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Chapter 4

Principal series components

We denote by B_n the Borel subgroup of $\mathrm{GL}_n(F)$ consisting of upper triangular matrices. Let T_n and U_n be the maximal torus and the unipotent radical of B_n respectively. In this chapter we will classify typical representations for the components $s = [T_n, \chi]$ where χ is a character of T_n. Let τ be a typical representation for the component s. The $\mathrm{GL}_n(\mathcal{O}_F)$-representation τ occurs in a $\mathrm{GL}_n(F)$-irreducible smooth representation π. Let B be a Borel subgroup, T the maximal torus of B and χ' be a character of T. If (T, χ') and (T_n, χ) are inertially equivalent then the representation π occurs as a sub-quotient of $i_B^{\mathrm{GL}_n(F)}(\chi'')$ where χ'' is obtained from χ' by twisting with an unramified character of T. Now to classify typical representations it is enough to say which $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of $i_B^{\mathrm{GL}_n(F)}(\chi')$ are typical for the component $[T_n, \chi]$. Let σ be a permutation of the set $\{1, 2, \dots, n\}$ and $\chi = \boxtimes_{i=1}^n \chi_i$ be a given characer of $T_n = \prod_{i=1}^n F^\times$. We denote by χ^σ the character $\boxtimes_{i=1}^n \chi_{\sigma(i)}$ of T_n. We observe that the pairs $(T_n, \sigma(\chi))$ and (T_n, χ) are inertially equivalent. We will use a convenient permutation σ which satisfies the condition in the following lemma. For a character χ of $F^\times$ we denote by $l(\chi)$ the level of χ, i.e. the least **positive integer m** such that $1 + \mathfrak{P}_F{}^m$ is contained in the kernel of χ.

Lemma 4.0.4. *Given any sequence of characters $x_i = \chi_i$ of $\mathcal{O}_F^\times$, there exists a permutation $\{y_i \mid 1 \le i \le n\}$ of $\{x_i \mid 1 \le i \le n\}$ such that*

$$l(y_i y_k^{-1}) \ge \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.$$

for all $1 \le i \le j \le k \le n$.

Proof. For any ultrametric space (X, d) and given any n points $x_1, x_2, x_3, \dots, x_n$ in X we may choose a permutation $y_1, y_2, \dots, y_n$ of the sequence $\{x_i | 1 \le i \le n\}$ such that

$$d(y_i, y_k) \ge \max\{d(y_i, y_j), d(y_j, y_k)\}$$

for all $i \le j \le k$. Now apply this fact to the space X consisting of characters of $\mathcal{O}_F^\times$ and the distance function $d(\chi_1, \chi_2)$ is defined as the level $l(\chi_1 \chi_2^{-1})$ if $\chi_1 \ne \chi_2$ and 0 otherwise. We point out that this ordering is not unique in general. We refer to [How73][lemma 1] for a proof of these results. □

Remark 4.0.5. *We note that the condition $l(y_i y_k^{-1}) \ge \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}$ is equivalent to an equality since we always have*

$$l(y_i y_k^{-1}) \le \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.$$

Given a principal series component $[T_n, \chi]$ we choose the representative (T_n, χ^σ) where σ is a permutation such that

$$l(\chi_{\sigma(i)}\chi_{\sigma(k)}^{-1}) \geq \max\{l(\chi_{\sigma(i)}\chi_{\sigma(j)}^{-1}), l(\chi_{\sigma(j)}\chi_{\sigma(k)}^{-1})\}.$$

From now on we assume that the pair $(T_n, \boxtimes_{i=1}^n \chi_i)$ satisfies the condition

$$l(\chi_i\chi_k^{-1}) \geq \max\{l(\chi_i\chi_j^{-1}), l(\chi_j\chi_k^{-1})\} \tag{4.1}$$

for all $i \leq j \leq k$.

4.1 Construction of compact open subgroups H_m

Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix of size $n \times n$ such that a_{ij} is non-negative for $i > j$ and

$$a_{ki} = \max\{a_{ji}, a_{kj}\} \tag{4.2}$$

for all i, j and k such that $1 \leq i < j < k \leq n$. We denote by $J(\mathcal{A})$ the set of $n \times n$ matrices (m_{pq}) such that $m_{pq} \in \mathcal{O}_F$ for $p < q$ and $m_{pq} \in \mathfrak{P}_F^{a_{pq}}$ for $p \geq q$. As a consequence of the condition $a_{ki} = \max\{a_{ji}, a_{kj}\}$ we get two important inequalities

$$a_{i1} \geq a_{i2} \geq \cdots \geq a_{ii-1} \tag{4.3}$$

and

$$a_{j+1j} \leq a_{j+2j} \leq \cdots \leq a_{nj}. \tag{4.4}$$

The first is a consequence of $a_{ik-1} = \max\{a_{kk-1}, a_{ik}\}$ for $k < i$ and the second is a consequence of $a_{k+1j} = \max\{a_{k+1k}, a_{kj}\}$ for $j < k$.

Lemma 4.1.1. *The set $\mathcal{J}(\mathcal{A})$ is an order in $M_n(\mathcal{O}_F)$*

Proof. The set $\mathcal{J}(\mathcal{A})$ is an additive group. We now check that the set $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from $J(\mathcal{A})$. If $i > j$ then the $i \times j$ term in the product matrix $(m_{ij})(m'_{ij})$ can be split into three terms: $t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji}$, $t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$ and $t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}$. The valuation of the term $m_{ik}m'_{kj}$ is greater or equal to a_{i1} for $k \leq j$. This shows that valuation of t_1 is greater or equal to $\min\{a_{i1}, a_{i2}, \ldots, a_{ij}\}$ and $\min\{a_{i1}, \ldots, a_{ij}\} \geq a_{ij}$. The valuation of the term $m_{ik}m'_{kj}$ is greater or equal to $a_{ik} + a_{kj}$ for all $j \leq k \leq i$ and $a_{ik} + a_{kj}$ is greater or equal to a_{ij}. This shows that the valuation of t_2 is greater or equal to a_{ij}. Finally the valuation of $m_{ik}m_{kj}$ is greater or equal to a_{kj} for $k > i$. The valuation of the term t_3 is greater or equal to $\min\{a_{i+1j}, \ldots, a_{nj}\}$ and $\min\{a_{i+1j}, \ldots, a_{nj}\} \geq a_{ij}$. Hence the additive group $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Since $\mathcal{J}(\mathcal{A})$ is an $\mathcal{O}_F$ lattice in $M_n(F)$ we get that $\mathcal{J}(\mathcal{A})$ is an order in $M_n(\mathcal{O}_F)$. $\square$

We denote by $J(\mathcal{A})$ the set of invertible elements of $\mathcal{J}(\mathcal{A})$. The following are a few examples of $J(\mathcal{A})$.

1. If $\mathcal{A} = 0$ then the group $J(\mathcal{A})$ is $\mathrm{GL}_n(\mathcal{O}_F)$.
2. If $\mathcal{A} = (a_{ij})$ with $a_{ij} = 1$ for $i > j$, then $J(\mathcal{A})$ is the Iwahori subgroup with respect to the standard Borel subgroup B_n.
3. Let $s = [T_n, \boxtimes_{i=1}^n \chi_i]$ be an inertial class satisfying the condition 4.1 and $\mathcal{A}_\chi$ be the lower nilpotent matrix (a_{ij}) where $a_{ij} = l(\chi_i \chi_j^{-1})$ for $i > j$.

The examples (2) and (3) satisfy Iwahori decomposition with respect to the standard Borel subgroup B_n. The next lemma concerns the Iwahori decomposition of $J(\mathcal{A})$ in general.

Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$ for $1 \leq i < j < k \leq n$. We define an ordered partition I of n by induction on the set of positive integers $m \leq n$. $I_1 := (1)$ now if we know $I_m = (n_1, n_2, \ldots, n_r)$ for some $m \leq n-1$ then I_{m+1} is the partition $(n_1, n_2, \ldots, n_r, 1)$ if $a_{m+1m} \neq 0$ and $(n_1, n_2, \ldots, n_r + 1)$ if $a_{m+1m} = 0$. We denote by $I(\mathcal{A})$ the partition I_n.

Lemma 4.1.2. *The group $J(\mathcal{A})$ satisfies Iwahori decomposition with respect to the parabolic subgroup $P_{I(\mathcal{A})}$ and the standard Levi-subgroup $M_{I(\mathcal{A})}$. We have $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = M_{I(\mathcal{A})}(\mathcal{O}_F)$, $J(\mathcal{A}) \cap U_{I(\mathcal{A})} = U_{I(\mathcal{A})}(\mathcal{O}_F)$.*

Proof. We use induction on the positive integer n. If $n = 1$ then $J(\mathcal{A})$ is $\mathcal{O}_F^\times$ and the lemma is vacuously true. We assume that the lemma is true for all positive integers less than n. Let $I(\mathcal{A})$ be the ordered partition $(n_1, n_2, \ldots n_r)$. If $r = 1$ then the lemma is true by default. We suppose $r > 1$. We will show below that every element $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Now j_1 can be written as $j_2 u_1^+$ where $u_1^+ \in U_{(n_1, n-n_1)}(\mathcal{O}_F)$ and $j_2 \in M_{(n_1, n-n_1)} \cap J(\mathcal{A})$. Now j_2 can be written as $j_3 u_2^+$ where $j_3 \in J(\mathcal{A}) \cap M_{I(\mathcal{A})}$ and $u_2^+ \in U_{I(\mathcal{A})}(\mathcal{O}_F)$. The group $J(\mathcal{A}) \cap M_{(n_1, n-n_1)}$ is equal to $\mathrm{GL}_{n_1}(\mathcal{O}_F) \times J(\mathcal{A}')$ where the nilpotent matrix $\mathcal{A}' = (a'_{ij})$ is given by $a'_{ij} = a_{i+n_1 j+n_1}$. By induction hypothesis $J(\mathcal{A}')$ satisfies Iwahori decomposition with respect to the standard parabolic subgroup $P_{I(\mathcal{A}')}$ and its Levi-subgroup $M_{I(\mathcal{A}')}$ and $I(\mathcal{A}') = (n_2, n_3, \ldots, n_r)$. Let $j_3 = (j_3^0, j_3^1)$ where $j_3^0 \in \mathrm{GL}_{n_1}(\mathcal{O}_F)$ and $j_3^1 \in J(\mathcal{A}')$. Now $j_3^1 = u_3^- j_4 u_3^+$ where $u_3^- \in \bar{U}_{I(\mathcal{A}')} \cap J(\mathcal{A}')$, $u_3^+ \in U_{I(\mathcal{A}')} \cap J(\mathcal{A}')$ and $j_4 \in M_{I(\mathcal{A}')} \cap J(\mathcal{A}')$. Hence $j = u_1 u_3^- (j_3^0, j_4) u_3^+ u_2^+$ (with a slight abuse of notation the elements u_3^- and u_3^+ are considered as elements of $\bar{U}_{I(\mathcal{A})}$ and $U_{I(\mathcal{A})}$ respectively and (j_3^0, j_4) is an element of $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = \mathrm{GL}_{n_1}(\mathcal{O}_F) \times (J(\mathcal{A}') \cap M_{I(\mathcal{A}')})$).

We now prove that $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Let $j = (j_{pq})$. Let C_i^1 be the i^{th}-column of the first diagonal block (of size $n_1 \times n_1$) on the diagonal. If every entry of C_i^1 has positive valuation then, we claim that the all the entries of the i^{th} column C_i have positive valuation. Suppose the k^{th} entry j_{ki} of C_i is an unit for some $k > n_1$. This shows that a_{ki} the ki^{th}-entry of $\mathcal{A}$ is zero. Now the inequality (4.3) gives $a_{ki} \geq a_{kn_1}$ and this implies that $a_{kn_1} = 0$. Now note that $a_{kn_1} \geq a_{n_1+1n_1}$ from the inequality (4.4). This shows that $a_{n_1+1n_1}$ is zero which gives a contradiction from the definition of $I(\mathcal{A})$. We now deduce that j_{ki} is not invertible. This shows the claim. Since j is invertible we conclude that at least one entry of C_i^1 is an unit. Let $E_{ij}(c) = I_n + e_{ij}(c)$ where $e_{ij}(c)$ is the matrix with its ij entry c and all other entries 0. The left multiplication of $E_{ij}(c)$ results in the row operation $R_j + cR_i$. Since at least one entry of C_i^1 is an unit we assume that its q^{th}-entry is an unit. We can perform row operations $R_p + cR_q$ for all $p \geq n_1$ to make the p^{th}-entry trivial. We also note that the elementary matrix corresponding to this row operation also belongs to the group $J(\mathcal{A})$ (note that $q \leq n_1 \leq p$). This completes the task of making j as the product $u_1 j_1$.

The uniqueness of the Iwahori decomposition is standard. The proof is included for the completeness. If $u_1^- j_1 u_1^+ = u_2^- j_2 u_2^+$ then $(u_2^-)^{-1} u_1^- j_1 u_1^+ (u_2^+)^{-1} = j_2$. Let $u_3^- = (u_2^-)^{-1} u_1^-$ and $u_3^+ = u_1^+ (u_2^+)^{-1}$. We then have $u_3^- j_1 u_3^+ = j_2$. The equality can be rewritten as

$$u_3^- j_1 j_2^{-1} = (j_2 u_3^+ j_2^{-1})^{-1}.$$

The right hand side of the above equality is an upper block matrix with identity on diagonal blocks and the left hand side is lower block matrix. This shows that right hand side is identity matrix. Similar reasoning shows that u_3^- and u_3^+ are both identity matrices and $j_1 = j_2$. This proves the uniqueness of the Iwahori decomposition. □

Let $s = [T_n, \chi]$ be an inertial equivalence class. Let m be a positive integer and $\mathcal{A}_\chi(m)$ be the lower nilpotent matrix (a_{ij}^m) where $a_{ij}^m = l(\chi_i \chi_j^{-1}) + m - 1$. As shown earlier the representative $(T_n, \chi = \boxtimes_{i=1}^n \chi_i)$ can be chosen such that

$$a_{ik} = \max\{a_{ij}, a_{jk}\}$$

for all $i < j < k$. We denote by $J_\chi(m)$ the group $J(\mathcal{A}_\chi(m))$. Note that $J_\chi(m') \subset J_\chi(m)$ for all $m' \geq m$. In our situation we have $I(\mathcal{A}_\chi(m))$ is $(1, 1, \ldots, 1)$ since none of a_{ii+1}^m are zero and hence by lemma 4.1.2 $J_\chi(m)$ satisfies Iwahori decomposition with respect to B_n.

Lemma 4.1.3. *The character $\chi = \boxtimes_{i=1}^n \chi_i$ of $T(\mathcal{O}_F)$ extends to a character of $J_\chi(1)$ such that $J_\chi(1) \cap U_n$ and $J_\chi(1) \cap \bar{U}_n$ are contained in the kernel of the extension.*

Proof. Let $m = (m_{ij})$ be an element of $J_\chi(1)$. We define $\tilde{\chi}(m) = \prod_{i=1}^n \chi_i(m_{ii})$. We verify that $\tilde{\chi}$ is a character of the group $J_\chi(1)$. This is very computational in nature. We sketch the proof here and for complete details see [Roc98, Section 3, Lemma 3.1, Lemma 3.2] or [How73, Pg 278-279]. The idea is to get an open normal subgroup U of $J_\chi(1)$ such that $J_\chi(1)/U$ is isomorphic to $T(\mathcal{O}_F)/T_\chi$ where T_χ is an open subgroup of $T(\mathcal{O}_F)$ which is contained in the kernel of χ. The subgroup U is generated by $J_\chi(1) \cap \bar{U}_n$ and $J_\chi(1) \cap U_n = U_n(\mathcal{O}_F)$. One shows that U satisfies Iwahori decomposition with respect to the Borel subgroup B_n and $U \cap T_n$ is given by $\prod_{\alpha \in \Phi} \alpha^\vee(1+\mathfrak{P}_F^{l(\chi\alpha^\vee)})$ where Φ is the set of roots of GL_n with respect to T_n and $\alpha^\vee$ stands for the dual root. We observe that $U \cap T_n$ is contained in the kernel of χ. □

We apply lemma 2.2.5 to the sequence of groups $H_m = J_\chi(m)$ for $m \geq 1$ and $\tau = \chi$ to get the isomorphism

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{B_n}^{\mathrm{GL}_n(F)}(\chi) = \bigcup_{m \geq 1} \mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

We denote by $\mathcal{A}_\chi(1,m)$ the lower nilpotent matrix (a_{ij}) where $a_{ij} = l(\chi_i\chi_j^{-1})$ for $j < i < n$, $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1$. Given a lower nilpotent matrix $\mathcal{A} = (a_{ij})$ such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$ we associated a compact subgroup $J(\mathcal{A})$. The condition $a_{ki} = \max\{a_{kj}, a_{ji}\}$ is a sufficient condition to define the group $J(\mathcal{A})$ in a similar recipe and this condition can be verified easily. The matrix $\mathcal{A}_\chi(1,m)$ need not satisfy this condition but we can still associate the group $J(\mathcal{A}_\chi(1,m))$ to the matrix $\mathcal{A}_\chi(1,m)$. We will prove this in the next lemma.

Lemma 4.1.4. *Let $\mathcal{J}(\mathcal{A}_\chi(1,m))$ be the set consisting of matrices (m_{ij}) such that $m_{ij} \in \mathfrak{P}_F^{a_{ij}}$ for all i,j. The set $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is an order in $M_n(\mathcal{O}_F)$.*

Proof. The set $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is clearly an additive group and is open. We have to verify that $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two elements of the set $\mathcal{J}(\mathcal{A}_\chi(1,m))$. We suppose $i > j$. The ij^{th}-term of the product $(m_{ij})(m'_{ij})$ is the sum of the terms:

$$t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji},$$

$$t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$$

and

$$t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}.$$

The valuation of the term t_1 is greater or equal to the minimum value among the valuation of $m_{ik}m'_{kj}$ for $1 \leq k \leq j$ and the valuation of $m_{ik}m'_{kj}$ is given by a_{ik}. If $i < n$ then $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{ij} \leq a_{ik}$ for all $k \leq j < i$. This shows that t_1 has valuation at least a_{ij}. If $i = n$ then the values of $a_{ik} = a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1 \geq l(\chi_i\chi_j^{-1}) + m - 1 = a_{ij}$ for all $k \leq j < n$. We conclude that in every possibility the valuation of t_1 is greater or equal to a_{ij}.

Consider the term t_2. The valuation of t_2 is at least the minimum among the valuation of $m_{ik}m'_{kj}$ for $j < k \leq i$. The valuation of $m_{ik}m'_{kj}$ is given by $a_{ik} + a_{kj}$ for $j < k \leq i$. If $i < n$ $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{kj} = l(\chi_k\chi_j^{-1})$. From our assumption on the arrangement of characters χ_i for $1 \leq i \leq n$, we get that $l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}$. At the same time $i < n$ implies $a_{ij} = l(\chi_i\chi_j^{-1})$. This shows that the valuation of the term is $m_{ik}m'_{kj}$ given by $a_{ik} + a_{kj}$ is at least a_{ij}. Consider the case $i = n$ and $a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1$. Now $a_{kj} = l(\chi_k\chi_j^{-1})$ and $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1$. From the equality $l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}$ we deduce that

$$l(\chi_i\chi_k^{-1}) + l(\chi_k\chi_j^{-1}) > l(\chi_i\chi_j^{-1})$$

and adding $m-1$ on both sides we get $a_{ij} > a_{ik} + a_{kj}$. We conclude that the valuation of the term t_2 is at least a_{ij}.

The valuation of the term $m_{ik}m'_{kj}$ for $i < k < n$ is given by $a_{kj} = l(\chi_k\chi_j^{-1})$ which is greater or equal to a_{ij} and $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1 \geq a_{ij}$ and we conclude that the valuation of t_3 is at least a_{ij}. This shows that the valuation of $t_1 + t_2 + t_3$ is at least a_{ij} which proves our result.

□

Let $J_\chi(1, m)$ be the group of units of $\mathcal{J}(\mathcal{A}_\chi(1, m)$. We will need the structure of the representation

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})$$

for the proof of our main theorem. We follow the strategy already established in the previous chapter. Let $K_\chi(1, m)$ be the set of matrices (m_{ij}) such that $m_{ij} \in \mathfrak{P}_F$ for $i < j < n$ and $m_{in} \in \mathcal{O}_F$ for $i < n$, $m_{ii} \in 1 + \mathfrak{P}_F$ for $i \leq n$ and $m_{ij} \in {\mathfrak{P}_F}^{a_{ij}}$ for $i > j$ and (a_{ij}) is the matrix $\mathcal{A}_\chi(1, m)$.

Lemma 4.1.5. *The set $K_\chi(1, m)$ is a normal subgroup of $J_\chi(1, m)$.*

Proof. We first check that $K_\chi(1, m)$ is closed under matrix multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from the set $K_\chi(1, m)$. Let $i < j < n$ the ij^{th} term is the sum of

$$t_1 = m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ii}m'_{ij},$$

$$t_2 = m_{ii+1}m'_{i+1j} + m_{ii+2}m'_{i+2j} + \cdots + m_{ij}m'_{jj}$$

and

$$t_3 = m_{ij+1}m'_{j+1j} + \cdots + m_{in}m'_{nj}.$$

The valuation of m'_{kj} is positive for $1 < k \leq i$ hence t_1 has positive valuation. The valuation of m_{ik} is positive for $i < k \leq j$ and hence t_2 is positive. The valuation of m'_{kj} is positive for $j < k \leq n$ hence valuation of t_3 is positive. This shows that ij^{th}-term of the matrix product has positive valuation. The rest of the verifications on congruence conditions are verified in lemma 4.1.4. The existence of inverse for an element in $K_\chi(1, m)$ follows from Gaussian elimination.

Now we establish the normality of $K_\chi(1,m)$. The group $K_\chi(1,m)$ satisfies Iwahori decomposition with respect to the subgroups $P_{(n-1,1)}$ and $M_{(n-1,1)}$. We also note that $K_\chi(1,m) \cap U_{(n-1,1)}$ is equal to $J_\chi(1,m) \cap U_{(n-1,1)}$ and $K_\chi(1,m) \cap \bar{U}_{(n-1,1)}$ is equal to $J_\chi(1,m) \cap \bar{U}_{(n-1,1)}$. To check the normality of $K_\chi(1,m)$ we have to check that $J_\chi(1,m) \cap M_{(n-1,1)}$ normalizes $K_\chi(1,m)$. This is equivalent to checking that $K_\chi(1,m) \cap M_{(n-1,1)}$ is a normal subgroup of $J_\chi(1,m) \cap M_{(n-1,1)}$.

We note that $J_\chi(1,m) \cap M_{(n-1,1)} = J_{\chi'}(1) \times \mathcal{O}_F^\times$ where $\chi' = \boxtimes_{i=1}^{n-1}\chi_i$. Let p_1 be the projection of $J_\chi(1,m) \cap M_{(n-1,1)}$ onto $J_{\chi'}(1)$ and π_1 be the reduction mod $\mathfrak{P}_F$ map. Note that $K_\chi(1,m) \cap M_{(n-1,1)}$ is the kernel of $\pi_1 \circ p_1$. □

From the above lemma the group $K_\chi(1,m)$ is a normal subgroup of $J_\chi(1,m)$. We also note that $J_\chi(1,m) \cap \bar{U}_{(n-1,1)}$ is contained in $K_\chi(1,m)$. From this we conclude that $J_\chi(1,m) = K_\chi(1,m)J_\chi(1,m+1)$. From the Mackey decomposition we get that

$$\mathrm{res}_{K_\chi(1,m)} \mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_\chi(1,m)\cap J_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}).$$

From the definition of $K_\chi(1,m)$ we get that $K_\chi(1,m) \cap J_\chi(1,m+1) = K_\chi(1,m+1)$ and

$$\mathrm{res}_{K_\chi(1,m)} \mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}). \tag{4.5}$$

Lemma 4.1.6. *The group $K_\chi(1,m+1)$ is a normal subgroup of $K_\chi(1,m)$.*

Proof. Since the groups $K_\chi(1,m)$ satisfy Iwahori decomposition, $K_\chi(1,m) \cap U_{(n-1,1)}$ is equal to $K_\chi(1,m+1) \cap U_{(n-1,1)}$ and $K_\chi(1,m) \cap M_{(n-1,1)}$ is equal to $K_\chi(1,m+1) \cap M_{(n-1,1)}$. We have to check that $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to $K_\chi(1,m+1)$ for all u^-, j and u^+ in

$K_\chi(1,m) \cap \bar{U}_{(n-1,1)}$,

$K_\chi(1,m) \cap M_{(n-1,1)}$ and

$K_\chi(1,m) \cap U_{(n-1,1)}$.

respectively.

We first consider the case $u^- j (u^-)^{-1}$. We can rewrite $u^- j (u^-)^{-1}$ as $j\{j^{-1}u^- j (u^-)^{-1}\}$. Since $j \in K_\chi(1,m) \cap M_{(n-1,1)} = K_\chi(1,m+1) \cap M_{(n-1,1)}$, it is enough to show that $j^{-1}u^- j (u^-)^{-1}$ belongs to the group $K_\chi(1,m+1)$. Let j and u^- be written in their block matrix form as follows.

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

The conjugation $j^{-1}u^- j (u^-)^{-1}$ in its block form is given by

$$\begin{pmatrix} 1_{n-1} & 0 \\ j_1^{-1} U^- J_1 - U^- & 1 \end{pmatrix}$$

Let $U^- = [u_1, u_2, \ldots, u_{n-1}]$ and $J_1 = (j_{ij})$. The k^{th} entry of the matrix $U^- J_1$ is the sum of $t_1 = u_1 j_{1k} + u_2 j_{2k} + \cdots + u_{k-1} j_{k-1k}$, $t_2 = u_k j_{kk}$ and $t_3 = u_{k+1} j_{k+1k} + \cdots + u_{n-1} j_{n-1k}$. Let $l(\chi_k \chi_n^{-1}) > 1$ then valuation of $u_t j_{tk}$ for $t < k$ is at least $l(\chi_t \chi_n^{-1}) + m - 1 + 1$ which is at least $l(\chi_k \chi_n^{-1}) + m$ this shows that valuation of the term t_1 is at least $l(\chi_k \chi_n^{-1}) + m - 1$. The valuation of $u_t a_{tk}$ for $k < t$ is at least $l(\chi_t \chi_n^{-1}) + l(\chi_t \chi_k^{-1}) + m - 1 > l(\chi_k \chi_n^{-1}) + m - 1$. This shows that $t_1 + t_2 + t_3 \equiv t_2 = u_k j_{kk} = u_2 \mod \mathfrak{P}_F{}^{l(\chi_k \chi_n^{-1}) + m}$. We note that $j_1^{-1} u_2 \equiv u_2$ hence the matrix

$$\begin{pmatrix} 1_n & 0 \\ j_1^{-1} U^- J_1 - U^- & 1 \end{pmatrix}$$

is contained in $K_\chi(1, m+1) \cap \bar{U}_{(n-1,1)}$

Let us consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in the block form as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix}$$

The conjugated matrix $u^- u^+ (u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_{n-1} - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1 \end{pmatrix}.$$

Let $1_{n-1} - U^+ U^- = (u_{ij})$. The valuation of u_{ij} for $i > j$ is greater or equal to $l(\chi_n \chi_j^{-1})$ and $l(\chi_n \chi_j^{-1})$ is greater or equal to $l(\chi_i \chi_j^{-1})$. From this we conclude that $u^- u^+ (u^-)^{-1} \in K_\chi(1, m+1)$. □

4.2 Preliminaries for main theorem

The inclusion map of $K_\chi(1, m) \cap \bar{U}_n$ in $K_\chi(1, m)$ induces an isomorphism of the quotient $K_\chi(1, m)/K_\chi(1, m+1)$ with the abelian quotient

$$\frac{K_\chi(1, m) \cap \bar{U}_{(n-1,1)}}{K_\chi(1, m+1) \cap \bar{U}_{(n-1,1)}}. \tag{4.6}$$

Hence the representation $\mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id})$ splits into characters η_k with $1 \leq k \leq p$. The group $J_\chi(1, m)$ acts on these characters and let $Z(\eta_k)$ be the $J_\chi(1, m)$-stabilizer of the character η_k. From Clifford theory we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1,m)}(U_{\eta_{n_k}}) \tag{4.7}$$

where η_{n_k} is a representative for an orbit under the action of $J_\chi(1, m)$ and $U_{\chi_{n_k}}$ is an irreducible representation of the group $Z(\eta_{n_k})$. Since

$$J_\chi(1, m) = (J_\chi(1, m) \cap M_{(n-1,1)}) K_\chi(1, m)$$

we get that $Z(\eta_k) = (Z(\eta_k) \cap M_{(n-1,1)})K_\chi(1,m)$.

The final step in our preliminaries is to understand the mod $\mathfrak{P}_F$ reduction of the group

$$Z(\eta_k) \cap M_{(n-1,1)}$$

for some non-trivial character η_k. The group $J_\chi(1,m) \cap M_{(n-1,1)}$ is equal to $J_{\chi'}(1) \times \mathcal{O}_F^\times$ acts on the quotient

$$\frac{K_\chi(1,m) \cap \bar{U}_{(n-1,1)}}{K_\chi(1,m+1) \cap \bar{U}_{(n-1,1)}} \tag{4.8}$$

by conjugation. Let j and u^- be two elements from $J_\chi(1,m) \cap M_{(n-1,1)}$ (which is $J_{\chi'}(1) \times \mathcal{O}_F^\times$ for $\chi' = \boxtimes_{i=1}^{n-1}\chi_i$) and $K_\chi(1,m) \cap \bar{U}$ respectively. We write the elements j and u^- written in their block diagonal form as

$$\begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}$$

respectively. The map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ induces an isomorphism between the group (4.8) and $M_{1\times(n-1)}(k_F)$. The group $J_\chi(1) \times \mathcal{O}_F^\times$ acts through its mod $\mathfrak{P}_{PF}$ reduction-$B_{n-1}(k_F) \times k_F^\times$ on the set of matrices $M_{1\times(n-1)}(k_F)$ by setting $(b,x)(A) = xAb^{-1}$ for all $b \in B_{n-1}(k_F), x \in k_F^\times$ and $A \in M_{1\times(n-1)}(k_F)$. Hence the map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ gives an $J_\chi(1) \times {\mathcal{O}_F}^\times$-equivariant map between $M_{1\times(n-1)}(k_F)$ and the group (4.8). We also have a $M_{(n-1,1)}(k_F)$-equivariant map between the group of characters of $M_{1\times(n-1)}(k_F)$ and $M_{(n-1)\times 1}(k_F)$ (see lemma 3.0.14). Hence we obtain a $J_\chi(1) \times {\mathcal{O}_F}^\times$ equivariant map between the group of characters of the quotient (4.8) and the group $M_{(n-1)\times 1}(k_F)$ where $J_\chi(1)$ acts through its mod $\mathfrak{P}_F$ reduction-$B_{(n-1)}(k_F) \times k_F^\times$ and the action is $(b,x)A = bAx^{-1}$ (see lemma 3.0.14). Hence the group $Z(\eta_k) \cap M_{(n-1,1)}$ for non-trivial η_k is equal to $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ for some non-zero matrix A in $M_{(n-1)\times 1}(k_F)$.

Let p be the projection of $B_{n-1}(k_F) \times k_F^\times$ onto the diagonal torus

$$T_{n-1}(k_F) \times k_F^\times = T_n(k_F),$$

let p_i be the i^{th} projection of $T_n(k_F)$ onto $k_F^\times$. The centralizer $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ of a non-zero matrix $A = [u_1, u_2, \ldots, u_{n-1}]^{tr}$ satisfies the following property: there exists $j < n$ such that $p_j(p(t)) = p_n(p(t))$ for all $t \in Z_{B_{n-1}(k_F)\times k_F^\times}(m)$ (see lemma 3.0.15). This shows that for any non-trivial character η_{n_k}, $Z(\eta_{n_k}) \cap T_n$ satisfies the property that

$$p_j(t) \equiv p_n(t)$$

mod $\mathfrak{P}_F$.

The character $\chi = \boxtimes_{i=1}^{n}\chi_i$ of $J_\chi(1)$ occurs with multiplicity one in the representation

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi).$$

We denote by $U_m^0(\chi)$ the complement of χ in $\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi)$. We denote by $U_m(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi)\}.$$

4.3 Main theorem

Theorem 4.3.1. *Let $\#k_F > 3$. The irreducible sub-representations of $U_m(\chi)$ are atypical for the component s.*

Proof. We prove the theorem by using induction on the positive integers n and m. For $n = 1$ the representation $U_m(\chi)$ is trivial and the theorem is vacuously true. Let n be a positive integer greater than one. We assume that the theorem is proved for all positive integers less than n. We will use the induction hypothesis to show the theorem for n. We note that $J_\chi(1,m)$ and $K_\chi(1,m)$ satisfy Iwahori decomposition with respect to the parabolic subgroup $P_{(n-1,1)}$ and standard Levi-subgroup $M_{(n-1,1)}$, $J_\chi(1,m)\cap U_{(n-1,1)} = K_\chi(1,m)\cap U_{(n-1,1)}$ and $J_\chi(1,m)\cap \bar{U}_{(n-1,1)} = K_\chi(1,m)\cap \bar{U}_{(n-1,1)}$. Now lemma 2.2.6 shows that $\mathrm{ind}_{J_\chi(m)\cap M_{(n-1)}}^{J_\chi(1,m)\cap M_{(n-1)}}(\chi)$ extends to a representation of $J_\chi(1,m)$ and this extension is given by

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1,m)}(\chi).$$

If we denote by χ' the character $\boxtimes_{i=1}^{n-1}\chi_i$ of $\prod_{i=1}^{n-1} F^\times$ then we have

$$\mathrm{ind}_{J_\chi(m)\cap M_{(n-1)}}^{J_\chi(1,m)\cap M_{(n-1)}}(\chi) \simeq \mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi')\boxtimes\chi_n.$$

We also have

$$\mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi')\boxtimes\chi_n \simeq U_m^0(\chi')\boxtimes\chi_n \oplus \chi.$$

Combining the above isomorphisms we get that

$$\mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi')\boxtimes\chi_n\}\bigoplus \mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi). \tag{4.9}$$

We will use the induction hypothesis to show that $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi')\boxtimes\chi_n\} \tag{4.10}$$

are atypical representations. By induction hypothesis any $\mathrm{GL}_{n-1}(\mathcal{O}_F)$-irreducible sub-representation of $U_m(\chi')$ occurs as sub-representation of some

$$i_{P_I}^{\mathrm{GL}_{n-1}(F)}(\sigma)$$

where $[T_{n-1}, \chi']$ and $[M_I, \sigma]$ are two distinct inertial classes. We now get that irreducible sub-representations of 4.10 occur as sub-representations of

$$i_{P_{I'}}^{\mathrm{GL}_n(F)}(\sigma \boxtimes \chi_n)$$

where I' is obtained from I by adding 1 at the end of the ordered partition I of $n-1$. If $I \neq (1, 1, \ldots, 1)$ then the Levi sub-groups $M_{I'}$ and T_n are not conjugate and hence the inertial classes $[M_{I'}, \sigma \boxtimes \chi_n]$ and $[T_n, \chi]$ are distinct inertial classes and this proves our claim in this case. Now let $M_I = T_{n-1}$ and $\sigma = \boxtimes_{i=1}^{n-1} \sigma_i$ be the tensor factorization of T_{n-1}. Since the components $[T_{n-1}, \chi']$ and $[T_{n-1}, \sigma]$ are distinct we get a character χ_t occurring with non-zero multiplicity in the multi-set $\{\chi_1, \chi_2, \ldots, \chi_{n-1}\}$ but with a different multiplicity in the multi-set $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$. Adding the character χ_n to both multi-sets above keeps the multiplicities of the character χ_t distinct and this shows that $[T_n, \chi]$ and $[T_n, \sigma \boxtimes \chi_n]$ are different inertial classes.

This shows that any typical representation must occur as a sub-representation of

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

The character χ occurs with multiplicity one in the representation $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}^0(\chi)$ the complement of the character χ in $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{1,m}^0(\chi)\}.$$

We first note that

$$U_m(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi') \boxtimes \chi_n\} \oplus U_{1,m}(\chi).$$

We already showed that the first summand on the right-hand side of the above equation has all its irreducible sub-representations atypical. We now show that irreducible sub-representations of $U_{1,m}(\chi)$ are atypical and this proves the main theorem.

We first note that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}\{\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \otimes \chi\}.$$

Using the decomposition 4.7 we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

Recall that η_{n_k} is a representative for the orbit under the action of the group $J_\chi(1,m)$ on the characters η_t of $K_\chi(1,m)$ which are trivial on $K_\chi(1,m+1)$

and $Z(\eta_{n_k})$ is the $J_\chi(1,m)$-stabilizer of the character η_{n_k}. There is exactly one orbit consisting of the id character and hence

$$\operatorname{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \operatorname{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \operatorname{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{4.11}$$

Consider the representation

$$\operatorname{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}$$

for some representative $\eta_{n_k} \neq \mathrm{id}$. Now recall that $Z(\eta_{n_k}) \cap T_n$ is a subgroup of $T_n(\mathcal{O}_F) = \prod_{i=1}^n \mathcal{O}_F^\times$ and there exists a positive integer $j < n$ such that $p_j(t) \equiv p_n(t) \bmod \mathfrak{P}_F$ for all $t \in Z(\eta_{n_k})$. Let κ be a character of $F^\times$ such that κ is ramified and $1 + \mathfrak{P}_F$ is contained in the kernel of κ. Let χ^κ be the character

$$\chi_1 \boxtimes \chi_2 \boxtimes \chi_j\kappa \boxtimes \cdots \boxtimes \chi_n\kappa^{-1}.$$

We observe that $\operatorname{res}_{Z(\eta_{n_k})}(\chi) = \operatorname{res}_{Z(\eta_{n_k})}(\chi^\kappa)$ and hence

$$\operatorname{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \operatorname{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}. \tag{4.12}$$

From the above paragraph we get that

$$U^0_{1,m+1}(\chi) \simeq U^0_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \operatorname{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

and from the above identity we conclude that

$$U_{1,m+1}(\chi) \simeq U_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \operatorname{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{4.13}$$

From the equation (4.12) we get that

$$\operatorname{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \operatorname{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}.$$

If we choose κ such that the components $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ are two distinct inertial classes then we can conclude that irreducible sub-representations of

$$\operatorname{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\}$$

are atypical and hence using the identity (4.13) recursively we get that irreducible sub-representations of $U_{1,m}(\chi)$ are atypical representations for all positive integers m.

To prove the theorem we have to justify that we can choose a character κ as in the previous paragraph. Now for any character κ non-trivial on $\mathcal{O}_F^\times$ (such a character exists since $\#k_F > 2$) and trivial on $1 + \mathfrak{P}_F$, the equality of the inertial classes $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ implies the equality of multiplicities of χ_j in the multi-sets $\{\chi_1, \chi_2, \dots, \chi_n\}$ and $\{\chi_1, \chi_2, \dots, \chi_j\kappa, \dots, \chi_n\kappa^{-1}\}$. The equality of multiplicities implies $\chi_j\chi_n^{-1} = \kappa$. Now if $\#k_F > 3$ we have at least two non-trivial tame characters and hence we can choose κ distinct from a possibly tame character $\chi_j\chi_n^{-1}$.

□

The pair $(J_\chi(1), \chi)$ is the Bushnell-Kutzko type for the component s (see [BK99, Section 8]). Hence from the above theorem we deduce that irreducible sub-representations of

$$\mathrm{ind}_{J_\chi}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi)$$

are precisely the typical representations for the component $s = [T_n, \chi]$ and $\#k_F > 3$. Moreover we have

Corollary 4.3.2. *Let $\#k_F > 3$. Let τ be a typical representation for the component $s = [T, \chi]$ then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, i_{B_n}^{\mathrm{GL}_n(F)}(\chi)) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \mathrm{ind}_{J_\chi}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi))$$

Remark 4.3.3. *When $\#k_F = 2$ and $n = 2$ Henniart showed in [BM02][A.2.6, A.2.7] that the Bushnell-Kutzko type for the component $s = [T_2, \chi_1 \boxtimes \chi_2]$, $\chi_1\chi_2^{-1} \neq \mathrm{id}$ has two typical representations one given by*

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

and the other representation turns out to be the (it can be shown a priori that there is a unique complement (see [Cas73][Proposition 1(b)])) complement of $\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$ in $\mathrm{ind}_{J_\chi(2)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$. But for $\#k_F > 2$ and $n > 3$ we expect that typical representations are precisely the irreducible sub-representations of

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

For $\#k_F = 2$ and $n > 2$ a typical representation may not be contained in the above representation as shown by Henniart for the case of $\mathrm{GL}_2(F)$.