entries of m is non-zero). If m is a matrix of full rank then $a = mbm^{-1}$ implies that $a \in mk_{E_2}^{-1}m^\times \cap k_{E_1}^\times$. Now $mk_{E_2}m^{-1} \cap k_{E_1}$ is a sub-field of k_{E_1} and there are two possibilities: either $mk_{E_2}m^{-1} \cap k_{E_1}$ is a proper sub-field or $mk_{E_2}m^{-1} = k_{E_1}$. The first one would imply that $a, b \in k_F^\times$ and $a = b$. We conclude that $Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$ has the form

$$\{(a, a) \mid a \in k_F^\times\}$$

or there exist a field isomorphism θ of k_{E_1} onto k_{E_2} such that

$$Z_{k_{E_1}^\times \times k_{E_2}^\times}(m) = \{(a, \theta(a)) \mid a \in k_{E_1}^\times\}.$$

In every possibility the mod $\mathfrak{P}_F$ reduction of $Z_{J_1^0 \times J_2^0}(m)$ is a subgroup of $\{(e, \theta(e) \mid e \in k_E^\times\}$ for some θ. Let $\chi_1 = (\theta \circ \chi_2)^{-1}$ and χ_2 be two non-trivial characters of J_1 and J_2 which are trivial on J_1^1 and J_2^1 respectively. Let σ_1' and σ_2' be two supercuspidal representations containing Bushnell-Kutzko types $(J_1, \lambda_1' = \lambda_1 \otimes \chi_1)$ and $(J_2, \lambda_2' = \lambda_2 \otimes \chi_2)$ respectively. If $(M, \sigma_1 \boxtimes \sigma_2)$ and $(M, \sigma_1' \boxtimes \sigma_2')$ are inertially equivalent for all χ_2 then we must have intertwining between (J_1, λ_1) and $(J_2, \lambda_2 \otimes \chi_2)$ (because (J_1, λ_1) and $(J_1, \lambda_1 \otimes \chi_1)$ cannot intertwine see [BK93, Chapter 5, Theorem 5.5.2(3)]) for any non-trivial character χ_2 of J_2^0/J_2^1. Now we use the intertwining implies conjugacy theorem [BK93, Chapter 6, 6.2.4] to get a $g \in \mathrm{GL}_2(F)$ such that $J_2 = gJ_1g^{-1}$ and $\lambda_1^g \simeq \lambda_2 \otimes \chi_2$. Since conjugacy is an equivalence relation we get that $(J_2, \lambda_2 \otimes \chi_2)$ are all conjugate for all non-trivial characters χ_2 of J_2/J_2^1. There exist two distinct non-trivial characters of $k_{E_2}^\times$ (since the cardinality of $k_{E_2}^\times$ is at least 3) hence we get a contradiction to the assumption that $(M, \sigma_1 \boxtimes \sigma_2)$ and $(M, \sigma_1' \boxtimes \sigma_2')$ are inertially equivalent. Moreover by definition we have

$$\mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1' \boxtimes \lambda_2').$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \lambda_2) \boxtimes U_{\eta_{n_k}}) \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1' \boxtimes \lambda_2') \boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical.

Case 3.2: Let us consider the case where $e_1 = 1$ and $e_2 = 2$. The group $J_1 \times J_2$ now acts via the quotient $k_{E_1}^\times \times U^0(E_2)J_2^1K_2(1)/K_2(1)$. Let m be the matrix associated to the character η_{n_k}. If m has rank one then every element of the first projection of $Z_{k_{E_1}^\times \times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ is contained in $k_F^\times$. If m is a full rank matrix then for all $(a, b) \in Z_{k_{E_1}^\times \times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ we have $a = mbm^{-1}$. Note that $U^0(E_2)J_2^1K_2(1)/K_2(1)$ is a product $k_F^\times X$ where X is a p-group. This shows that the first projection is contained in $k_F^\times$. In each case the first projection of $Z_{k_{E_1}^\times \times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ is contained in $k_F^\times$. Let χ be a character of $k_{E_1}^\times$ which is trivial on $k_F^\times$. Such a character exists since the cardinality of $k_{E_1}^\times/k_F^\times$ is $q+1 \geq 3$. Let σ_1' be a cuspidal representation of $\mathrm{GL}_2(F)$ containing the type $(J_1, \lambda_1 \otimes \chi)$. We note that σ_1 and σ_2 are not inertial twist of each other (see [BK93, Chapter 5, Theorem 5.5.2(3)]). Hence $(M, \sigma_1 \boxtimes \sigma_2)$ and $(M, \sigma_1' \boxtimes \sigma_2)$ are not inertially equivalent. Moreover we have

$$\mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1' \boxtimes \lambda_2).$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \lambda_2) \boxtimes U_{\eta_{n_k}}) \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1' \boxtimes \lambda_2) \boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical for $s = [M, \sigma_1 \boxtimes \sigma_2]$.

Case 3.3: We are left with the case where $e_1 = 2$ and $e_2 = 2$. The group $(J_1 \times J_2)$ acts through its quotient

$$U^0(E_1)J_1^1K_2(1)/K_2(1) \times U^0(E_2)J_2^1K_2(1)/K_2(1).$$

We write this quotient as $k_F^\times X_1 \times k_F^\times X_2$ where X_1 and X_2 are two p-groups. The group $X_1 \times X_2$ is a p-group and there is a decreasing filtration of $M_2(k_F)$ by sub-spaces $\mathcal{F}^i$ such that $X_1 \times X_2$ acts trivially on $\mathcal{F}^i/\mathcal{F}^{i+1}$ and $\cap_i \mathcal{F} = 0$. Let k be the largest positive integer such that $m \in \mathcal{F}^k$. Let $\bar{m}$ be the image of m in $\mathcal{F}^k/\mathcal{F}^{k+1}$. If $(ax_1, bx_2) \in Z_{k_F^\times X_1, k_F^\times X_2}(m)$ then (ax_1, bx_2) fixes the element $\bar{m}$. The group $k_F^\times \times k_F^\times$ acts on $M_2(k_F)$ by the character ψ given by $\psi(m) = bma^{-1}$. The group $k_F^\times \times k_F^\times$ has cardinality relatively prime to p. Hence we get that the action of $k_F^\times \times k_F^\times$ on $\mathcal{F}^k/\mathcal{F}^{k+1}$ decomposes as a direct sum of isomorphic copies of ψ. From this we conclude that $a = b$.

Let η be a non-trivial character of the group $J_1/J_1^1 = k_F^\times$. Let σ_1' and σ_2' be two cuspidal representations containing the Bushnell-Kutzko types $(J_1, \lambda_1 \otimes \eta)$ and $(J_1, \lambda_1 \otimes \eta^{-1})$. If the pairs $(M, \sigma_1 \boxtimes \sigma_2)$ and $(M, \sigma_1' \boxtimes \sigma_2')$ are inertially equivalent for every η then we have (J_1, λ_1) and $(J_2, \lambda_2 \otimes \eta^{-1})$ must intertwine and hence they should be G-conjugate which implies that $(J_2, \lambda_2 \otimes \eta^{-1})$ are all G conjugate. Now by our assumption that $\#k_F > 3$, we can find two distinct non-trivial characters of $k_F^\times$ which is a contradiction by [BK93, Chapter 5, Theorem 5.5.2(3)]. Hence there is a non-trivial character η of J_1^0/J_1^1 such that $[M, \sigma_1 \boxtimes \sigma_2]$ and $[M, \sigma_1' \boxtimes \sigma_2']$ are distinct inertial classes and

$$\mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1}).$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \lambda_2) \boxtimes U_{\eta_{n_k}}) \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1\eta \boxtimes \lambda_2\eta^{-1}) \boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical.

Now by using induction on the positive integer r we prove that $U_r(\lambda_s)$ does not contain any atypical representations. □

The previous theorem reduces the problem of classifying typical representations for the component $s = [M, \sigma_1 \boxtimes \sigma_2]$ to classifying typical representations occurring in the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \tag{6.7}$$

where λ_s is a $J_1 \times J_2$ representation isomorphic to $\lambda_1 \boxtimes \lambda_2$. The representation (6.7) may still contain atypical representations. To examine this we need the Bushnell-Kutzko semi-simple type (J_s, λ_s) for the component s. To write explicitly the structure of the semi-simple type (J_s, λ_s) we need to recall the characteristic polynomial associated to the strata $[\mathfrak{A}, n, 0, \beta]$. For any given strata $[\mathfrak{A}, n, 0, \beta]$ in $M_2(F)$, define g to be $\gcd(n, e)$. The element $\varpi_F^{n/g}\beta^{e/g}$ lies in the ring $\mathfrak{A}$. Since we assume that our hereditary order $\mathfrak{A}$ is defined by a lattice chain $\mathcal{L}$ such that $\mathcal{L}(0) = \mathcal{O}_F \oplus \mathcal{O}_F$, the element $\varpi_F^{n/g}\beta^{e/g}$ belongs to (the maximal hereditary order containing $\mathfrak{A}$) $M_2(\mathcal{O}_F)$. The characteristic polynomial associated to the class $\overline{\varpi_F^{n/g}\beta^{e/g}}$ in $M_2(k_F)$ will be called the characteristic polynomial associated to the strata $[\mathfrak{A}, n, 0, \beta]$.

Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining the maximal simple types (J_1^0, λ_1) and (J_2^0, λ_2) respectively. In our situation the characteristic polynomials associated to the above strata are powers of irreducible polynomials ϕ_1 and ϕ_2 respectively (see [BK93][2.3.11]). The underlying compact group J_s of the semi-simple type depends on the data $n_1/e_1, n_2/e_2$ and ϕ_1 and ϕ_2. We have two possibilities:

1. $n_1/e_1 \neq n_2/e_2$ or $n_1/e_1 = n_2/e_2$ but $\phi_1 \neq \phi_2$
2. $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$.

In the first case σ_1 and σ_2 are said to be completely distinct. In the second case σ_1 and σ_2 are said to have common approximation. We will classify typical representations in two important cases: the first case when σ_1 and σ_2 have a common approximation of level zero, they are also called homogenous inertial classes and the second case where σ_1 and σ_2 are completely distinct.

6.2 Homogenous inertial classes

In this section we assume that $\#k_F > 3$. So far we have shown that typical representations occur as sub-representations of

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

This may not (although we believe that this is never the complete classification when σ_1 and σ_2 have common approximation) be the complete classification in the case where σ_1 and σ_2 admit common approximation. In this section we treat the case where $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$; and σ_1 and σ_2 have level-zero common approximation. This means that the simple characters of σ_1 and σ_2 intertwine and hence they are conjugate. We may as well assume that $\mathfrak{A}_1$ is equal to $\mathfrak{A}_2$, n_1 is equal to n_2, β_1 is equal to β_2. We henceforth assume that σ_1 and σ_2 contain the simple strata $[\mathfrak{A}, n, 0, \alpha]$. Moreover the simple characters defining σ_1 and σ_2 are the same. We denote by $E = F[\alpha]$ and $[E : F] > 1$. We refer to [BK99][Section 4.3] for further details.

Let σ_1 and σ_2 be two supercuspidal representations of $\mathrm{GL}_2(F)$ containing the simple stratum $[\mathfrak{A}, n, 0, \alpha]$. Let (J^0, λ_1) and (J^0, λ_2) be Bushnell-Kutzko type,s associated to $[\mathfrak{A}, n, 0, \alpha]$, contained in σ_1 and σ_2 respectively. We also define a non-negative integer $t = [n/2]$. After twisting by a character χ of $\mathrm{GL}_4(F)$ we may assume that α is minimal in the sense [BK93][1.4.14]. The group J_s in the Bushnell-Kutzko type (J_s, λ_s) for the component $s = [M, \sigma_1 \boxtimes \sigma_2]$ is given by

$$\begin{pmatrix} J^0 & \mathfrak{J}^0 \\ \mathfrak{H}^1 & J^0 \end{pmatrix}.$$

We refer to [BK99][Section 7.2] for this construction. We also refer to the article [Blo06][Corollaire 1] for an exposition. It follows from the minimality of β that

$$\mathfrak{H}^1 = \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} \text{ and } \mathfrak{J}^0 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}.$$

We refer to [BK93][Definition 3.1.7] for the definition of the lattices $\mathfrak{H}^1$ and $\mathfrak{J}^0$. From the above description the group J_s is of the form

$$J_s = \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t} \\ \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}.$$

The representation $\lambda_1 \boxtimes \lambda_2$ of $J^0 \times J^0$ extends to a representation of J_s such that $J_s \cap U$ and $J_s \cap \bar{U}$ are contained in the kernel of the extension. We denote by λ_s the extension of $\lambda_1 \boxtimes \lambda_2$. The pair (J_s, λ_s) is the Bushnell-Kutzko semi-simple type for the component s.

6.2.1 The complete classification when E is unramified

We first understand the case when E is an unramified extension of F. In particular $\mathfrak{A} = M_2(\mathcal{O}_F)$.

Recall that the compact groups $P^0(n+1)$ and $P^0(t+1)$ represented in block form are as follows:

$$\begin{pmatrix} J^0 & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}$$

and

$$\begin{pmatrix} J^0 & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{(n+1)} & J^0 \end{pmatrix}$$

respectively. We also define an auxiliary subgroup J'_s:

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}.$$

Lemma 6.2.1. *The representation* $\mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s)$ *is an irreducible representation of* $P^0(t+1)$.

Proof. The double coset representatives for $J'_s \backslash P^0(t+1)/J'_s$ can be chosen from $U(\mathcal{O}_F) = P^0(t+1) \cap U$. Let u^+ be a coset representative represented in the block diagonal form as

$$u^+ = \begin{pmatrix} \mathrm{id} & U \\ 0 & \mathrm{id} \end{pmatrix}.$$

Suppose T is a non-zero operator in the space

$$\mathrm{Hom}_{J'_s \cap (J'_s)^{u^+}}(\lambda_s, \lambda_s^{u^+}).$$

The operator T satisfies the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + UC & -UCU \\ C & -CU + \mathrm{id} \end{pmatrix} T(v).$$

Further we take C in $\mathfrak{P}_{\mathfrak{A}}^{t+1}$. Now we get that

$$\psi_\alpha(\mathrm{id} + UC)\psi_\alpha(-CU + \mathrm{id}) = 1.$$

Hence we have $\psi_{(\alpha U - U\alpha)}(1 + C) = 1$. This shows us that U belongs to $\mathfrak{N}_{-t}(\alpha, \mathfrak{A})$ which is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}$ (See [BK93, Remark page 42]). Since $J'_s \cap U$ is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{[(n+1)/2]}$, u^+ is equivalent to id. This shows the lemma with Mackey criterion. □

We observe that $P^0(t+1)$ can be decomposed as $(J'_s)P^0(n+1)$ and Mackey decomposition applied to this decomposition shows that the space of intertwining operators

$$\mathrm{Hom}_{P^0(t+1)}(\mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s), \mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s))$$

has dimension 1. Now we need to find the complement of the representation $\pi_2 := \mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s)$ in $\pi_1 = \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s)$.

Let f be an element of the representation π_1. Let $I(f)$ be a function defined by the equation

$$I(f)(p) = \int_{u^- \in \ P^0(t+1) \cap \bar{U}} f(u^- p) du^-$$

for all $p \in P^0(t+1)$.

Lemma 6.2.2. *The operator I is a non-zero intertwining operator between π_1 and π_2.*

Proof. Let $p \in P^0(t+1)$ and $u^+ \in J'_s \cap U$. It is enough to show that $I(f)(u^+p) = \lambda_s(u^+)I(f)(p) = I(f)(p)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}, \; u^+ = \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively. Now observe that

$$\int_{u^- \in \; P^0(t+1)\cap \bar{U}} f(u^-u^+p)du^- = \int_{u^- \in \; P^0(t+1)\cap \bar{U}} f(u^-u^+(u^-)^{-1}u^-p)du^- \tag{6.8}$$

The above integral can be written as

$$\int_{u^- \in \; P^0(t+1)\cap \bar{U}} \psi_{(\alpha U^+ - U^+\alpha)}(1+U^-)f(u^-p)du^-. \tag{6.9}$$

Since $U^+ \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}$ we get that $\alpha U^+ - U^+\alpha \in \mathfrak{P}_{\mathfrak{A}}^{-t}$ since valuation of α with respect to the filtration $\mathfrak{P}_{\mathfrak{A}}^{k}$, $k \in \mathbb{Z}$, is $-n$. This shows that $I(f) \in \pi_2$. To see that I is non-zero we can take a function $f \in \pi_1$ which is constant on $P^0(t+1) \cap \bar{U}$ and observe that $I(f)(1_4) \neq 0$. □

Notation 6.2. *For an element $u^+ \in U$ the 2×2 block matrix form is always denoted by*

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}.$$

Similarly, for any element $u^- \in \bar{U}$ the 2×2 block matrix form is always denoted by

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

We note that I is surjective and $\ker(I)$ is the complement of π_1 in π_2. If f is in the kernel of I then the above integral vanishes for all $U^+ \in \mathfrak{A}$. Hence the representation $\ker I$ is contained in the space $S(\alpha)$ given by

$$\left\{ f \in \pi_1 | \int_{u^- \in \; P^0(t+1)\cap \bar{U}} \psi_{(\alpha U^+ - U^+\alpha)}(1+U^-)f(u^-)du^- = 0 \; \forall \; \mathrm{U}^+ \in \mathfrak{A} \right\}. \tag{6.10}$$

Lemma 6.2.3. *Let W be an irreducible sub-representation of* $\ker(I)$. *Then irreducible sub-representations of* $\mathrm{ind}_{P^0(t+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(W)$ *are atypical representations.*

Proof. We first define a subgroup

$$H_t = \begin{pmatrix} 1+\mathfrak{P}_{\mathfrak{A}}^{t+1} & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & 1+\mathfrak{P}_{\mathfrak{A}}^{t+1} \end{pmatrix}.$$

of $P^0(t+1)$. Now H_t is a normal subgroup of $P^0(t+1)$ and $P^0(t+1) = H_t P^0(n+1)$. Mackey decomposition gives us

$$\begin{aligned} &\mathrm{res}_{H_t} \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s) \\ &\simeq \mathrm{ind}_{P^0(n+1)\cap H_t}^{H_t}(\mathrm{res}_{P^0(n+1)\cap H_t} \lambda_s) \\ &\simeq \mathrm{ind}_{P^0(n+1)\cap H_t}^{H_t}((\psi_\alpha \boxtimes \psi_\alpha)^{\dim \lambda_s}). \end{aligned}$$

The character ψ_α is defined in [BK93][1.1.6]. We first describe the irreducible sub-representations of $\mathrm{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$.

The group $H_t \cap M$ acts by the character $\psi_\alpha \boxtimes \psi_\alpha$ on the representation $\mathrm{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$. Note that $(H_t \cap \bar{U})P^0(n+1) = P^0(t+1)$ and by Mackey decomposition we have

$$\mathrm{res}_{H_t\cap\bar{U}} \mathrm{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha) = \mathrm{ind}_{P^0(n+1)\cap\bar{U}}^{H_t\cap\bar{U}}(\mathrm{id}).$$

Hence the restriction splits as distinct characters of $(H_t\cap\bar{U})/(P^0(n+1)\cap\bar{U})$. The map $u^- \mapsto U^-$ gives us the isomorphism

$$(H_t\cap\bar{U})/(P^0(n+1)\cap\bar{U}) \simeq \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}.$$

The group of characters of $\mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$ is identified in the standard (as in [BK93][1.1.6]) way with the group $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$.

We can choose a basis $\{f_V | V \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}\}$ for the space of functions $\mathrm{ind}_{P^0(n+1)\cap\bar{U}}^{H_t\cap\bar{U}}(\mathrm{id})$ such that

$$\begin{pmatrix} \mathrm{id} & 0 \\ U_1 & \mathrm{id} \end{pmatrix} f_V = \psi_V(U_1) f_V$$

where ψ_{U^-} is the character of $\mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$ corresponding to $U^- \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. Let e_X be the characteristic function for the coset $X + \mathfrak{P}_{\mathfrak{A}}^{n+1}$. The function f_V can be written as

$$f_V = \sum_{X \in \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X) e_X.$$

Let $u^+ \in H_t \cap U$. We first observe that

$$u^+ e_X = \psi_\alpha(1+U^+X)\psi_\alpha(1-XU^+)e_X = \psi_{[\alpha,U^+]}(1+X)e_X.$$

Now

$$\begin{aligned} & u^+ f_V \\ = & \sum_{X \in \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X) u^+ e_X \\ = & \sum_{X \in \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X)\psi_{[\alpha,U^+]}(1+X)e_X = f_{V+[\alpha,U^+]}. \end{aligned}$$

Any irreducible sub-representation W of $\mathrm{ind}_{H_t \cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$ contains a character ψ_V for some $V \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. Now consider the space spanned by the set

$$\{f_V, f_{V+[\alpha,U^+]} \mid \forall\, U^+ \in \mathfrak{A}\}.$$

By the observation on the action of the element u^+, we get that the span is stable under the action of $H_t \cap U$ and by construction it is stable under the action of $H_t \cap \bar{U}$. Moreover the group $H_t \cap M$ acts by a character $\psi_\alpha \boxtimes \psi_\alpha$. Hence $W = < f_V, f_{V+[\alpha,U^+]} | \forall\, U^+ \in \mathfrak{A} >$. Let us denote the representation W by $W(V)$ where V is a coset representative for $\mathfrak{P}_{\mathfrak{A}}^{-n}/\operatorname{img}([\alpha,\ ])$ This description will be sufficient for our present purpose.

Now we return to the proof of the lemma 6.2.3. The subgroup H_t is a normal subgroup of $P^0(t+1)$. Using Clifford theory we can write $\ker(I)$ as a direct sum of irreducible sub-representations:

$$\Gamma := \mathrm{ind}_{Z_{P^0(t+1)}(W(V))}^{P^0(t+1)}\{\widetilde{W(V)}\}$$

for some $V \in \mathfrak{P}_{\mathfrak{A}}^{-n}$ and $\widetilde{W(V)}$ is an irreducible representation of $Z_{P^0(t+1)}(W(V))$. (The representation $\widetilde{W(V)}$ is the isotopic component of $W(V)$ in the representation $\mathrm{ind}_{H_t \cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$. But we do not make use of this.). Let $s_{E/F}$ be a tame co-restriction of A with respect to E (see [BK93][Definition 1.3.3]). Suppose $s_{E/F}(V)$ belongs to $\mathfrak{P}_E^{-t}$ for such a V then $V = [\alpha, U]$ (since the kernel of the map induced by $s_{E/F}$ on $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$ is given by the image of $[\alpha, \cdot]$ see [BK93][corollary1.4.10]) and hence a contradiction to the identity (6.10). This shows $s_{E/F}(V)$ does not belong to $\mathfrak{P}_E^{-t}$ for all V such that $W(V)$ is contained in Γ.

Note that $(Z_{P(t+1))}(W(V)))(P^0(t+1) \cap P)$ is equal to $P^0(t+1)$. Mackey decomposition shows that

$$\mathrm{res}_{P^0(t+1)\cap P}\ \mathrm{ind}_{Z_{P^0(t+1)}(W(V))}^{P^0(t+1)}\{\widetilde{W(V)}\} \simeq \mathrm{ind}_{Z_{P^0(t+1)}(W(V))\cap P}^{P^0(t+1)\cap P}\widetilde{W(V)}.$$

It follows from Frobenius reciprocity that $\mathrm{Hom}_{P^0(n+1)}(\Gamma, \lambda_s) \neq 0$. In particular $\mathrm{Hom}_{P^0(n+1)\cap P}(\Gamma, \lambda_s) \neq 0$. We note that $P^0(n+1) \cap P$ is equal to

$P^0(t+1)\cap P$ and as a consequence $\mathrm{Hom}_{P^0(t+1)\cap P}(\Gamma,\lambda_s)\neq 0$. Again applying Frobenius reciprocity we note that

$$\mathrm{Hom}_{Z_{P^0(t+1)}(W(V))\cap P}(\widetilde{W(V)},\lambda_s)\neq 0.$$

We will construct in the next paragraph another typical representation $\lambda_{s'}$ such that $s'\neq s$ and

$$\mathrm{res}_{Z_{P^0(t+1)}(W(V))\cap P}(\lambda_s)\simeq \mathrm{res}_{Z_{P^0(t+1)}(W(V))\cap P}(\lambda_{s'}). \tag{6.11}$$

The group $P^0(n+1)\cap\bar U$ acts trivially on Γ and on $\lambda_{s'}$. The condition (6.11) now shows that

$$\mathrm{Hom}_{P^0(n+1)}(\Gamma,\lambda_{s'})$$

is non-zero. Which shows that the representation Γ occurs in the representation

$$\mathrm{ind}_{P^0(t+1)}^{P^0(n+1)}(\lambda_{s'}).$$

Hence the representation $\mathrm{ind}_{P^0(t+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma)$ is atypical for the component s.

Now we give the construction of $\lambda_{s'}$ satisfying equation (6.11). We first bound the group $Z_{P^0(t+1)}(W(V))\cap M$. The elements of $Z_{P^0(t+1)}(W(V))\cap M$ act on the characters ψ_V such that

$$\begin{pmatrix} j_1 & 0\\ 0 & j_2\end{pmatrix}\psi_V=\psi_{j_1Vj_2^{-1}}=\psi_{V+[\alpha,U^+]}$$

for some $U^+\in\mathfrak{A}$. We now get that

$$j_1Vj_2^{-1}+\mathfrak{P}_{\mathfrak{A}}^{-t}=V+[\alpha,U^+]+\mathfrak{P}_{\mathfrak{A}}^{-t}. \tag{6.12}$$

Recall that $V\in\mathfrak{P}_{\mathfrak{A}}$, valuation of α with respect to the filtration $\mathfrak{P}_{\mathfrak{A}}^k$, $k\in\mathbb{Z}$, is $-n$ and $U^+\in\mathfrak{A}$. Applying a tame co-restriction $s_{E/F}$ on both sides of (6.12) and taking j_1 and j_2 in $\mathcal{O}_E^\times$ we get that

$$j_1s_{E/F}(V)j_2^{-1}+\mathfrak{P}_E^{-t}=s_{E/F}(V)+\mathfrak{P}_E^{-t}.$$

The above equation implies that $j_1\equiv j_2$ modulo $\mathfrak{P}_E$.

The representation λ_s is $\lambda_1\boxtimes\lambda_2$ where (J^0,λ_i) are typical representations of σ_i for $i\in\{1,2\}$. Now J^0/J^1 is isomorphic to $k_E^\times$. Since $\#k_E>3$ we can choose a nontrivial character η of $k_E^\times$ such that the multi-set of types $\{\lambda_1,\lambda_2\}$ and $\{\lambda_1\eta,\lambda_2\eta^{-1}\}$ are distinct. Let $\lambda_{s'}=\lambda_1\eta\boxtimes\lambda_2\eta^{-1}$. The pair (J_s,λ_s') is a type for the component $s'=[\mathrm{GL}_2(F)\times\mathrm{GL}_2(F),\sigma_1'\boxtimes\sigma_2']$ where σ_1' and σ_2' are supercuspidal representations containing $(J^0,\lambda\eta)$ and $(J^0,\lambda\eta^{-1})$ respectively. Moreover we observe that $s'\neq s$ and

$$\mathrm{res}_{Z_{P^0(t+1)}(W(V))\cap P}(\lambda_s)\simeq \mathrm{res}_{Z_{P^0(t+1)}(W(V))\cap P}(\lambda_{s'}).$$

□

From the above lemma typical representations for the component s occur as sub-representations of

$$\mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Since J_s contains the group J'_s we have

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id}) \otimes \lambda_s\} \simeq \mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Frobenius reciprocity implies that $\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id})$ contains id with multiplicity one. Let $\epsilon(s)$ be the complement of id in $\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id})$.

Lemma 6.2.4. *Irreducible sub-representations of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\epsilon(s) \otimes \lambda_s\}$$

are not typical representations.

Proof. The subset $\mathfrak{P}_E \cap \mathfrak{P}_{\mathfrak{A}}^{t+1}$ is an ideal in $\mathcal{O}_E$ and $\mathfrak{P}_E \cap \mathfrak{P}_{\mathfrak{A}}^{t+1} = \mathfrak{P}_E^{t+1}$. We first define a sequence of subgroups of J_s as follows:

$$H_i = \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t} \\ \mathfrak{P}_E^i + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix} \ \forall\ 1 \leq i \leq t+1.$$

We note that that $H_1 = J_s$ and $H_{t+1} = J'_s$. Now

$$\mathrm{ind}_{H_{(i+1)}}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) = \mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \oplus \mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\epsilon_i(s) \otimes \lambda_s).$$

Where $\epsilon_i(s)$ is the complement of id in the representation $\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id})$. If $\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ admits a complement of $\mathrm{ind}_{H_1}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ containing only atypical representations then we show the same for $i+1$. For this we verify that irreducible sub-representations of $\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\epsilon_i(s) \otimes \lambda_s)$ are atypical for the component s. Hence by induction we show the above lemma. Consider the representation

$$\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}).$$

The group $H_i \cap U$ acts trivially on this representation. To see this let u^+ be an element of $H_i \cap U$. We can choose coset representatives u^- for $H_i/H_{(i+1)}$ from $H_i \cap \bar{U}$. Let u^+ and u^- represented in their block form be

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}$$

respectively. Moreover we may assume that $U^- \in \mathfrak{P}_E^i$. We observe that $(u^-)^{-1}u^+u^-$ is of the form

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ -U^-U^+U^- & 1_2 + U^-U^+ \end{pmatrix}$$

and this is clearly in the group $H_{(i+1)}$. Hence u^+ acts trivially on $\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id})$.

Define $S(i) = H_i \cap \bar{P}$ where $\bar{P}$ is the opposite group of P with respect to the Levi-subgroup M. Observe that

$$\mathrm{res}_{H_i \cap \bar{U}} \mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}) = \oplus \eta_i$$

where η_i are distinct characters of $(H_i \cap \bar{U})/(H_{(i+1)} \cap \bar{U})$. Now the group $S(i) \cap M$ acts on these characters and splits them into two different orbits id and the rest of the characters. Applying Clifford theory we have

$$\mathrm{res}_{S(i)} \mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}) = \mathrm{id} \oplus \mathrm{ind}_{Z_{S(i)}(\eta)}^{S(i)}(U_\eta)$$

where η is a nontrivial character of $(H_i \cap \bar{U})/(H_{(i+1)} \cap \bar{U})$ and U_η is an irreducible representation of the group $Z_{S(i)}(\eta)$. This decomposition extends to a representation of H_i.

For all $U^- \in \mathfrak{P}_E^i$, the element

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}$$

is contained in the group $H_i \cap \bar{U}$. Now the map $U^- \mapsto u^-$ gives us an isomorphism

$$\mathfrak{P}_E^i/\mathfrak{P}_E^{i+1} \simeq (H_i \cap \bar{U})/(H_{i+1} \cap \bar{U}).$$

The above isomorphism is $\mathcal{O}_E^\times \times \mathcal{O}_E^\times$ (considered as a subgroup of M) equivariant.

Let

$$\begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix}$$

be an element in $Z_{S(i)}(\eta) \cap M$ and j_1, j_2 belong to $\mathcal{O}_E^\times$. Since η is non-trivial we have $j_1 \equiv j_2$ modulo $\mathfrak{P}_E$. As in the previous lemma we can construct another component s' such that $s' \neq s$ and

$$\mathrm{res}_{Z_{S(i)}(\eta)}(\lambda_s) = \mathrm{res}_{Z_{S(i)}(\eta)}(\lambda_{s'}).$$

This shows us that the irreducible sub-representations of

$$\mathrm{ind}_{H_i}^{\mathrm{GL}_2(\mathcal{O}_F)}(\mathrm{ind}_{Z_{S(i)}(\eta)}^{S(i)}(U_\eta) \otimes \lambda_s) \simeq \mathrm{ind}_{H_i}^{\mathrm{GL}_2(\mathcal{O}_F)}(\epsilon_i(s) \otimes \lambda_s)$$

are not typical representations. Hence we show the lemma. □

With this discussion we conclude that all typical representations for the component s occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

6.2.2 The complete classification when E is ramified

Although ideas in this section are essentially inspired from previous section, we have to do additional work at various instances to finish the complete classification of typical representations when $e(\mathfrak{A}) = 2$.

Let $\mathfrak{M}$ and $\mathfrak{J}$ be two hereditary orders corresponding to the lattice chains $\mathcal{L}_1(i) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^i$ and $\mathcal{L}_2(2i) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^i, \mathcal{L}_2(2i+1) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^{i+1}$ respectively. Let $[\mathfrak{J}, 2n-1, 0, \alpha]$ be a simple strata contained in σ_1 and σ_2. We have $v_{\mathfrak{J}}(\alpha) = -(2n-1)$ (the valuation given by the filtration $\mathfrak{P}_{\mathfrak{J}}^k$, $k \in \mathbb{Z}$) and from the inclusions

$$\mathfrak{P}_{\mathfrak{M}}^{-(n-1)} \subset \mathfrak{P}_{\mathfrak{J}}^{-(2n-2)} \subset \mathfrak{P}_{\mathfrak{J}}^{-(2n-1)} \subset \mathfrak{P}_{\mathfrak{A}}^{-n}$$

we get that $v_{\mathfrak{M}}(\alpha) = -n$. Let $s_{E/F}$ be a tame co-restriction map. Now

$$s_{E/F}(\mathfrak{P}_{\mathfrak{M}}^i) = s_{E/F}(\varpi_{\mathfrak{J}}^{2i}\mathfrak{M}) = \varpi_E^{2i+r}\mathcal{O}_E$$

where r is given by $s_{E/F}(\mathfrak{M}) = \mathfrak{P}_E^r$. Hence we have a sequence

$$\frac{\mathfrak{M}}{\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{M}}^{-n}}{\mathfrak{P}_{\mathfrak{M}}^{-(n-1)}} \xrightarrow{s_{E/F}} \frac{\mathfrak{P}_E^{-2n+r}}{\mathfrak{P}_E^{-2(n-1)+r}} \tag{6.13}$$

Lemma 6.2.5. *The sequence (6.13) is exact.*

Proof. The composition of the maps

$$\frac{\mathfrak{M}}{\mathfrak{P}_{\mathfrak{M}}} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{M}}^{-n}}{\mathfrak{P}_{\mathfrak{M}}^{-(n-1)}} \xrightarrow{\varpi_F^n} \frac{\mathfrak{M}}{\mathfrak{P}_{\mathfrak{M}}}$$

is given by a map $m \mapsto [\overline{\varpi_F^n\alpha}, m]$ from $M_2(k_F)$ to $M_2(k_F)$. Here $\overline{\varpi^n\alpha}$ is the class of $\varpi_F^n\alpha$ in $\mathfrak{M}/\mathfrak{P}_{\mathfrak{M}}$. We have $\varpi_F^{2n}\alpha^2 = \varpi_{\mathfrak{J}}^{4n}\alpha^2 = \varpi_F(\varpi_{\mathfrak{J}}^{4n-2}\alpha^2)$ and $\varpi_{\mathfrak{J}}^{4n-2}\alpha^2 \in \mathfrak{J}$. This shows that the $\varpi_F^n\alpha$ is a nontrivial nilpotent matrix. The dimension of the commutator of a nontrivial nilpotent element is 2. Now $\mathcal{O}_E$ is in the kernel and we observe that $\mathcal{O}_E/\varpi_F\mathcal{O}_E$ is a 2 dimensional vector space over k_F. This shows that kernel is exactly $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. Since the image of $[\alpha,\ ]$ in the sequence (6.13) is of dimension 2 we get that $\operatorname{img}[\alpha,\ ] = \ker s_{E/F}$. This concludes the proof of the lemma. □

The proof of the above lemma shows that $\mathfrak{N} = \{u \in \mathfrak{M} \mid [\alpha, u] \in \mathfrak{P}_{\mathfrak{M}}^{-(n-1)}\} = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We denote by H_1, H_2 the groups

$$\begin{pmatrix} J^0 & \mathfrak{M} \\ \mathfrak{P}_{\mathfrak{M}}^{n+1} & J^0 \end{pmatrix} \text{ and } \begin{pmatrix} J^0 & \mathfrak{M} \\ \mathfrak{P}_{\mathfrak{M}}^{n} & J^0 \end{pmatrix}.$$

Note that these groups are $P^0(n+1)$ and $P^0(n)$ defined in the previous section. We note that the group $J^0 \times J^0$ (considered as a subgroup of $M(\mathcal{O}_F)$) normalizes the group

$$U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}) = \left\{ \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix} \mid B \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \right\}$$

hence we have the semi-direct product $(J^0 \times J^0)U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}})$. Now we define H_3 to be the subgroup $K_4(n)(J^0 \times J^0)U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}})$. The group H_3 in the block form is as follows :

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \\ \mathfrak{P}_{\mathfrak{M}}^n & J^0 \end{pmatrix}.$$

Lemma 6.2.6. *The representation λ_s of $H_3 \cap M$ extends to a representation of H_3 such that $H_3 \cap U$ and $H_3 \cap \bar{U}$ are contained in the kernel of the extended representation.*

Proof. The representation λ_s extends to a representation of

$$H_3' = \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \\ \mathfrak{P}_{\mathfrak{M}}^{n+1} & J^0 \end{pmatrix}$$

since $\mathrm{res}_{1_2 + \mathfrak{P}_{\mathfrak{M}}^{n+1}} \lambda_s = \mathrm{id}$. Now $H_3 = H_3'(H_3 \cap \bar{U})$. Let

$$u^+ = \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix}$$

be an element of $(H_3' \cap U) = (H_3 \cap U)$ and

$$u^- = \begin{pmatrix} 1_2 & 0 \\ C & 1_2 \end{pmatrix}$$

be an element of $H_3 \cap \bar{U}$. We observe that $u^- u^+ {u^-}^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - BC & B \\ -CBC & CB + 1_2 \end{pmatrix}.$$

The above element belongs to the group H_3' and

$$\lambda_s(u^- u^+ {u^-}^{-1}) = \psi_{[\alpha, B]}(1 + C) = 1.$$

Hence the representation λ_s extends to a representation of H_3 such that $H_3 \cap \bar{U}$ is contained in the kernel of the extension. □

By Mackey decomposition we get that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_2}(\mathrm{ind}_{H_1}^{H_2}(\lambda_s), \mathrm{ind}_{H_3}^{H_2}(\lambda_s)) = 1.$$

Let π_1 and π_2 be the representations $\mathrm{ind}_{H_1}^{H_2}(\lambda_s)$ and $\mathrm{ind}_{H_3}^{H_2}(\lambda_s)$ respectively. Let I be an operator from the space π_1 to the space of functions on H_1 given by

$$I(f)(h) = \int_{H_1 \cap \bar{U}} f(uh)du.$$

Lemma 6.2.7. *The operator I is a non-trivial intertwining operator from π_1 to π_2.*

Lemma 6.2.8. *The representation $\mathrm{ind}_{H_3}^{H_2}(\lambda_s)$ is an irreducible representation of H_2.*

Lemma 6.2.9. *The irreducible sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I))$ are not typical representations.*

We prove lemmas 6.2.7, 6.2.8 and 6.2.9 in an axiomatic way as the same argument is used in various contexts. Let us first recall some definitions (see [BK93][1.1.4]). Let ψ be an additive character of F which is trivial on $\mathfrak{P}_F$ but not on $\mathcal{O}_F$ and A be the set of matrices $M_2(F)$. We denote by ψ_A the character $x \mapsto \psi(\mathrm{tr}(x))$. For a given subset $S \subset A$, we denote by S^* the set

$$\{x \in A \mid \psi_A(xs) = 0 \ \forall \ s \in S\}.$$

Let $(H_1, H_2, H_3, \lambda_s)$ be a tuple consisting of three groups H_i for $1 \leq i \leq 3$ and λ_s a representation of a common subgroup of H_i for $1 \leq i \leq 3$. We assume that the tuple satisfies the following conditions

1. $H_1 = \begin{pmatrix} J^0 & \mathfrak{N}_1 \\ \mathfrak{P}_2 & J^0 \end{pmatrix}$.
2. $H_2 = \begin{pmatrix} J^0 & \mathfrak{N}_1 \\ \mathfrak{P}_1 & J^0 \end{pmatrix}$.
3. $H_3 = \begin{pmatrix} J^0 & \mathfrak{N}_2 \\ \mathfrak{P}_1 & J^0 \end{pmatrix}$.
4. J^0 is the group J^0_α for the simple strata $[\mathfrak{A}, n, 0, \alpha]$ and λ_s is a representation $\lambda_1 \boxtimes \lambda_2$ of $J^0 \times J^0$ such that (J^0, λ_1) and (J^0, λ_2) are Bushnell-Kutzko types for some supercuspidal representations σ_1 and σ_2 respectively.
5. The lattices $\mathfrak{P}_i$ and $\mathfrak{N}_i$ for i in $\{1, 2\}$ satisfy the inclusion relations $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$, $\mathfrak{P}_1\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$ and $\mathfrak{P}_1\mathfrak{N}_1\mathfrak{P}_1 \subseteq \mathfrak{P}_2$. The map ψ_α sending x to $\mathrm{tr}(\alpha x)$ is a character of the quotient $\mathfrak{P}_1/\mathfrak{P}_2$ where α is in $\mathfrak{P}_2^*$. λ_s extends to a representation of H_1 and H_3.
6. λ_1 and λ_2 are representations of J^0 such that the restriction to $1_2 + \mathfrak{P}_1$ is a multiple of the character of the form $x \mapsto \psi_\alpha(1 + x)$

7. We have an exact sequence

$$0 \to \frac{\mathfrak{N}_1}{\mathfrak{N}_2} \to \frac{\mathfrak{P}_2^*}{\mathfrak{P}_1^*} \to \frac{s(\mathfrak{P}_2^*)}{s(\mathfrak{P}_1^*)} \to 0 \tag{6.14}$$

such that the first arrow is given by $[\alpha, \ ]$ and the second by s the tame co-restriction with respect to the field $F[\alpha]/F$.

It follows from Mackey decomposition that up to scalars there exists a unique non-trivial intertwining operator between the representations $\pi_1 = \mathrm{ind}_{H_1}^{H_2}(\lambda_s)$ and $\pi_2 = \mathrm{ind}_{H_3}^{H_2}(\lambda_s)$. Let f be a function in the space π_1. Let I be the operator from π_1 to the space of functions on H_1 given by

$$I(f)(h) = \int_{H_2 \cap \bar{U}} f(uh) du$$

for all $h \in H_1$.

Lemma 6.2.10. *The operator I is a non-trivial intertwining operator between the space π_1 and π_2.*

Proof. The proof is a repetition of the proof of 6.2.2. Let $h \in H_2$ and $u^+ \in H_3 \cap U$. It is enough to show that $I(f)(u^+h) = \lambda_s(u^+)I(f)(h) = I(f)(h)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}, \ u^+ = \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively.

$$\int_{u^- \in \ H_1 \cap \bar{U}} f(u^-u^+p)du^- = \int_{u^- \in \ H_1 \cap \bar{U}} f(u^-u^+(u^-)^{-1}u^-p)du^- \tag{6.15}$$

Using the axiom 6 the above integral can be written as

$$\int_{u^- \in \ H_1 \cap \bar{U}} \psi_{(\alpha U^+ - U^+\alpha)}(1 + U^-)f(u^-p)du^-. \tag{6.16}$$

Since $U^+ \in \mathfrak{N}_2$ and by the exact sequence in axiom (7) we get that $\alpha U^+ - U^+\alpha$ belongs to $\mathfrak{P}_1^*$. This shows that $I(f) \in \pi_2$. To see that I is a non-zero operator we can take a function $f \in \pi_1$ which is a non-zero constant on $H_1 \cap \bar{U}$ and observe that $I(f)(1_4) \neq 0$. □

Lemma 6.2.11. *The representation π_2 is an irreducible representation of H_2. Any irreducible sub-representation in the kernel of this intertwining operator I also occurs as a sub-representation of*

$$\mathrm{ind}_{H_1}^{H_2}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$$

for any tame character η of $\mathcal{O}_E^\times$.

Proof. Except for few changes the proof is a repetition of lemma 6.2.3. We first prove that π_2 is irreducible. The double coset representatives for $H_3\backslash H_2/H_3$ can be chosen from $H_2 \cap U$. Let u^+ be a coset representative. We write u^+ in block diagonal form as

$$u^+ = \begin{pmatrix} \mathrm{id} & U \\ 0 & \mathrm{id} \end{pmatrix}.$$

Suppose T be a non-zero operator in the space

$$\mathrm{Hom}_{H_3 \cap (H_3)^{u^+}}(\lambda_s, \lambda_s^{u^+}).$$

The operator T satisfies the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + UC & -UCU \\ C & -CU + \mathrm{id} \end{pmatrix} T(v).$$

where $C \in \mathfrak{P}_1$. Now we get that

$$\psi_\alpha(\mathrm{id} + UC)\psi_\alpha(-CU + \mathrm{id}) = 1.$$

Hence we have $\psi_{(\alpha U - U\alpha)}(1 + C) = 1$ for all $C \in \mathfrak{P}_1$. The first arrow of the exact sequence (6.14) shows that $U \in \mathfrak{N}_2$ and hence u^+ belongs the double coset represented by id. By Mackey irreducibility criteria π_2 is irreducible. For the second part let f be a function in $\ker(I)$ then we have

$$I(f)(u^+) = \int_{u^- \in H_2 \cap \bar{U}} f(u^- u^+) du^- = 0$$

for all u^+ in $H_2 \cap U$. We write u^+ and u^- in their block diagonal form as

$$\begin{pmatrix} \mathrm{id} & U^+ \\ 0 & \mathrm{id} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & U^- \\ 0 & \mathrm{id} \end{pmatrix}$$

respectively. We observe that $u^- u^+ {u^-}^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1_2 \end{pmatrix}.$$

The above matrix is an element of H_1. We now have

$$\begin{aligned} &\int_{H_2 \cap \bar{U}} f(u^- u^+) du^- \\ &= \int_{H_2 \cap \bar{U}} \lambda_s(u^- u^+ {u^-}^{-1}) f(u^-) du^- \\ &= \int_{U^- \in \mathfrak{P}_1} \psi_\alpha(1_2 - U^+U^-)\psi_\alpha(1_2 + U^-U^+) f(U^-) dU^-. \end{aligned}$$

This shows that $\ker(I)$ consists of functions which satisfy the identity

$$\int_{U^-\in\mathfrak{P}_1} \psi_{(\alpha U^+-U^+\alpha)}(1+U^-)f(U^-)dU^- = 0$$

for all $U^+ \in \mathfrak{N}_1$. We denote by H' the group

$$\begin{pmatrix} 1_2+\mathfrak{P}_1 & \mathfrak{N}_1 \\ \mathfrak{P}_1 & 1_2+\mathfrak{P}_1 \end{pmatrix}.$$

We have $H'H_1 = H_2$ hence by Mackey decomposition we get that

$$\operatorname{res}_{H'} \operatorname{ind}_{H_1}^{H_2}(\lambda_s) = \operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s).$$

Let $\operatorname{res}_{H'\cap M} \lambda_s$ be $(\psi_\alpha \boxtimes \psi_\alpha)^n$. Now we get that

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s) \simeq (\operatorname{ind}_{H_1\cap H'}^{H'}(\psi_\alpha \boxtimes \psi_\alpha))^n.$$

The representation

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\psi_\alpha \boxtimes \psi_\alpha)$$

can be realised as space of functions on the abelian group

$$\frac{H'}{H_1\cap H'} \simeq \frac{\mathfrak{P}_1}{\mathfrak{P}_2} \tag{6.17}$$

If e_{u^-} is the characteristic function for the coset representative u^-, the element u^+ acts by the constant $\psi_{\alpha U^+-U^+\alpha}(U^-)$ on e_{u^-}. The element $j = \operatorname{diag}(j_1, j_2)$ of $H'\cap M$ acts by sending e_{U^-} to

$\psi_\alpha(j_1j_2)e_{j_1u^-j_2^{-1}} = \psi_\alpha(j_1j_2)e_{u^-}$ (see 6.12). The space of functions on the group (6.17) are spanned by the characters on the abelian quotient (6.17). The set of characters of the group (6.17) can be identified with the standard isomorphism

$$\frac{\mathfrak{P}_2^*}{\mathfrak{P}_1^*} \simeq \widehat{\frac{\mathfrak{P}_1}{\mathfrak{P}_2}}$$

sending U to ψ_U. The action of the element u^+ on ψ_V is given by

$$\psi_V \mapsto \psi_{V+[\alpha,U^+]}.$$

Let $W(V)$ be a space spanned by the functions of the form $\psi_{V+[\alpha,U^+]}$ for all $U^+ \in \mathfrak{N}_1$. This space is stable for the action of H' and is irreducible. Hence we have

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s) \simeq \bigoplus_{V\in\mathfrak{P}_2^*/\operatorname{img}([\alpha,\ ])} (W(V))^n$$

Now the group H_2 acts on the set of representations $W(V)$ of H' and we get that

$$\operatorname{ind}_{H_1}^{H_2}(\lambda_s) \simeq \bigoplus_{V_i}(\operatorname{ind}_{Z(W(V_i))}^{H_2}(\widetilde{W(V_i)}))^n$$

where $V_i|1 \leq i \leq l$ is the representatives for the action of H_2, $Z(W(V_i))$ is the stabilizer of the group $W(V_i)$ and $\widetilde{W(V_i)}$ is the isotopic component of $W(V_i)$.

Let $s_{E/F}$ be a tame co-restriction on A with respect to E (see [BK93][1.3.3]) If $s_{E/F}(V_i) = 0$ then by the exact sequence (6.14) we get that $V_i = [\alpha, U^+]$ for some $U^+ \in \mathfrak{N}_1/\mathfrak{N}_2$. This shows that the representation

$$\Gamma := \mathrm{ind}_{Z(W(V_i))}^{H_2}(\widetilde{W(V_i)})$$

contains a function $\psi_{[\alpha,U^+]}$. Hence the above representation is not contained in the space $\ker(I)$. Hence the kernel consists of representations $W(V)$ with $s_{E/F}(V) \neq 0$ Now $Z(W(V)) = \{(J^0 \times J^0) \cap Z(W(V))\}H'$. Note that $J^0 = \mathcal{O}_E^\times J^1$ and $(\mathcal{O}_E^\times \times \mathcal{O}_E^\times) \cap \{(J^0 \times J^0) \cap Z(W(V))\}$ is contained in the group of the form $\{(a,b) \mid a, b \in \mathcal{O}_E^\times; a \equiv b \bmod \mathfrak{P}_E\}$. Let η be a character of $J^0/J^1 \simeq k_E^\times$. We observe that

$$\mathrm{res}_{Z(W(V))}\, \lambda_1 \boxtimes \lambda_2 \simeq \mathrm{res}_{Z(W(V))}\, \lambda_1\eta \boxtimes \lambda_2\eta^{-1}$$

Now we note that $(Z(W(V)) \cap P)H_1 = H_2$ and $H_1 \cap P = H_2 \cap P$. From Frobenius reciprocity we get that $\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_s) \neq 0$. We have

$$\begin{aligned}
&\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_{s'})\\
&= \mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_{s'})\\
&= \mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_s).
\end{aligned}$$

and $\mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_s)$ is equal to $\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_s)$ which shows that $\mathrm{Hom}_{H_1}(\Gamma, \lambda_{s'})$ is non-zero. □

Proof of lemma 6.2.8 and 6.2.9. We apply lemma 6.2.11 for the tuple

$$(H_1, H_2, H_3, \lambda_s)$$

defined at the beginning of this subsection with (J^0, λ_1) and (J^0, λ_2) being the Bushnell-Kutzko types for the representations σ_1 and σ_2 respectively. The exact sequence is provided by (6.13). The irreducible sub-representations of $\ker(I)$ also occur in the representation

$$\mathrm{ind}_{H_1}^{H_2}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1}).$$

for any tame character η of $\mathcal{O}_E^\times$. If $\#k_F > 3$ then we can choose a character η of $k_E^\times = k_F^\times$ such that the multi-sets of types $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ are distinct. If σ_1' and σ_2' are two supercuspidal representations containing $(J^0, \lambda_1\eta)$ and $(J^0, \lambda_2\eta^{-1})$ respectively then $s' = [M, \sigma_1' \boxtimes \sigma_2']$ and $s = [M, \sigma_1 \boxtimes \sigma_2]$ are different inertial classes and hence the irreducible sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I))$ occur as sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$ and hence are atypical. □

Now typical representations occur as sub-representations of $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ or rather the complement of this representation in the parabolic induction

$$i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$$

contains only atypical representations for the component s.

Lemma 6.2.12. *The group $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}$ is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$.*

Proof. Since the group group $\mathfrak{P}_{\mathfrak{M}}$ is contained in $\mathfrak{P}_{\mathfrak{J}}$, $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}$ is a subset of $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We will now show that $\mathfrak{P}_{\mathfrak{J}} \subset \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We recall that the groups $\mathfrak{P}_{\mathfrak{J}}$ and $\mathfrak{P}_{\mathfrak{M}}$ are given by

$$\begin{pmatrix} \mathfrak{P}_F & \mathcal{O}_F \\ \mathfrak{P}_F & \mathfrak{P}_F \end{pmatrix} \text{ and } \begin{pmatrix} \mathfrak{P}_F & \mathfrak{P}_F \\ \mathfrak{P}_F & \mathfrak{P}_F \end{pmatrix}$$

respectively. Let Π be the matrix of the form

$$\begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}.$$

We note that

$$\mathcal{O}_F \Pi + \mathfrak{P}_{\mathfrak{M}} = \mathfrak{P}_{\mathfrak{J}}. \tag{6.18}$$

Since the element Π normalizes the hereditary order $\mathfrak{J}$, we have $\Pi = \varpi_E j$ for some $j \in U(\mathfrak{J})$. Now multiplying j^{-1} on both sides of the equation (6.18) we get that $\varpi_E \mathcal{O}_F + \mathfrak{P}_{\mathfrak{M}} = \mathfrak{P}_{\mathfrak{J}}$. This shows the lemma. □

Hence the group H_3 is of the following form.

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}} \\ \mathfrak{P}_{\mathfrak{M}}^n & J^0 \end{pmatrix}.$$

Let a, b be integers such that $a + b = 2n$, $a \geq n$ and $b \geq 0$. We denote by $\mathcal{H}(a, b)$ the set consisting of the matrices

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} A, D \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n;\ B \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b; C \in \mathfrak{P}_{\mathfrak{J}}^a \right\}.$$

Lemma 6.2.13. *The set $\mathcal{H}(a, b)$ is an order.*

Proof. Let h_1 and h_2 be two matrices from the set $\mathcal{H}(a, b)$ we write h_1 and h_2 in its block form as

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

The product of $h_1 h_2$ has the 1×1 entry $A_1 A_2 + B_1 C_2$ Since $a + b = 2n$ we can see that $B_1 C_2 \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n + \mathfrak{P}_{\mathfrak{J}}^a$ since we assumed $a \geq n$ we have $B_1 C_2 \in \mathfrak{P}_{\mathfrak{J}}^n$ and hence $A_1 A_2 + B_1 C_2 \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n$. The 1×2 term is contained in $(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n)(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b)$ and this product set is contained in $(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b)$. The 2×1 term of the product is easily seen to be contained in $\mathfrak{P}_{\mathfrak{J}}^a$. The 2×2 term is similar to 1×1. □

We denote by $H(a,b)$ the group of units of $\mathcal{H}(a,b)$. In the block form the group $H(a,b)$ is as follows:

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b \\ \mathfrak{P}_{\mathfrak{J}}^a & J^0 \end{pmatrix}.$$

Lemma 6.2.14. *The representation λ_s of $H(a,b) \cap M$ extends to a representation of $H(a,b)$ such that $H(a,b) \cap U$ and $H(a,b) \cap \bar{U}$ are contained in the kernel of the extension.*

Proof. The proof is similar to that of 6.2.6. We note that $H(a,b)$ is equal to $H(2n,b)(H(a,b) \cap \bar{U})$. The representation λ_s extends to a representation of $H(2n,b)$ since $\mathrm{res}_{\mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^{2n}}(\lambda_s) = \mathrm{id}$. Let u^+ and u^- be two elements of $H(a,b) \cap U$ and $H(a,b) \cap \bar{U}$ respectively and we take them in the block form as in lemma 6.2.6. We observe that $\lambda_s(u^- u^+ {u^-}^{-1}) = \psi_{[\alpha,B]}(\mathrm{id}+C)$. Now $B \in \mathcal{O}_E^\times + \mathfrak{P}_{\mathfrak{J}}^b$ hence $[\alpha, B] \in \mathfrak{P}_{\mathfrak{J}}^{-(2n-1)+b}$ and hence $\psi_{[\alpha,B]}(\mathrm{id}+C) = 1$ for all $C \in \mathfrak{P}_{\mathfrak{J}}^a$. This shows that $\lambda_s(u^- u^+ {u^-}^{-1}) = 1$. Hence the representation λ_s extends to $H(a,b)$ such that $H(a,b) \cap \bar{U}$ is in the kernel of the extended representation. □

Since $\mathfrak{P}_{\mathfrak{M}}^n \subset \mathfrak{P}_{\mathfrak{J}}^{2n-1}$ we get that $H_3 \subset H(2n-1,1)$. The induction $\mathrm{ind}_{H_3}^{H(2n-1,1)}(\lambda_s)$ has a unique complement of the representation λ_s (here λ_s is considered as a representation of $H(2n-1,1)$). Let $U(3,2n-1)$ be the complement of λ_s in $\mathrm{ind}_{H_3}^{H(2n-1,1)}(\lambda_s)$.

Lemma 6.2.15. *The irreducible sub-representations of*

$$\mathrm{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U(3,2n-1))$$

are not typical representations.

Proof. The proof is similar to the proof of 6.1.2. We denote by H' the group

$$\begin{pmatrix} \mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^n & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}} \\ \mathfrak{P}_{\mathfrak{J}}^{2n-1} & \mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^n \end{pmatrix}.$$

We observe that H' is a normal subgroup of $H(2n-1,1)$ and $H'H_3 = H(2n-1,1)$. Using Mackey decomposition we get that

$$\mathrm{res}_{H'} \mathrm{ind}_{H_3}^{H(2n-1,1)}(\mathrm{id}) \simeq \oplus_{j=1}^p \eta_j$$

where η_j are characters of H' which are trivial on $H' \cap H_3$. The quotient is given by $\mathfrak{P}_{\mathfrak{J}}^{2n-1}/\mathfrak{P}_{\mathfrak{M}}^n$. Now the group $H(2n-1,1)$ acts on these characters η_i and let $Z(\eta)$ be the $H(2n-1,1)$ stabilizer of η. Note that $Z(\eta) = (\mathcal{O}_E^\times \times \mathcal{O}_E^\times)H(2n-1,1)$. From Clifford theory we get that

$$\mathrm{ind}_{H_3}^{H(2n-1,1)}(\mathrm{id}) \simeq \oplus_{i\in\Lambda} \mathrm{ind}_{Z(\eta_i)}^{H(2n-1,1)}(U_{\eta_i})$$

where Λ is a set of representatives for the action of $H(2n-1,1)$ on the characters η_j for $1 \leq j \leq p$ and U_{η_i} is an irreducible representation of $Z(\eta_i)$. The action of $\mathcal{O}_E^\times \times \mathcal{O}_E^\times$ on the quotient $\mathfrak{P}_{\mathfrak{J}}^{2n-1}/\mathfrak{P}_{\mathfrak{M}}^{n}$ factors through $U_2(E)$ and if η is non-trivial character, from the arguments of **Case 3.3** in theorem 6.1.2 we get that modulo $\mathfrak{P}_E$ reduction of the group $Z(\eta)$ is of the form $\{(e,e) \mid e \in k_F^\times\}$. This shows that for $\#k_F > 3$ we can choose a character η of $k_F^\times$ such that the multi-sets $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ and $\{\lambda_1, \lambda_2\}$ are distinct. Moreover

$$\mathrm{ind}_{Z(\eta_i)}^{H(2n-1,1)}\{U_{\eta_i} \otimes (\lambda_1 \boxtimes \lambda_2)\} \simeq \mathrm{ind}_{Z(\eta_i)}^{H(2n-1,1)}\{U_{\eta_i} \otimes (\lambda_1\eta \boxtimes \lambda_2\eta^{-1})\}$$

for any $\eta \neq \mathrm{id}$. This shows that irreducible sub-representations of

$$\mathrm{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U(3,2n-1))$$

are atypical representations. □

The above lemma shows that typical representations can only occur as sub-representations of $\mathrm{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. We suppose that typical representations occur as sub-representations of $\mathrm{ind}_{H(a,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ for some positive integer b such that $1 \leq b < n-1$ then we show that typical representations occur as sub-representation of $\mathrm{ind}_{H(a-1,b+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. By Mackey decomposition we note that

$$\dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_s), \mathrm{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)) = 1.$$

Let $I(a,b)$ be the non-trivial intertwining operator between $\mathrm{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_s)$ and $\mathrm{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)$.

Lemma 6.2.16. *The representation* $\mathrm{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)$ *is irreducible.*

Lemma 6.2.17. *The irreducible sub-representations of* $\mathrm{ind}_{H(a-1,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I(a,b)))$ *are not-typical representations.*

Proof of lemma 6.2.16 and 6.2.17. We apply the formalism developed in lemma 6.2.11 for the tuple $(H(a,b), H(a-1,b), H(a-1,b+1), \lambda_s)$. We have $\mathfrak{N}_1 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^{b}$ and $\mathfrak{N}_2 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^{b+1}$. In the language of Bushnell-Kutzko $\mathfrak{N}_1 = \mathfrak{N}_{-2n+b+1}(\beta, \mathfrak{J})$ and $\mathfrak{N}_2 = \mathfrak{N}_{-2n+b+2}(\beta, \mathfrak{J})$. We note that $\mathfrak{P}_2^* = \mathfrak{P}_{\mathfrak{J}}^{1-a} = \mathfrak{P}_{\mathfrak{J}}^{-2n+b+1}$ and $\mathfrak{P}_1^* = \mathfrak{P}_{\mathfrak{J}}^{-2n+b+2}$. The exact sequence 6.14 is given by [BK93, corollary 1.4.10] which says that the sequence

$$\frac{\mathfrak{N}_k(\beta,\mathfrak{J})}{\mathfrak{N}_{k+1}(\beta,\mathfrak{J})} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{J}}^{k}}{\mathfrak{P}_{\mathfrak{J}}^{k+1}} \xrightarrow{s_{E/F}} \frac{\mathfrak{P}_E^{k}}{\mathfrak{P}_E^{k+1}}$$

is exact for all $k \geq k_0(\beta, \mathfrak{J})$ and in our context $k_0(\beta, \mathfrak{J}) = -2n+1$ and $k = -2n+b+1$. The irreducible sub-representations of $\ker(I(a,b))$ also occur in the representation

$$\mathrm{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1}).$$

for any tame character η of $\mathcal{O}_E^\times$. If $\#k_F > 3$ then we can choose a character η of $k_E^\times = k_F^\times$ such that the multi-sets types $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ are distinct. If σ_1' and σ_2' are two supercuspidal representations containing $(J^0, \lambda_1\eta)$ and $(J^0, \lambda_2\eta^{-1})$ respectively then $s' = [M, \sigma_1' \boxtimes \sigma_2']$ and $s = [M, \sigma_1 \boxtimes \sigma_2]$ are different inertial classes and hence the irreducible sub-representations of $\mathrm{ind}_{H(a-1,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I(a,b)))$ occur as sub-representations of $\mathrm{ind}_{H(a,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$ and hence are atypical. $\square$

This brings us to the final step of this section from the lemma 6.2.17 we conclude that typical representations occur as sub-representations of

$$\mathrm{ind}_{H(n+1,n-1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

From lemma 6.2.4 we get that typical representations occur as sub-representations of $\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ where (J_s, λ_s) is the Bushnell-Kutzko type for the component s.

6.3 Completely distinct inertial classes

In this section we assume that $\#k_F > 3$ We fix Bushnell-Kutzko types (J_1^0, λ_1) and (J_2^0, λ_2) contained in the supercuspidal representations σ_1 and σ_2 respectively. Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining the types (J_1^0, λ_1) and (J_2^0, λ_2) respectively. Let $e_i = e(\mathfrak{A}_i | \mathcal{O}_F)$ for $i \in \{1, 2\}$. In this section we consider the case where σ_1 and σ_2 are completely distinct. **We will always assume that** $n_1/e_1 \geq n_2/e_2$. We denote by Λ_i the lattice sequence $\Lambda_i(r) = \mathcal{L}_i(-[-r])$ for all $r \in \mathbb{R}$ where $\mathcal{L}_i$ is the lattice chain defining the hereditary order $\mathfrak{A}_i$ and moreover we assume that $\mathcal{L}_i(0) = \mathcal{O}_F \oplus \mathcal{O}_F$ for $i \in \{1, 2\}$.

Let U and $\bar{U}$ be the unipotent radicals of P and the opposite parabolic subgroup of P with respect to M respectively. We denote by e the least common multiple of $e(\Lambda_1)$ and $e(\Lambda_2)$ respectively. Let l be the positive integer such that $l/e = \max\{n_1/e_1, n_2/e_2\}$. Let Λ be the direct sum of lattice sequences Λ_1 and Λ_2. The Bushnell-Kutzko type J_s satisfies the Iwahori decomposition with respect to the parabolic subgroup P and the Levi-subgroup M. The pair (J_s, λ_s) is characterised by the following properties

1. $J_s \cap U = u_0(\Lambda) \cap U$,
2. $J_s \cap \bar{U} = u_{l+1}(\Lambda) \cap \bar{U}$,

3. $J_s \cap M = J_1^0 \times J_2^0$.

4. The restriction of the representation λ_s to the subgroup $J_s \cap M$ is isomorphic to $\lambda_1 \boxtimes \lambda_2$ and $J_s \cap U$ and $J_s \cap \bar{U}$ are contained in the kernel of λ_s.

We refer to [BK99][Section 8, paragraph 8.3.1] for the construction of the pair (J_s, λ_s). We will explicitly compute $u_0(\Lambda) \cap U$ and $u_{l+1}(\Lambda) \cap \bar{U}$ in the following possibilities $e_1 = e_2 = 1$; $e_1 = 1, e_2 = 2$; $e_1 = 2, e_2 = 1$; and $e_1 = 2, e_2 = 2$.

We first consider the case $e_1 = e_2 = 1$. In this case

$$\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$$

and $\Lambda(i+1) = \varpi_F \Lambda(i)$. Hence $u_0(\Lambda)$ is $\mathrm{GL}_4(\mathcal{O}_F)$ and $u_{l+1}(\Lambda)$ is the principal congruence subgroup of level $l+1$ inside $\mathrm{GL}_4(\mathcal{O}_F)$. This shows that $u_0(\Lambda) \cap U = U(\mathcal{O}_F)$ and $u_{l+1}(\Lambda) \cap \bar{U} = \bar{U}(\mathfrak{P}_F^{l+1})$. Moreover $N = l$ Hence we observe that in this case $J_s = P^0(N+1)$. Hence the theorem 6.1.2 completes classification of typical representations in the case where $e_1 = e_2 = 1$ and σ_1 and σ_2 are completely distinct.

We will now consider the case $e_1 = 1, e_2 = 2$. In this situation

$$\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$$
$$\Lambda(1) = \Lambda_1(1/2) \oplus \Lambda_2(1) = \mathfrak{P}_F \oplus \mathfrak{P}_F \oplus \mathcal{O}_F \oplus \mathfrak{P}_F$$

and $\Lambda(i+2) = \varpi_F \Lambda(i)$. Let $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ be the upper and lower nilpotent matrices of the type $(2,2)$ i.e the Lie algebras of U and $\bar{U}$ respectively. Now $u_i(\Lambda) \cap U = 1 + (a_i(\Lambda) \cap \mathfrak{n})$ and $u_i(\Lambda) \cap \bar{U} = 1 + (a_i(\Lambda) \cap \bar{\mathfrak{n}})$ ($a_i(\Lambda)$ is defined in the section 5.1). We note that $a_0 \cap \mathfrak{n}$ is the set

$$\{x \in M_4(F) \cap \mathfrak{n} \mid x\Lambda(i) \subset \Lambda(i) \forall i \in \mathbb{Z}\}$$

In our case it is given by the set

$$\begin{pmatrix} 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly the groups $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ are given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}$$

respectively. Let $l+1 = 2l'+r$ with $r \in \{0,1\}$ then $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}})$. We recall that $N = \max\{n_1, [(n_2+1)/2]\}$ and $l/2 = \max\{n_1, n_2/2\} = n_1$ since we assume that $n_1 \geq n_2/2$. Since n_1 is a positive integer, $n_1 \geq n_2/2$ if and only if $n_1 \geq [(n_2+1)/2]$. In this case $N = n_1$. Hence we deduce that $2N = l$. From this we get that $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^N(a_1(\Lambda) \cap \bar{\mathfrak{n}})$. The first observation is that the dimensions of the representations

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \quad \text{and} \quad \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

are the same and moreover the intertwining of the second representation from the lemma [BK98][lemma 11.5] is bounded by the cardinality of $N_{\mathrm{GL}_4(F)}(s)/M$. Since the representations σ_1 and σ_2 are not inertially equivalent we get that the cardinality of $N_{\mathrm{GL}_4(F)}(s)/M$ is one. Hence the representation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

is irreducible. Now at least one typical representation must be contained in $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$ and lemma 2.2.4, theorem 6.1.2 gives the inclusion of the above representation in

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

Hence we have the isomorphism

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2). \tag{6.19}$$

Let us consider the case where $e_1 = 2$ and $e_2 = 1$. In this case $\Lambda(i+2) = \varpi_F \Lambda(i)$ and

$$\begin{aligned}
\Lambda(0) &= \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F, \\
\Lambda(1) &= \Lambda_1(1) \oplus \Lambda_2(1/2) = \mathcal{O}_F \oplus \mathfrak{P}_F \oplus \mathfrak{P}_F \oplus \mathfrak{P}_F\,.
\end{aligned}$$

The group $a_0(\Lambda) \cap \mathfrak{n}$ is given by

$$\begin{pmatrix} 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The group $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}.$$

In this situation $n_1/2 \geq n_2$ and hence $l/2 = \max\{n_1/2, n_2\} = n_1/2$. Now consider the positive integer $N = \max\{[(n_1+1)/2], n_2\}$. Since $n_1/2 \geq n_2$ and $[(n_1+1)/2] \geq n_1/2$ we get that $N = [(n_1+1)/2]$. Here we can use the fact that

n_1 is odd and we have $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{(n_1+1)/2}(a_0(\Lambda) \cap \bar{\mathfrak{n}}) = \varpi_F^N(a_0(\Lambda) \cap \bar{\mathfrak{n}})$. Observe that $P^0(N+1) \subset J_s$.

The representation λ_s occurs with multiplicity one in the representation $\mathrm{ind}_{P^0(N+1)}^{J_s}(\lambda_s)$. We denote by $U_N^0(\lambda_s)$ the unique complement of λ_s in the representation $\mathrm{ind}_{P^0(N+1)}^{J_s}(\lambda_s)$. We denote by $U_N(\lambda_s)$ the representation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(U_N^0(\lambda_s)).$$

Lemma 6.3.1. *The irreducible sub-representations of $U_N(\lambda_s)$ are atypical.*

Proof. The proof of this lemma is similar to 5.3.2. The first step is to split the representation

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}).$$

To begin with we will show that the group $U(\mathcal{O}_F)$ acts trivially on the above representation. Let u^+ be an element of $U(\mathcal{O}_F)$ and u^- be an element of $J_s \cap \bar{U}$. We denote u^+ and u^- in their respective block form as

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix} \text{ and } \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

Now the conjugation $u^-u^+(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ U^-U^+U^- & 1_2 + U^-U^+ \end{pmatrix}.$$

We observe that $U^-U^+U^- \in \varpi^{N+1}M_2(\mathcal{O}_F)$. This shows that the conjugation $u^-u^+(u^-)^{-1}$ lies in the group $P^0(N+1)$. Hence the group $U(\mathcal{O}_F)$ acts trivially on

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}). \tag{6.20}$$

From the Iwahori decomposition of the group J_s we get that J_s is equal to $(J_s \cap \bar{P})P^0(N+1)$. Hence we get that

$$\mathrm{res}_{J_s \cap \bar{P}_I} \mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}) \simeq \mathrm{ind}_{P^0(N+1)\cap\bar{P}}^{J_s\cap\bar{P}}(\mathrm{id}).$$

Note that $J_s \cap \bar{P}$ is a semi-direct product of the groups $(J_s \cap M)$ and $(J_s \cap \bar{U})$. Let η_k for $1 \leq k \leq t$ (we mean counting them with their multiplicity, but in our case the multiplicity is one) be all the characters of the group $J_s \cap \bar{U}$ which are trivial on the group $P^0(N+1) \cap \bar{U}$. The group $J_s \cap \bar{P}$ acts on these characters and let $\{\eta_{k_p}\}$ be a set of representatives for the orbits under this action. We denote by $Z(\eta_{k_p})$ the $J_s \cap \bar{P}$ stabiliser of the character η_{k_p}. Now Clifford theory gives the decomposition

$$\mathrm{ind}_{P^0(N+1)\cap\bar{P}}^{J_s\cap\bar{P}}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})}^{J_s\cap\bar{P}}(U_{\eta_{k_p}})$$

where $U_{\eta_{k_p}}$ is an irreducible representation of $Z(\eta_{k_p})$. We note that the character id occurs with a multiplicity one in the list of characters η_k.

The representation $U_{\eta_{k_p}}$ is the isotypic component of the character η_{k_p} in the representation

$$\mathrm{ind}_{P^0(N+1)\cap\bar{P}}^{J_s\cap\bar{P}}(\mathrm{id}).$$

which naturally has the action of $Z(\eta_{k_p})$. Now if K_s is the kernel of the representation (6.20) then $K_s \cap Z(\eta_{k_p})$ acts trivially on $U_{\eta_{k_p}}$. Hence we can extend the representation $U_{\eta_{k_p}}$ to the group $Z(\eta_{k_p})K_s$ such that K_s acts trivially on the extended representation. Now consider the representation

$$\pi = \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}.$$

Note that $K_s \cap \bar{P}$ is contained in the group $Z(\eta_{k_p}) \cap \bar{P}$ and moreover $U(\mathcal{O}_F)$ is contained in K_s hence $J_s = (J_s \cap \bar{P})Z(\eta_{k_p})K_s$. Using Mackey decomposition we have

$$\mathrm{res}_{J_s\cap\bar{P}}\,\mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}} \simeq \mathrm{ind}_{Z(\eta_{k_p})K_s\cap(J_s\cap\bar{P})}^{J_s\cap\bar{P}}(U_{\eta_{k_p}}) \simeq \mathrm{ind}_{Z(\eta_{k_p})}^{J_s\cap\bar{P}}(U_{\eta_{k_p}}).$$

We hence obtain

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}. \tag{6.21}$$

Now using the decomposition (6.21) we get that the decomposition

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda_1 \boxtimes \lambda_2)\}.$$

Note that the character id occurs with multiplicity one among the characters η_k and the fact that $Z(\mathrm{id})K_s = (J_s \cap \bar{P}_I)K_s = J_s$ implies the following isomorphism

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\boxtimes\lambda_2) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\boxtimes\lambda_2) \oplus \bigoplus_{\eta_{k_p}\neq\mathrm{id}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}}\otimes(\lambda_1\boxtimes\lambda_2)\}. \tag{6.22}$$

Let Γ be an irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda_1 \boxtimes \lambda_2)\}.$$

From the reasoning given in **Case 3.2** of theorem 6.1.2, for $\#k_F > 3$, we can find two types (J_1^0, λ_1') and (J_2^0, λ_2') such that $[M, \sigma_1 \boxtimes \sigma_2] \neq [M, \sigma_1 \boxtimes \sigma_2]$ where σ_1' and σ_2' are two supercuspidal representations containing (J_1^0, λ_1') and (J_2^0, λ_2') respectively,

$$\mathrm{res}_{Z(\eta_{k_p})}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z(\eta_{k_p})}(\lambda_1' \boxtimes \lambda_2').$$

This shows that the representation Γ is not a typical representation. □

Now we look at our last possibility $e_1 = e_2 = 2$. In this case $\Lambda(i+2) = \varpi_F \Lambda(i)$, $\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$ and $\Lambda(1) = \mathcal{O}_F \oplus \mathfrak{P}_F \oplus \mathcal{O}_F \oplus \mathfrak{P}_F$. The group $a_0(\Lambda) \cap \mathfrak{n}$ is given by

$$\begin{pmatrix} 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The groups $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ are given by the groups

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}$$

respectively. We have $l/2 = \max\{n_1/2, n_2/2\} = n_1/2$ and hence $l = n_1$. At the same time $N = \max\{[(n_1+1)/2], [(n_2+1)/2]\} = [(n_1+1)/2]$. We use the fact that n_1 is odd and get that $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{(n_1+1)/2}(a_0(\Lambda) \cap \bar{\mathfrak{n}}) = \varpi_F^N(a_0(\Lambda) \cap \bar{\mathfrak{n}})$.

Let H_1 be the group

$$\begin{pmatrix} J_1^0 & \varpi_F M_2(\mathcal{O}_F) \\ \varpi_F^N M_2(\mathcal{O}_F) & J_2^0 \end{pmatrix}.$$

The group H_1 is contained in the group $P^0(N)$.

Lemma 6.3.2. *The representation λ_s of $J_1^0 \times J_2^0$ extends to a representation of H_1 such that $H_1 \cap U$ and $H_1 \cap \bar{U}$ are contained in the kernel of this extension.*

Proof. The representation $\lambda_s = \lambda_1 \boxtimes \lambda_2$ extends to the on the group $P^0(N+1)$ such that $P^0(N+1) \cap U$ and $P^0(N+1) \cap \bar{U}$ are contained in the kernel of the extension. Hence the representation λ_s extends to the group $H_1 \cap P^0(N+1)$. We note that $H_1 = (P^0(N) \cap \bar{U})(H_1 \cap P^0(N+1))$. Let u^- be an element of $P^0(N) \cap \bar{U}$. To prove the lemma it is enough to verify that $(u^-)^{-1}u^+u^-$ belongs to $H_1 \cap P^0(N+1)$ and $\lambda_s((u^-)^{-1}u^+u^-) = 1$ for all $u^- \in H_1 \cap \bar{U}$ and $u^+ \in H_1 \cap U$ respectively.

We write u^- and u^+ in the block form as

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix} \text{ and } \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively and $U^- \in \varpi_F^N M_2(\mathcal{O}_F)$ and $U^+ \in \varpi_F M_2(\mathcal{O}_F)$. The conjugation $(u^-)^{-1}u^+u^-$ in the block form is given by

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1_2 \end{pmatrix}.$$

Since $U^-U^+U^- \in \varpi_F^{2N+1}M_2(\mathcal{O}_F)$ we get that the above matrix belongs to $H_1 \cap P^0(N+1)$. The element $1_2 + U^-U^+$ and $1_2 - U^+U^-$ are contained in the kernel of λ_2 and λ_1 respectively. Hence $\lambda_s((u^-)^{-1}u^+u^-) = 1$. $\square$

From the observation $P^0(N) = P^0(N+1)H_1$ we get that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{P^0(N)}(\operatorname{ind}_{P^0(N+1)}^{P^0(N)}(\lambda_s), \operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)) = 1.$$

Lemma 6.3.3. *The representation* $\operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ *is an irreducible representation of* $P^0(N)$*.*

Proof. We prove the lemma by showing that the support of the intertwining of the representation considered in the lemma is contained in the double coset containing identity. The double coset representatives for $H_1 \backslash P^0(N)/H_1$ can be chosen from the group $P^0(N) \cap U$. Let u^+ be an element from the group $P^0(N) \cap U$. Let the following matrix be the block from of u^+:

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

where $U^+ \in M_2(\mathcal{O}_F)$. Let T be a non-zero operator in the space

$$\operatorname{Hom}_{H_1 \cap (H_1)^{u^+}}(\lambda_s, (\lambda_s)^{u^+}).$$

Now T must satisfy the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + U^+C & -U^+CU^+ \\ C & -CU^+ + \mathrm{id} \end{pmatrix} T(v).$$

for all $v \in \operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ and $C \in \varpi_F^N M_2(\mathcal{O}_F)$. This implies that the character $\psi_{\beta_1 U^+ - U^+ \beta_2}(1 + C) = 1$. Since characteristic polynomials of β_1 and β_2 are relatively prime we get that $U^+ \in \varpi_F M_2(\mathcal{O}_F)$ (see [BK99][lemma 4 p.72]). Hence the intertwining is supported only on the double coset containing identity. This shows the lemma. $\square$

We note that $\operatorname{ind}_{P^0(N+1)}^{P^0(N)}(\lambda_s)$ and $\operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ are of the same dimension. The above lemma together with the transitivity of induction we get that

$$\operatorname{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \operatorname{ind}_{H_1}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Let H_2 be the group

$$\begin{pmatrix} J_1^0 & \mathfrak{P}_{\mathfrak{A}_1} \\ \varpi_F^N M_2(\mathcal{O}_F) & J_2^0 \end{pmatrix}.$$

(Note that $\mathfrak{A}_1$ is a hereditary order defined by the lattice sequence Λ_1 with periodicity 2 and $\Lambda_1(0) = \mathcal{O}_F \oplus \mathcal{O}_F$ and $\Lambda_1(1) = \mathcal{O}_F \oplus \mathfrak{P}_F$.) We observe that

$H_1 \subset H_2$ and moreover $H_1 \cap \bar{U} = H_2 \cap \bar{U}$ and $H_1 \cap M = H_2 \cap M$. We will also need another group H_3 given by

$$\begin{pmatrix} J_1^0 & \mathfrak{P}_{\mathfrak{A}_1} \\ \mathfrak{P}_{\mathfrak{A}_1}^{2N-1} & J_2^0 \end{pmatrix}.$$

The representation λ_s of $H_3 \cap M = J_1^0 \times J_2^0$ extends to a representation of H_3 such that λ_s extends to a representation of H_3 such that $H_3 \cap U$ and $H_3 \cap \bar{U}$ are contained in the kernel of the extension the proof is similar to 6.3.2, the only important fact is that kernels of λ_1 and λ_2 contain the group $U^{2N}(\mathfrak{A}_1)$. Since $\varpi_F^N M_2(\mathcal{O}_F) \subset \mathfrak{P}_{\mathfrak{A}_1}^{2N-1}$, we get that $H_2 \subset H_3$. From this we also get that the representation λ_s of $H_2 \cap M$ extends to a representation of H_2 such that $H_2 \cap U$ and $H_2 \cap \bar{U}$ are contained in the kernel of this extension.

It is by now a standard practice (see lemma 6.3.1) to decompose $\mathrm{ind}_{H_i}^{H_{i+1}}(\mathrm{id})$ as a direct sum

$$\mathrm{id} \oplus \bigoplus_j \mathrm{ind}_{Z_j^i}^{H_{i+1}}(U_j)$$

where the mod $\mathfrak{P}_F$ reduction of $Z_j^i \cap M$ is contained in the $M(k_F)$ stabiliser of a non-zero matrix A in $M_{2\times 2}(k_F)$ for $i \in \{1, 2\}$. From the arguments in the **Case 3.3** of theorem 6.1.2, for $\#k_F > 3$, we can find two types (J_1^0, λ_1') and (J_2^0, λ_2') such that $[M, \sigma_1 \boxtimes \sigma_2] \neq [M, \sigma_1 \boxtimes \sigma_2]$ where σ_1' and σ_2' are two supercuspidal representations containing (J_1^0, λ_1') and (J_2^0, λ_2') respectively,

$$\mathrm{res}_{Z_j^i}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_j^i}(\lambda_1' \boxtimes \lambda_2').$$

This shows that irreducible sub-representations of

$$\mathrm{ind}_{Z_j^i}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\lambda_s \otimes U_j\}$$

are atypical for $\#k_F > 3$ and $i \in \{1, 2\}$.

Now observe that the dimensions of $\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ and $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ are the same. Note that the type (J_s, λ_s) occurs in the smooth representation $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$. This shows that at least one irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \tag{6.23}$$

must be contained in $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$. The representation (6.23) is irreducible and typical. From our results so far typical representations for $s = [M, \sigma_1 \boxtimes \sigma_2]$ must occur as sub-representations of $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. Hence we have an isomorphism

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

This concludes the classification of typical representations in this case.

The results of this chapter can be collected in the following theorem.

Theorem 6.3.4. *Let $s = [M, \sigma_1 \boxtimes \sigma_2]$ be an inertial class such that σ_1 and σ_2 are either completely distinct or homogenous. Let $\#k_F > 3$. If Γ is a typical representation for the component s then Γ is a sub-representation of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$$

where (J_s, λ_s) is the Bushnell-Kutzko semi-simple type for s and moreover

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma, \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma, i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2))$$

where P is any parabolic subgroup containing M as its Levi-subgroup.