Chapter 5

The inertial class with Levi-subgroup of type (n,1)

Let V and V_1 be two F-vector spaces of dimensions $n > 1$ and 1 respectively. Let P be the parabolic subgroup of $GL(V \oplus V_1)$ fixing the flag $V \subset V \oplus V_1$. We denote by M its Levi-subgroup fixing the decomposition $V \oplus V_1$ and hence we have $M = GL(V) \times GL(V_1)$. In this chapter we are interested in the classification of typical representations for the component $[M, \sigma \boxtimes \chi]$ where σ is a cuspidal representation of $GL(V)$ and χ is a character of $GL(V_1)$. Let (J^0, λ) be a maximal simple (Bushnell-Kutzko's) type contained in the representation σ. We recall certain important features of this type for our purpose.

5.1 Bushnell-Kutzko semi-simple type

We denote by A the algebra $\mathrm{End}_F(V)$. Let $[\mathfrak{A}, l, 0, \beta]$ be a simple strata in A defining the maximal simple type (J^0, λ). We denote by B the algebra of endomorphisms commuting with $E = F[\beta]$. Let $\mathfrak{B} = \mathfrak{A} \cap B$. We denote by $\mathfrak{P}$ and $\mathfrak{Q}$ the radicals of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Given any hereditary order $\mathfrak{A}$, we define the filtration $U^i(\mathfrak{A})$ by setting

$$U^i(\mathfrak{A}) = 1_n + \mathfrak{P}^i$$

for all $i \geq 1$ and $U^0(\mathfrak{A})$ is the set of units of $\mathfrak{A}$. The type (J^0, λ) is called maximal if $\mathfrak{B}$ is a maximal hereditary order in B. The group J^0 contains $U^0(\mathfrak{B})$. There is a normal subgroup J^1 such that $J^1 \cap U^0(\mathfrak{B}) = U^1(\mathfrak{B})$ and

$$\frac{U^0(\mathfrak{B})}{U^1(\mathfrak{B})} \simeq \frac{J^0}{J^1}.$$

The group $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ is a general linear group of a vector space over a finite field. The representation λ is an irreducible representation which is is given by a tensor product $\kappa \otimes \rho$ where κ is a representation of J^0, called β-extension (see [BK93, Chapter 5, Definition 5.2.1]) and ρ is a cuspidal representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ (considered as a representation of J^0 through the quotient J^0/J^1). We refer to [BK93, Chapter 5] for complete details of these constructions. For the precise definition and description see [BK93, chapter 5, Definition 5.5.10].

We fix the following conventions. Let e and f be the ramification index and inertial index of E. We fix a lattice chain $\mathcal{L}$ defining the order $\mathfrak{B}$. Let

$\mathfrak{A}$ be the hereditary order defined by the lattice chain $\mathcal{L}$. We fix a $\mathcal{O}_E$-basis $(w_1, w_2, \dots w_{n/ef})$ for the lattice chain $\mathcal{L}$ and then a $\mathcal{O}_F$-basis for $\mathcal{O}_E w_i$ for $1 \leq i \leq ef$. (see [BK93, Chapter 1, 1.1.7]). We can now extract a F-basis $(v_1, v_2, \dots v_{n+1})$ for the vector space $V \oplus V_1$ where V_1 is a one dimensional vector space over F. In this basis we write all our endomorphisms as matrices of $M_{n+1}(F)$. This also provides $J^0 \subset \mathrm{GL}_n(\mathcal{O}_F)$.

Let I be the ordered partition $(n, 1)$ of $n+1$. We are interested in the classification of typical representations for the component $[M_I, \sigma \boxtimes \chi]$. From the lemma 2.2.7 it is enough to classify the typical representations for the component $s = [M_I, \sigma \boxtimes \mathrm{id}]$.

To classify typical representations for the component $s = [M_I, \sigma \boxtimes \mathrm{id}]$ it is enough to examine which $\mathrm{GL}_{n+1}(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \mathrm{id})$$

are typical for the component $[M, \sigma \boxtimes \mathrm{id}]$. Let τ be the unique typical representation contained in the representation σ. It follows from lemma 2.2.4 that the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id})$$

has a complement in

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi)$$

whose irreducible sub-representations are atypical.

Now we have to look for typical representations occurring in the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

For this purpose we will define the groups H_m for $m \geq N_0$ for some positive integer N_0, $\cap_{m \geq N_0} H_m = P_I(\mathcal{O}_F)$, H_m has Iwahori decomposition with respect to P_I and its Levi-subgroup M_I and $\tau \boxtimes \mathrm{id}$ admits an extension to H_{N_0} with $H_{N_0} \cap \bar{U}_I$ and $H_{N_0} \cap U_I$ in the kernel of this extension. The construction of H_m would give us

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_0} \mathrm{ind}_{H_m}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

Before we start this construction it is instructive to first examine the Bushnell-Kutzko semi-simple type for the component $[M_I, \sigma \boxtimes \mathrm{id}]$.

Let us recall some standard material required from [BK99]. First let us begin with lattice sequences. A lattice sequence is a function Λ from $\mathbb{Z}$ to the set of $\mathcal{O}_F$-lattices in a F-vector space V such that $\Lambda(n+1) \subseteq \Lambda(n)$ for $n \in \mathbb{Z}$

and there exists an $e(\Lambda) \in \mathbb{Z}$ such that $\Lambda(n + e(\Lambda)) = \mathfrak{P}_F \Lambda(n)$ for all $n \in \mathbb{Z}$. A lattice chain is a lattice sequence with the strict inclusion between $\Lambda(n+1)$ and $\Lambda(n)$ for all $n \in \mathbb{Z}$. One extends the function Λ to the set of real numbers by setting

$$\Lambda(r) = \Lambda(-[-r])$$

for all $r \in \mathbb{R}$ and $[x]$ is the greatest integer less than or equal to x. Given two lattice sequences Λ_1 and Λ_2 in the vector spaces V_1 and V_2 over F, Bushnell and Kutzko defined the direct sum say Λ of Λ_1 and Λ_2, a lattice sequence, in the vector space $V_1 \oplus V_2$. Let $e = \text{lcm}(e(\Lambda_1), e(\Lambda_2))$ Then

$$\Lambda(er) = \Lambda_1(e_1 r) \oplus \Lambda(e_2 r).$$

Given a lattice sequence Λ in a vector space V one can define a filtration $\{a_r(\Lambda) \mid r \in \mathbb{R}\}$ on the algebra $\text{End}_F(V)$ given by the equation

$$a_r(\Lambda) = \{x \in \text{End}_F(V) |\ x\Lambda(i) \subseteq \Lambda(i+r)\ \forall\ i \in \mathbb{Z}\}.$$

We also define $u_r(\Lambda)$ for $r > 0$ and $r \in \mathbb{Z}$ to be $1 + a_r(\Lambda)$ and $u_0(\Lambda)$ is the group of units in the order $a_0(\Lambda)$.

Let (J_s, λ_s) be the Bushnell-Kutzko type for the component

$$[\text{GL}_n(F) \times \text{GL}_1(F), \sigma \boxtimes \text{id}].$$

The group J_s satisfies Iwahori decomposition with respect to the parabolic subgroup P_I and the Levi-subgroup M_I. Let us recall that we have the stratum $[\mathfrak{A}, l, 0, \beta]$ defining the simple type (J^0, λ) for the inertial class $[\text{GL}_n(F), \sigma]$. The order $\mathfrak{A}$ is defined by a lattice chain Λ_1 with values in sub-lattices of $\mathcal{O}_F^n$. We denote by Λ_2 the lattice chain defined by $\Lambda_2(i) = \mathfrak{P}_F^i$ for all $i \in \mathbb{Z}$. Then we have

1. $J_s \cap U_{(n,1)} = u_0(\Lambda_1 \oplus \Lambda_2) \cap U_{(n,1)}$.
2. $J_s \cap M$ is $J^0 \times \mathcal{O}_F^\times$.
3. $J_s \cap \bar{U}_{(n,1)} = u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_{(n,1)}$.
4. The restriction of λ_s to $J_s \cap M$ is isomorphic to $\lambda \boxtimes \text{id}$, the groups $J_s \cap \bar{U}_{(n,1)}$ and $J_s \cap U_{(n,1)}$ are contained in the kernel of λ_s.

We refer to [BK99][Section 8, paragraph 8.3.1] for the construction of the above Bushnell-Kutzko's type.

Now we make an explicit calculation of the terms $u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_{(n,1)}$ and $u_0(\Lambda_1 \oplus \Lambda_2) \cap U_{(n,1)}$. Note that the periodicity of the direct sum $\Lambda_1 \oplus \Lambda_2$ is the least common multiple of the periodicity of the two lattice sequences Λ_1 and Λ_2. We hence deduce that the periodicity of the lattice sequence Λ is e

where e is the period of the lattice chain Λ_1. Let t be an integer such that $0 \leq t \leq e-1$ and L_0 be the free $\mathcal{O}_F$ module $\mathcal{O}_F^{n/e}$. The lattice chain Λ_1 is given by :

$$\Lambda_1(t) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0)$$

where the L_0 is repeated $e-t$ times and $\varpi_F L_0$ is repeated t times in the first and second direct summand respectively. Hence the lattice chain Λ is given by

$$\Lambda(0) = \Lambda_1(0) \oplus \Lambda_2(0) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus \mathcal{O}_F$$

and

$$\Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t/e) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0) \oplus \mathfrak{P}_F\,.$$

for $0 \leq t \leq e-1$. We observe that the lattice sequence Λ is a lattice chain and the units of the hereditary order $a_0(\Lambda)$ corresponding to Λ, in the notation of chapter 3, are given by $P_J(1)$ where $J = (n/e, n/e, \ldots (e-1 \text{ times}) \ldots, n/e, n/e+1)$.

We note that $u_0(\Lambda) \cap U_{(n,1)} = U_{(n,1)}(\mathcal{O}_F)$. Let $\bar{\mathfrak{n}}_{(n,1)}$ be the lower nilpotent matrices of the type $(n,1)$ i.e the Lie-algebra of $\bar{U}_{(n,1)}$. We have the identity

$$u_{l+1}(\Lambda) \cap \bar{U}_{(n,1)} = 1 + (a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}).$$

Let $l+1 = el' + r$ where $0 \leq r < e$. Then from the observation that Λ is a lattice chain of periodicity e we deduce that

$$a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)} = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}).$$

Finally it remains to calculate the group $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$. We note that $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$ is the following set

$$\{x \in M_{n+1}(F) \cap \bar{\mathfrak{n}}_{(n,1)} \mid x\Lambda(i) \subseteq \Lambda(i+r)\ \forall\ i \in \mathbb{Z}\}.$$

For $r \geq 1$ the matrix $A = [M_1, M_2, \ldots, M_e, 0]$ in $\bar{\mathfrak{n}}_{(n,1)}$ (M_i is a matrix of type $1 \times n/e$ for $1 \leq i \leq e$) belongs to the set $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$ if and only if the following conditions are satisfied.

1. $M_i \in \varpi_F^2 M_{1\times e}(\mathcal{O}_F)$ for $i \leq r-1$ and
2. $M_i \in \varpi_F M_{1\times e}(\mathcal{O}_F)$ for $i > r-1$.

If $r = 0$ and $e > 1$ then we know that $M_i \in \varpi_F M_{1\times n/e}(\mathcal{O}_F)$ for $1 \leq i \leq e-1$ and $M_e \in M_{1\times n/e}(\mathcal{O}_F)$. If $r = 0$ and $e = 1$ then we have $A \in M_{1\times n}(\mathcal{O}_F)$. This completes the description of the Bushnell-Kutzko semi-simple type.

5.2 Preliminaries.

Let m be a positive integer and $P_I(m)$ be the inverse image of the group $P_I(\mathcal{O}_F / \mathfrak{P}_F{}^m)$ under the mod-$\mathfrak{P}_F{}^m$ reduction of $\mathrm{GL}_{n+1}(\mathcal{O}_F)$. There exists a positive integer N_1 such that the principal congruence sub-group of level N_1 is contained in the kernel of the representation τ. The representation $\tau \boxtimes \mathrm{id}$ of $M_I(\mathcal{O}_F / \mathfrak{P}_F{}^m)$ now extends to a representation of $P_I(m)$ by inflation for all $m > N_1$. We note that $P_I(m) \cap \bar{U}_I$ and $P_I(m) \cap U_I$ are both contained in the kernel of this extension. Now applying lemma 2.2.5 to the sequence of groups $P_I(m)$ and $\tau \boxtimes \mathrm{id}$ for $m \geq N_1$ we get that

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_1+1} \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

This concludes that the typical representations occur as sub-representations of

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id})$$

for some positive integer $m \geq N_1 + 1$.

For making Mackey decompositions easier and other reasons, it is convenient to work with a smaller subgroup $P_I^0(m)$ of $P_I(m)$. We now modify the representation

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

We recall that $K_n(p)$ is the principal congruence subgroup of level p of $\mathrm{GL}_n(\mathcal{O}_F)$. The group J^0 contains the group $U^{[l/2]+1}(\mathfrak{A})$ ($U^{[l/2]+1}(\mathfrak{A}) \subset H^0 \subset J^0$). The representation λ restricted to the group $U^{[l/2]+1}(\mathfrak{A})$ is a direct sum of the same character ψ_β which is trivial on the group $U^{l+1}(\mathfrak{A})$. We also recall the notation that $l+1 = el' + r$ where $0 \leq r \leq e-1$. We note that $U^{l+1}(\mathfrak{A}) = 1_n + \varpi_F^{l'}\mathfrak{P}_{\mathfrak{A}}^r$. If $r = 0$ then $K_n(1) \subset \mathfrak{P}_{\mathfrak{A}}^r$. If $r > 1$ then from the formulas [BK93][2.5.2] we get that $K_n(2) \subset \mathfrak{P}_{\mathfrak{A}}^r$ for $0 \leq r < e$. This shows that the representation λ is trivial on $K_n(N_s)$ where N_s is given by:

Notation 5.1. *From now we fix $N_s = [(l+1)/e] + 1$ if $r = 0$ and $e > 1$, if $r = 0$ and $e = 1$ then $N_s = l + 1$ and $N_s = [(l+1)/e] + 2$ if $r \geq 1$.*

Let π be the projection map

$$P_I(\mathcal{O}_F) \to M_I(\mathcal{O}_F).$$

For $m \geq N_s$ we denote by $P_I^0(m)$ the group $K_{n+1}(m)\pi^{-1}(J^0 \times \mathcal{O}_F^\times)$. Since $K_{n+1}(m) \cap P_I \subset \pi^{-1}(J^0 \times \mathcal{O}_F^\times)$ the group $P_I^0(m)$ satisfies Iwahori decomposition with respect to the subgroup P_I and its Levi-subgroup M_I i.e we have

$$P_I^0(m) = (P_I^0(m) \cap U_I)(P_I^0(m) \cap M_I)(P_I^0(m) \cap \bar{U}_I)$$

where $P_I^0(m) \cap U_I$ is $U_I(\mathcal{O}_F)$, $P_I^0(m) \cap M_I$ is $J^0 \times \mathcal{O}_F^\times$ and $(P_I^0(m) \cap \bar{U}_I)$ is $K_{n+1}(m) \cap \bar{U}_I$. We observe that $\lambda \boxtimes \mathrm{id}$ extends to a representation of $P^0(m)$ for all $m \geq N_s$. Now the representation $\tau \boxtimes \mathrm{id}$ of $\mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_F^\times$ is isomorphic to

$$\{\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda)\} \boxtimes \mathrm{id}\,.$$

Applying lemma 2.2.6 to the groups $J_1 = P_I(m)$ and $J_2 = P_I^0(m)$ and $\lambda = \lambda \boxtimes \mathrm{id}$ we get that

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id})$$

for all $m \geq N_s$. Hence we have

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_s} \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}).$$

Now a typical representation occurs as a sub-representation of

$$\mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id})$$

for some $m \geq N_s$.

As we did in the previous chapters we first have to understand the representation

$$\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id})$$

for $m \geq N_s$. The strategy is fairly standard by now. We will first consider a convenient normal subgroup $K_I(m)$ of $P_I(m)$ such that $P_I^0(m)$ is equal to $K_I(m)P_I^0(m+1)$ and $K_I(m) \cap P_I^0(m+1) = K_I(m+1)$ for $m \geq N_s$. For $m \geq N_s$ we define $K_I(m)$ to be the group $K_{n+1}(m)\pi^{-1}(K_n(N_s) \times (1 + \mathfrak{P}_F^{N_s}))$. This group does satisfy the above two properties.

Lemma 5.2.1. *The group $K_I(m)$ is a normal subgroup of $P_I^0(m)$ and $K_I(m+1)$ is a normal subgroup of $K_I(m)$ for all $m \geq N_s$.*

Proof. By definition of the groups $K_I(m)$ we have $K_I(m) \cap U_I = P_I^0(m) \cap U_I$ and $K_I(m) \cap \bar{U}_I = P_I^0(m) \cap \bar{U}_I$. To show the normality of $K_I(m)$ in $P_I(m)$ we have to verify that $P_I^0(m) \cap M_I$ normalize the group $K_I(m)$. But $P_I^0(m) \cap M_I$ normalizes the group $K_I(m) \cap U_I = U_I(\mathcal{O}_F)$ and $K_I(m) \cap \bar{U}_I = \bar{U}_I(\varpi_F^m \mathcal{O}_F)$. The group $K_I(m) \cap M_I$ is a normal subgroup of $M_I(\mathcal{O}_F)$ and hence $P_I^0(m) \cap M_I$ normalizes $K_I(m) \cap M_I$. This shows the first part of the lemma.

Since $K_I(m) \cap P_I = K_I(m+1) \cap P_I$, we have to check that $K_I(m) \cap \bar{U}_I$ normalizes the group $K_I(m+1)$. We note that $\bar{U}_I$ is abelian hence we have to check that the conjugations $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $K_I(m+1)$ for all $u^- \in K_I(m) \cap \bar{U}_I$, $j \in K_I(m+1) \cap M_I = K_I(m) \cap M_I$ and

$u^+ \in K_I(m+1) \cap U_I = U_I(\mathcal{O}_F)$. Let us begin with the element $u^- j (u^-)^{-1}$. We have $u^- j (u^-)^- = j\{j^{-1}u^- j(u^-)^{-1}\}$. Let

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

be the block diagonal form of j and u^-; $J_1 \in K_n(N_s), j_1 \in 1 + \mathfrak{P}_F{}^{N_s}$ and $U^- \in \varpi_F^m M_{1\times n}(\mathcal{O}_F)$. The element $j^{-1}u^- j(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_n & 0 \\ j_1^{-1}U^- J_1 - U^- & 1 \end{pmatrix}$$

We note that the matrix $j_1^{-1}U^- J_1 - U^-$ belongs to $\varpi_F^{m+1} M_{1\times n}(\mathcal{O}_F)$. This shows that $j^{-1}u^- j(u^-)^{-1} \in K_I(m+1) \cap \bar{U}_I$. Hence we get that

$$u^- j(u^-)^- j\{j^{-1}u^- j(u^-)^{-1}\} \in K_I(m+1).$$

We now consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in its block matrix form as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix}$$

where $U^+ \in M_{n\times 1}(\mathcal{O}_F)$. Now the conjugation $u^- u^+ (u^-)^{-1}$ in the block matrix from is as follows

$$\begin{pmatrix} 1_n - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1 \end{pmatrix}.$$

Since $U^- U^+ U^- \in \varpi^{m+1} M_{1\times n}(\mathcal{O}_F)$, we conclude that $u^- u^+ (u^-)^{-1} \in K_I(m+1)$. This ends the proof of this lemma. □

We use Mackey decomposition to the double coset decomposition $P_I^0(m) = K_I(m) P_I^0(m+1)$ to get that

$$\mathrm{res}_{K_I(m)} \mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(m+1)}^{K_I(m)}(\mathrm{id}).$$

It follows from Iwahori decomposition that the inclusion of $K_I(m) \cap \bar{U}_I$ in $K_I(m)$ induces an isomorphism between $K_I(m)/K_I(m+1)$ and the abelian group

$$\frac{K_I(m) \cap \bar{U}_I}{K_I(m+1) \cap \bar{U}_I}.$$

Hence the representation $\mathrm{ind}_{K_I(m+1)}^{K_I(m)}(\mathrm{id})$ decomposes as a direct sum of characters η_k for $1 \leq k \leq p$ where η_k is trivial on $K_I(m+1)$. The group $P_I^0(m)$

acts on these characters and let η_{n_k} be the set of representatives for the orbits under this action. We also denote by $Z(\eta)$ the $P_I^0(m)$-stabilizer of the character η. Now Clifford theory gives us the isomorphism

$$\operatorname{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) = \bigoplus_{\eta_{n_k}} \operatorname{ind}_{Z(\eta_{n_k})}^{P_I^0(m)}(U_{\eta_{n_k}}) \tag{5.1}$$

where η_{n_k} is a representative for the action of $P_I^0(m)$ on the set of characters η_k and $U_{\eta_{n_k}}$ is an irreducible representation of $Z(\eta_{n_k})$.

Now we have to bound the group $Z(\eta)$. We note that $P_I^0(m)$ is equal to $(P_I^0(m) \cap M_I)K_I(m)$ and hence $Z(\eta) = (Z(\eta) \cap M_I)K_I(m)$. To bound the group $Z(\eta)$ we can only need to control $Z(\eta) \cap M_I$. Let $u^- \in K_I(m) \cap \bar{I}_I$ and

$$\begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

be the block form of u^- where U^- is a matrix in $\varpi_F^m M_{1\times n}(\mathcal{O}_F)$. The map $u^- \mapsto \varpi_F^{-m} U^-$ induces an $M_I(\mathcal{O}_F)$-equivariant isomorphism between $M_{1\times n}(k_F)$ and the quotient

$$\frac{K_I(m) \cap \bar{U}_I}{K_I(m+1) \cap \bar{U}_I}.$$

We also have an $M_I(\mathcal{O}_F)$-equivariant isomorphism between $M_{n\times 1}(k_F)$ and $\widehat{M_{1\times n}(k_F)}$ (see lemma 3.0.14). We note that $P_I^0(m) \cap M_I = J^0 \times \mathcal{O}_F^\times$.

Let η be a non-trivial character of $K_I(m)$ which is trivial on $K_I(m+1)$. We will now bound the subgroup $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$ for $\eta \neq \mathrm{id}$ and this will be enough for our purpose. Since we have a $M_I(\mathcal{O}_F)$-equivariant isomorphism between the group of characters on the quotient $K_I(m)/K_I(m+1)$ with $M_{n\times 1}(k_F)$, we can as well study the group $Z(A) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$ where $Z(A)$ is the $M_I(\mathcal{O}_F)$-stabilizer of a non-zero matrix $A \in M_{n\times 1}(k_F)$. The action of the group $M_I(\mathcal{O}_F)$ factorizes through $K_n(1) \times (1 + \mathfrak{P}_F)$ from which we conclude that $(1_n + \mathfrak{D}^e) \times (1 + \mathfrak{P}_F)$ is contained in the kernel of the action of $M_I(\mathcal{O}_F)$. This reduces our situation to the following setting. The group $\mathrm{GL}_n(k_F) \times k_F^\times$ acts on $M_{n\times 1}$ by setting

$$(g_1, g_2)A = g_1 A g_2^{-1}$$

where $g_1 \in \mathrm{GL}_n(k_F)$, $g_2 \in k_F^\times$ and $A \in M_{n\times 1}(k_F)$. If we fix a $\mathcal{O}_E$-basis as in the previous paragraph then a k_F-basis for the vector space

$$(\mathcal{O}_E/\varpi_F\mathcal{O}_E)^{n/ef} = (\mathcal{O}_E/\mathfrak{P}_E^e)^{n/ef}$$

we get the inclusion

$$\mathrm{GL}_{n/ef}(\mathcal{O}_E \,/\, \mathfrak{P}_E{}^e) = U^0(\mathfrak{B})/U^e(\mathfrak{B}) \hookrightarrow \mathrm{GL}_n(k_F).$$

We are interested in the mod $\mathfrak{P}_E$ reduction of the first projection of

$$Z_{\mathrm{GL}_{n/ef}(\mathcal{O}_E\,/\,\mathfrak{P}_E{}^e)\times k_F^\times}(A)$$

for some non-zero matrix A in $M_{n\times 1}(k_F)$ (recall that $\lambda = \kappa\otimes\rho$ where ρ is a cuspidal representation of $\mathrm{GL}_{n/ef}(k_E) = U^0(\mathfrak{B})/U^1(\mathfrak{B})$). **We put** $n_0 = n/ef$.

Let ϖ_E be a uniformizer of $\mathcal{O}_E$. Let N be the operator on the k_E-vector space $V := (\mathcal{O}_E/\mathfrak{P}_E^e)^{n_0}$ given by

$$N(v) = \varpi_E.v.$$

Since $\mathcal{O}_E/\mathfrak{P}_E^e = k_E \oplus k_E\overline{\varpi_E} \oplus k_E\overline{\varpi_E}^2 \oplus \cdots \oplus k_E\overline{\varpi_E}^{e-1}$, we obtain a decomposition of $V = V_1 \oplus V_2 \oplus \cdots \oplus V_e$ such that N restricted to V_i is an isomorphism onto V_{i+1} for $i < e$ and N acts trivially on V_e. The mod $\mathfrak{P}_E$-reduction of V is the projection onto the first factor V_1. Any $k_E[N]$-linear map T is determined by its restriction to the space V_1. Given a map $T \in \mathrm{Hom}_{k_E}(V_1, V)$ we obtain an extension $\tilde{T} \in \mathrm{End}_{k_E[N]}(V)$ by setting

$$\tilde{T}(v) = N^{(i-1)}T(N^{-(i-1)}v)$$

for all $v \in V_i$ and $1 \le i \le e$. The map $T \mapsto \tilde{T}$ gives us an isomorphism of vector spaces

$$\mathrm{Hom}_{k_E}(V_1, V) \simeq \mathrm{End}_{k_E[N]}(V, V). \tag{5.2}$$

We may write $V = V_1 \oplus NV$. This shows that the mod $\mathfrak{P}_E$ reduction map say π_E is given by sending $\tilde{T}$ to $p_1 \circ \tilde{T}_{|V_1}$ where p_1 is the projection onto the first factor of the direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_e$. Now $\widetilde{\mathrm{End}_{k_E}}(V_1)$ is a subspace of $\mathrm{Hom}_{k_E}(V_1, V)$ and mod $\mathfrak{P}_E$ reduction of $\widetilde{\mathrm{End}_{k_E}}(V_1)$ (the image of $\mathrm{End}_{k_E}(V_1)$ under the map $T \mapsto \tilde{T}$) is identity on $\mathrm{End}_{k_E}(V_1)$. Hence $\mathrm{Aut}_{k_E[N]}(V)$ is the semi-direct product $\widetilde{\mathrm{Aut}_{k_E}}(V_1)\ker(\pi_E)$.

Now we have the embedding of $\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e)$ in $\mathrm{GL}_n(k_F)$ by considering V as a k_F-vector space. Let P be a parabolic subgroup fixing the flag $\mathcal{F}^i = \oplus_{j=1}^i V_i$ and M be its Levi-subgroup fixing the decomposition $V_1 \oplus V_2 \oplus \cdots \oplus V_e$. Now $\widetilde{\mathrm{Aut}_{k_E}}(V_1)$ diagonally embeds in M and $\ker(\pi_E)$ is a subgroup of the radical of P. The group $\mathrm{GL}_n(k_F) \times k_F^\times$ acts on $M_{n\times 1}(k_F)$ by the map $(g_1, g_2)A \mapsto g_1 A g_2^{-1}$ where $g_1 \in \mathrm{GL}_n(k_F)$, $g_2 \in k_F^\times$ and $A \in M_{n\times 1}(k_F)$. We now have the action of $\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e) \times k_F^\times$ on $M_{n\times 1}(k_F)$ by restriction from $\mathrm{GL}_n(k_F) \times k_F^\times$. We are interested in

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\}$$

for some $A \in M_{n\times 1}(k_F)\backslash\{0\}$. We first look at $Z_{P\times k_F^\times}(A)$. Let (A_{ij}) be an element of P in its block form. Let $(A_1, A_2, \ldots, A_e)^{tr}$ be the block form of

A where A_j is a block of size $1 \times n_0$. If k is the largest positive integer such that $A_k \neq 0$ and $A_k = 0$ then we get that $A_{kk} A_k a^{-1} = A_k$ for all $((A_{ij}), a) \in Z_{P \times k_F^\times}(A)$. Hence $\{A_{kk} \mid ((A_{ij}), a) \in Z_{P \times k_F^\times}(A)\}$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(V_k)$. We now conclude that

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e) \times k_F^\times}(A)\}$$

is a subgroup of $H \times k_F^\times$ where H is a subgroup of $\mathrm{Aut}_{k_E}(V_1)$ whose image under the inclusion map $\mathrm{Aut}_{k_E}(V_1) \hookrightarrow \mathrm{Aut}_{k_F}(V_1)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{Aut}_{k_F}(V_1)$.

We recall the following proposition due to Paskunas (see [Pas05, Definition 6.2, lemma 6.5, Proposition 6.8]).

Proposition 5.2.2. *Let V be a k_E-vector space with (finite) dimension greater than one. Let ρ be a cuspidal representation of $\mathrm{Aut}_{k_E}(V)$. Let H be a subgroup of $\mathrm{Aut}_{k_E}(V)$ such that the image of H under the inclusion map $\mathrm{Aut}_{k_E}(V) \hookrightarrow \mathrm{Aut}_{k_F}(V)$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(V)$. For every H-irreducible sub-representation ξ of $\mathrm{res}_H(\rho)$ there exists an irreducible representation ρ' of $\mathrm{Aut}_{k_E}(V)$ such that $\rho' \not\simeq \rho$ and $\mathrm{Hom}_H(\xi, \rho') \neq 0$.*

Going back to $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$, for $n_0 > 1$, we get that for every irreducible sub-representation ξ of

$$\mathrm{res}_{Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)}((\kappa \otimes \rho) \boxtimes \mathrm{id})$$

there exists an irreducible representation ρ' of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{Hom}_{Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)}(\xi, (\kappa \otimes \rho') \boxtimes \mathrm{id}) \neq 0.$$

For the case $n_0 = 1$ and $\# k_F > 2$, we have to look at

$$(\pi_E \times \mathrm{id})\{Z_{\mathcal{O}_E/\mathfrak{P}_E^e{}^\times \times k_F^\times}(A)\} \tag{5.3}$$

for some nonzero matrix $A \in M_{n \times 1}(k_F)$. We notice that the group (5.3) is of the form $\{(a, a) | a \in k_F^\times\}$ if $k_E = k_F$. Let k_E be a proper extension of k_F. If (a, b) be an element of the centralizer (5.3) then $a A_k b^{-1} = A_k$ (A_k is defined in the previous paragraph). This shows that a lies in a proper parabolic subgroup of $\mathrm{GL}_f(k_F)$. This shows that the group (5.3) is of the form $\{(a, b) \ | a \in \mathbb{F}^\times, b \in k_F^\times\}$ where $\mathbb{F}$ is a proper sub-field of k_E. In the first case we consider a non-trivial character η of $U^0(\mathfrak{B})/U^1(\mathfrak{B}) = k_F^\times$. We observe that

$$\mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda\eta \boxtimes \eta^{-1}) \simeq \mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda \boxtimes \mathrm{id})$$

and moreover $[M, \sigma \boxtimes \mathrm{id}]$ and $[M, \sigma' \boxtimes \eta^{-1}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda \otimes \eta)$.

In the second case where k_E is a proper extension of k_F, we consider a non-trivial character η of $k_E^\times$ which is trivial on $\mathbb{F}^\times$. We note that

$$\mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda\eta \boxtimes \mathrm{id}) \simeq \mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda \boxtimes \mathrm{id})$$

and moreover $[M, \sigma \boxtimes \mathrm{id}]$ and $[M, \sigma' \boxtimes \mathrm{id}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda \otimes \eta)$.
With this we finish our preliminaries.

5.3 Main result

In this section we will prove the main result of this chapter. By Frobenius reciprocity we get that $\lambda \boxtimes \mathrm{id}$ occurs with multiplicity one in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda \boxtimes \mathrm{id})$ for all $m \geq N_s$. We denote by $U_m^0(\lambda \boxtimes \mathrm{id})$ the complement of $\lambda \boxtimes \mathrm{id}$ in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda \boxtimes \mathrm{id})$. We use the notation $U_m(\lambda \boxtimes \mathrm{id})$ for the representation

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m^0(\lambda \boxtimes \mathrm{id})\}.$$

Theorem 5.3.1. *Let $\#k_F > 2$. The $\mathrm{GL}_{n+1}(\mathcal{O}_F)$-irreducible sub-representations of $U_m(\lambda \boxtimes \mathrm{id})$ are atypical for the component $[\mathrm{GL}_n(F) \times F^\times, \sigma \boxtimes \mathrm{id}]$ for all $m \geq N_s$.*

Proof. We prove the theorem by induction on the positive integer $m \geq N_s$. For $m = N_s$ the representation $U_m(\lambda \boxtimes \mathrm{id})$ is trivial hence the theorem is vacuously true. We suppose the theorem is true for some positive integer $m > N_s$ we will show the same for $m+1$. We first note that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \otimes (\lambda \boxtimes \mathrm{id})\}.$$

From the decomposition (5.1) we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Since there is a unique orbit of characters η_k consisting of the identity we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.4}$$

Let Γ be an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.5}$$

We have two cases $n_0 = 1$ and $n_0 > 1$. If $n_0 = 1$ we have seen that we can find a non-trivial character η of $k_E^\times = U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda\eta \boxtimes \eta^{-1}) \otimes U_{\eta_{n_k}}\}$$

or

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda\eta \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Hence in this case the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}$$

are atypical representations.

Now consider the case $n_0 > 1$. In this case there exists an irreducible representation ξ of $p_1(Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times))$ such that Γ is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{((\xi \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.6}$$

Now proposition 5.2.2 gives us an irreducible representation $\rho' \not\simeq \rho$ of $U^0(\mathfrak{B})$ obtained by inflation of an irreducible representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that ξ is contained in ρ'. Now the representation 5.6 is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{((\rho' \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

The above representation is contained in

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\rho' \boxtimes \kappa) \boxtimes \mathrm{id}).$$

The above representation by lemma 2.2.6 is isomorphic to the representation

$$\mathrm{ind}_{P_I(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id}) \tag{5.7}$$

where τ' is

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\rho' \boxtimes \kappa).$$

We will show that irreducible sub-representations of (5.7) are atypical for the component

$$[\mathrm{GL}_{n+1}(F), \sigma \boxtimes \mathrm{id}].$$

Any irreducible sub-representation of (5.7) occurs as a sub-representation of

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

where γ is an irreducible sub-representation of τ'. Now γ is contained in an irreducible smooth representation say π of $\mathrm{GL}_n(F)$. By Frobenius reciprocity this is possible only if the representation $\rho' \otimes \kappa$ of J^0 is contained in π. We have two possible situations either ρ' is cuspidal or otherwise. If ρ' is cuspidal then we can say that π is a supercuspidal representation such that $\pi \not\simeq \sigma$ hence the representation

$$\operatorname{ind}_{P_I(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

occurs in

$$\operatorname{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)} i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\pi \boxtimes \mathrm{id})$$

at the same time $[\mathrm{GL}_n(F) \times F^\times, \pi \boxtimes \mathrm{id}] \neq [\mathrm{GL}_n(F) \times F^\times, \sigma \boxtimes \mathrm{id}]$. This shows that irreducible sub-representations of 5.7 are atypical representations.

If ρ' is not cuspidal the representation $\rho' \boxtimes \kappa$ is still irreducible [BK93, Chapter 5, Proposition 5.3.2(3)]. If $(J^0, \rho' \boxtimes \kappa)$ is contained in a smooth irreducible representation π then π also contains a simple-type $(J_1^0, \rho_1 \boxtimes \kappa_1)$ which is not maximal [BK93, Chapter 8, 8.3.5] (we also refer to the article [BH13, Lemma 2, Proposition 1] for quick reference). From this we conclude that π is not a cuspidal representation hence (5.7) is contained in a

$$\operatorname{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)} i_{P_J}^{\mathrm{GL}_{n+1}(F)}(\sigma_J)$$

where J is a strict refinement of the partition I. Hence we get that the irreducible sub-representations of (5.7) are atypical representations. $\square$

The above theorem shows that typical representations occur as sub-representations of

$$\operatorname{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}).$$

The above representation may still contain atypical representations. We will indeed show that this is the case and complete the classification.

The first observation is that the (the semi-simple type after fixing J^0) Bushnell-Kutzko type J_s for $s = [M_I, \sigma \boxtimes \mathrm{id}]$ is contains the group $P^0(N_s)$. Hence we will try to decompose the representation

$$\operatorname{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}). \tag{5.8}$$

We also note that $P^0(N_s) \cap P_I = J_s \cap P_I$. Let $l + 1 = el' + r$ where $0 \leq r < e$. If $e = 1$ then $J_s = P_I^0(N_s)$ hence we have nothing to analyse further the theorem 5.3.1 completes the classification of typical representations. From now we assume that $e > 1$. We will first verify that the group $U_I(\mathcal{O}_F)$ acts trivially on the representation (5.8).

Let u^+ and u^- be two matrices from $J_s \cap U_I = U_I(\mathcal{O}_F)$ and $J_s \cap \bar{U}_I$ respectively. Let u^+ and u^- in block form be written as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}.$$

The block form of the conjugation $u^- u^+ (u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_n - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1 \end{pmatrix}.$$

Now $U^- \in a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_I)$. If $r \geq 1$ the valuation of each entry of a matrix in $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is at least one. This shows that the valuation of each entry in $U^-U^+U^-$ is at least $l' + 2$ this shows that the conjugation $u^- u^+ (u^-)^{-1}$ lies in the group $P^0(N_s)$. If $r = 0$ and $l' = 0$ we are in the case where σ is a level-zero cuspidal representation and in this case $J_s = P_I^0(N_s)$. If $r = 0$ and $l' > 0$ then valuation of each entry in $U^-U^+U^-$ has valuation $2l' > l' + 1$ and hence $u^- u^+ (u^-)^{-1} \in P_I^0(N_s)$. Hence the group $U_I(\mathcal{O}_F)$ acts trivially on the representation (5.8).

From the Iwahori decomposition of the group J_s we get that J_s is equal to $(J_s \cap \bar{P}_I)P_I^0(N_s)$. Hence we get that

$$\operatorname{res}_{J_s \cap \bar{P}_I} \operatorname{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \operatorname{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}).$$

Note that $J_s \cap \bar{P}_I$ is a semi-direct product of the groups $(J_s \cap M_I)$ and $(J_s \cap \bar{U}_I)$. Let η_k for $1 \leq k \leq t$ (we mean counting them with their multiplicity, but in our case the multiplicity is one) be all the characters of the group $J_s \cap \bar{U}_I$ which are trivial on the group $P_I^0(N_s) \cap \bar{U}_I$. The group $J_s \cap \bar{P}_I$ acts on these characters and let $\{\eta_{k_p}\}$ be a set of representatives for the orbits under this action. We denote by $Z(\eta_{k_p})$ the $J_s \cap \bar{P}_I$ stabiliser of the character η_{k_P}. Now Clifford theory gives the decomposition

$$\operatorname{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \operatorname{ind}_{Z(\eta_{k_p})}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}})$$

where $U_{\eta_{k_p}}$ is an irreducible representation of $Z(\eta_{k_p})$. We note that the character id occurs with a multiplicity one in the list of characters η_k.

The representation $U_{\eta_{k_p}}$ is the isotypic component of the character η_{k_p} in the representation

$$\operatorname{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}).$$

which naturally has the action of $Z(\eta_{k_p})$. Now if K_s is the kernel of the representation (5.8) then $K_s \cap Z(\eta_{k_p})$ acts trivially on $U_{\eta_{k_p}}$. Hence we can extend the representation $U_{\eta_{k_p}}$ to the group $Z(\eta_{k_P})K_s$ such that K_s acts trivially on the entered representation. Now consider the representation

$$\pi = \operatorname{ind}_{Z(\eta_{k_P})K_s}^{J_s} U_{\eta_{k_p}}.$$

Note that $K_s \cap \bar{P}_I$ is contained in the group $Z(\eta_{k_p}) \cap \bar{P}_I$ and moreover $U_I(\mathcal{O}_F)$ is contained in K_s hence $J_s = (J_s \cap \bar{P}_I)Z(\eta_{k_p})K_s$ hence from Mackey decomposition we have

$$\operatorname{res}_{J_s \cap \bar{P}_I} \operatorname{ind}_{Z(\eta_{k_P})K_s}^{J_s} U_{\eta_{k_p}} \simeq \operatorname{ind}_{Z(\eta_{k_P})K_s \cap (J_s \cap \bar{P}_I)}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}}) \simeq \operatorname{ind}_{Z(\eta_{k_p})}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}}).$$

We hence have

$$\operatorname{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \operatorname{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}. \tag{5.9}$$

Now using the decomposition (5.9) we get the decomposition

$$\operatorname{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \operatorname{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}.$$

Note that the character id occurs with multiplicity one among the characters η_k and the fact that $Z(\mathrm{id})K_s = (J_s \cap \bar{P}_I)K_s = J_s$ implies the following isomorphism

$$\operatorname{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda\boxtimes\mathrm{id}) \simeq \operatorname{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda\boxtimes\mathrm{id}) \oplus \bigoplus_{\eta_{k_p}\neq\mathrm{id}} \operatorname{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}}\otimes(\lambda\boxtimes\mathrm{id})\}. \tag{5.10}$$

Lemma 5.3.2. *Let $\#k_F > 2$ and η_{k_p} be a non-trivial character. The irreducible sub-representations of*

$$\operatorname{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

are atypical.

Proof. We observe that $Z(\eta_{k_p}) = (Z(\eta_{k_P}) \cap M_I)(J_s \cap \bar{U}_I)$. This shows that we have to bound the group $Z(\eta_{k_P}) \cap M_I$ for $\eta_{k_p} \neq$ id. Recall that η_k for $1 \leq k \leq t$ are the characters on the quotient group

$$\frac{(J_s \cap \bar{U}_I)}{(P_I^0(N_s) \cap \bar{U}_I)} \tag{5.11}$$

Now let u^- be a matrix from the group $J_s \cap \bar{U}_I$. In the block form the matrix u^- is of the form

$$\begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

where $U^- = [M_1, M_2, \ldots, M_e]$, M_i is a matrix of size $(1 \times n/e)$. Let $\delta = N_s - 1$ then the map Φ

$$[M_1, M_2, \ldots, M_e] \mapsto [\varpi_F^\delta M_1, \varpi_F^\delta M_2, \ldots, \varpi_F^\delta M_e]$$

identifies the quotient 5.11 with the subspace of $M_{1\times n}(k_F)$. We have a $M_I(\mathcal{O}_F)$ equivariant map from group of characters of $M_{1\times n}(k_F)$ and $M_{n\times 1}(k_F)$ moreover $M_I(\mathcal{O}_F)$ acts through the quotient $M_I(k_F)$ (see lemma 3.0.14). The map Φ commutes with the action of $M_I \cap J_s$ since Φ is none other than conjugation by an element from the $Z(M_I)$ (The centre of M_I). Now the group $(U^0(\mathfrak{B}) \times \mathcal{O}_F^\times) \subset J_s \cap M_I$ acts on a non-zero matrix A in the space $M_{n\times 1}(k_F)$ through the quotient by $(1_n + \mathfrak{D}^e) \times (1 + \mathfrak{P}_F)$. Now recall that we denote by

π_E by mod $\mathfrak{P}_E$ reduction map. We have seen that (The paragraph above the proposition 5.2.2)

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_r(\mathcal{O}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\}$$

is a subgroup of $H \times k_F^\times$ where H is a subgroup of $\mathrm{GL}_{n/ef}(k_E)$ whose image under the inclusion map $\mathrm{GL}_{n/ef}(k_E) \hookrightarrow \mathrm{GL}_n(k_F)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{GL}_n(k_F)$. From the result of Paskunas 5.2.2 we get that for every irreducible representation ξ of

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho) \boxtimes \mathrm{id})\}$$

we can find an irreducible representation $\rho' \not\simeq \rho$ such that ξ occurs in the representation

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Hence irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

occur as a sub-representation of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Now the above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\kappa \otimes \rho') \boxtimes \mathrm{id}\} \simeq \mathrm{ind}_{P_I(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id})\}$$

where τ' is given by

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\kappa \otimes \rho' \boxtimes \mathrm{id}).$$

Any irreducible representation γ of τ' occurs in an irreducible smooth representation π of $\mathrm{GL}_n(F)$. If ρ' is cuspidal then $\kappa \otimes \rho'$ is contained in the representation γ and hence is contained in π which gives that π is cuspidal but it is not isomorphic to an unramified twist of σ. Now if ρ' is not cuspidal π is not cuspidal. Hence in every case π is not inertially equivalent to σ. This shows that irreducible sub-representations of

$$\mathrm{ind}_{P_I(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

are atypical. This shows the lemma. □

Theorem 5.3.3. *Let* Γ *be a typical representation for the component*

$$s = [M_I, \sigma \boxtimes \chi]$$

then Γ *is isomorphic to the representation*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda_s)$$

where (J_s, λ_s) is the Bushnell-Kutzko semi-simple type for the component s. If P is a parabolic subgroup containing M_I as a Levi-factor then Γ occurs with a multiplicity one in the representation

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_P^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi).$$

Proof. The representation $\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda_s)$ is irreducible since the intertwining of this representation is bounded by the group W_s where W_s is the set of representatives for $N_G(s)/M_I$. We can see that in our case W_s is trivial. We refer to [BK98][Lemma 11.5] for these results. Hence the uniqueness of the typical representation. The multiplicity follows from the results 2.2.4, 5.3.1 and 5.3.2. □