Author: Nadimpalli, Santosh
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Chapter 3

Level zero inertial classes

Definition 3.0.8. *Let $I = (n_1, n_2, \ldots, n_r)$ be an ordered partition of n. An inertial class $s = [M_I, \boxtimes_{i=1}^r \sigma_i]$ is called a level-zero inertial class if for every σ_i there exists an irreducible representation τ_i of $\mathrm{GL}_{n_i}(\mathcal{O}_F)$ such that τ_i is the inflation of an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(k_F)$ and $\mathrm{Hom}_{\mathrm{GL}_{n_i}(\mathcal{O}_F)}(\tau_i, \sigma_i) \neq 0$.*

We fix a level-zero inertial class $s = [M_I, \sigma_I]$ with the pairs $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ as in the above definition. **The pair $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ is the Bushnell-Kutzko type for the inertial class** $[\mathrm{GL}_{n_i}(F), \sigma_i]$. Let m be a positive integer and $P_I(m)$ be the inverse image of $P_I(\mathcal{O}_F/\mathfrak{P}_F^m)$ under the mod-$\mathfrak{P}_F^m$ reduction map

$$\pi_m : \mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{P}_F^m).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $M_I(k_F)$ extends to a representation of $P_I(k_F)$ by inflation via the quotient map

$$P_I(k_F) \to P_I(k_F)/U_I(k_F) \simeq M_I(k_F).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $P_I(k_F)$ extends to a representation of $P_I(1)$ by inflation via the map π_1. We note that $P_I(1) \cap U_I$ and $P_I(1) \cap \bar{U}_I$ are contained in the kernel of this extension. The pair $(P_I(1), \tau_I)$ is the Bushnell-Kutzko type for the component s (see [BK99][Section 8.3.1]). The irreducible sub-representations of

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

are thus typical for s.

We note that the groups $P_I(m)$ satisfy Iwahori decomposition with respect to P_I and M_I. The $M_I(\mathcal{O}_F)$ representation τ_I extends to a representation of $P_I(m)$ such that $P_I(m) \cap U_I$ and $P_i(m) \cap \bar{U}_I$ are contained in the kernel of the extension. This shows that the sequence of groups $\{P_I(m) \mid m \geq 1\}$ and τ_I satisfy the hypothesis for the groups $\{H_m \mid m \geq 1\}$ and τ in lemma 2.2.5 hence we have the isomorphism

$$\bigcup_{m \geq 1} \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) \simeq \mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I).$$

We recall that the lemma 2.2.4 shows that typical representations for the component s can only occur in the above representation.

Using Frobenius reciprocity we get that the representation τ_I occurs in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $m \geq 1$ and $U_m^0(\tau_I)$ be the $P_I(1)$-stable complement of the representation τ_I in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$. Let $U_m(\tau_I)$ be the representation

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_m^0(\tau_I)).$$

We note that

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) \oplus U_m(\tau_I) \simeq \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

We will show that irreducible sub-representations of $U_m(\tau_I)$ are atypical.

Theorem 3.0.9 (Main). *Let $m \geq 1$. The $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible subrepresentations of $U_m(\tau_I)$ are atypical.*

The classification of typical representations for the inertial class s is given by the following corollary.

Corollary 3.0.10. *The irreducible sub-representations of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ are precisely the typical representations for the level-zero inertial class $[M_I, \sigma_I]$. Moreover if Γ is a typical representation then*

$$\dim_{\mathbb{C}}\ \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)) = \dim_{\mathbb{C}}\ \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)).$$

Proof. Given a typical representation Γ for the inertial class s, the theorem shows that Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ and the multiplicity formula follows from 2.2.4 and the above theorem. Conversely if Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ then by Frobenius reciprocity we get that $\mathrm{Hom}_{P_I(1)}(\tau_I, \Gamma) \neq 0$. If Γ is contained as a $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representation in an irreducible smooth representation π of $\mathrm{GL}_n(F)$ then the restriction of π to $P_I(1)$ contains the representation τ_I. The pair $(P_I(1), \tau_I)$ is the Bushnell-Kutzko type for the inertial class $s = [M_I, \sigma_I]$ hence the inertial support of π is s. Hence Γ is a typical representation and this proves the corollary. $\square$

We will need a few lemmas before the proof of this theorem. Let $I = (n_1, n_2, \ldots, n_r)$ be the ordered partition of the positive integer n as fixed at the beginning of this chapter. **Until the beginning of the section 3.1 we assume that** $r > 1$. We denote by I' the ordered partition $(n_1, n_2, \ldots, n_{r-1})$ of $n - n_r$. Let m be a positive integer and $P_I(1, m)$ be the following set

$$\left\{ \begin{pmatrix} A & B \\ \varpi_F^m C & D \end{pmatrix} | A \in P_{I'}(1); B, C^t \in M_{n_r \times (n-n_r)}(\mathcal{O}_F); D \in \mathrm{GL}_{n_r}(\mathcal{O}_F) \right\}.$$

Note that $P_I(1,1) = P_I(1)$.

Lemma 3.0.11. *The set $P_I(1,m)$ is a subgroup of $P_I(1)$.*

Proof. The group $\mathrm{GL}_n(\mathcal{O}_F)$ acts on the set of lattices of F^n contained in the lattice ${\mathcal{O}_F}^n$. If $r-1=1$ the set $P_I(1,m)$ is the $\mathrm{GL}_n(\mathcal{O}_F)$-stabilizer of the lattice $(\mathcal{O}_F)^{n_1} \oplus (\varpi_F^m \mathcal{O}_F)^{n_2}$. In the case $r-1>1$ the set $P_I(1,m)$ is the $\mathrm{GL}_n(\mathcal{O}_F)$-stabilizer of lattices L_k for $1<k\le r-1$ defined as:

$$L_k = (\mathcal{O}_F)^{n_1} \oplus \cdots \oplus (\mathcal{O}_F)^{n_{k-1}} \oplus (\varpi_F \mathcal{O}_F)^{n_k} \oplus \cdots \oplus (\varpi_F \mathcal{O}_F)^{n_{r-1}} \oplus (\varpi_F^m \mathcal{O}_F)^{n_r}.$$

This shows that $P_I(1,m)$ is a subgroup and is contained in $P_I(1)$ from the definition. □

The structure of the representation

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id})$$

will be used in the proof of the main theorem. Using Clifford theory we decompose the above representation. Let $K_I(m)$ be the group $K_n(m)U_{(n-n_r,n_r)}(\mathcal{O}_F)$.

Lemma 3.0.12. *The group $K_I(m)$ is a normal subgroup of $P_I(1,m)$ and $K_I(m)\cap P_I(1,m+1)$ is a normal subgroup of $K_I(m)$.*

Proof. The groups $K_I(m)$ and $P_I(1,m)$ satisfy Iwahori decomposition with respect to $U_{(n-n_r,n_r)}$, $\bar{U}_{(n-n_r,n_r)}$ and $M_{(n-n_r,n_r)}$. We also note that

$$K_I(m)\cap U_{(n-n_r,n_r)} = P_I(1,m)\cap U_{(n-n_r,n_r)}$$

and

$$K_I(m)\cap \bar{U}_{(n-n_r,n_r)} = P_I(1,m)\cap \bar{U}_{(n-n_r,n_r)}.$$

Hence $P_I(1,m)\cap U_{(n-n_r,n_r)}$ and $P_I(1,m)\cap\bar{U}_{(n-n_r,n_r)}$ normalize $K_I(m)$. Since $K_I(m)$ is a product of the group $K_n(m)$ and $U_{(n-n_r)}(\mathcal{O}_F)$ the group $P_I(1,m)\cap M_{(n-n_r,n_r)}$ normalizes the group $K_I(m)$. This shows the first part.

Notice that $K_I(m)\cap U_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)}$ and $K_I(m)\cap M_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ hence it is enough to check that $K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$ normalizes the group $K_I(m)\cap P_I(1,m+1)$. Since $K_I(m)\cap P_I(1,m+1)\cap \bar{U}_{(n-n_r,n_r)}$ is abelian and is contained in $K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$ hence we need to check that $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ are contained in $K_I(m)\cap P_I(1,m+1)$ for all u^-, j and u^+ in

$K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$,

$K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ and

$K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)} = U_{(n-n_r,n_r)}(\mathcal{O}_F)$

respectively.

Let u^+, u^- and j be three elements from $U_{n-n_r,n_r}(\mathcal{O}_F)$, $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ and $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ respectively. We write them in their block form as:

$$u^+ = \begin{pmatrix} 1_{n-n_r} & B \\ 0 & 1_{n_r} \end{pmatrix}$$

where $B \in M_{(n-n_r)\times n_r}(\mathcal{O}_F)$,

$$u^- = \begin{pmatrix} 1_{n-n_r} & 0 \\ \varpi_F^m C & 1_{n_r} \end{pmatrix}$$

where $C \in M_{n_r\times(n-n_r)}(\mathcal{O}_F)$ and

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

We observe that $u^-j(u^-)^{-1} = j\{j^{-1}u^-j(u^-)^{-1}\}$ and the commutator $\{j^{-1}u^-j(u^-)^{-1}\}$ in its block form is as follows:

$$\begin{pmatrix} 1_{n-n_r} & 0 \\ J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C) & 1_{n_r} \end{pmatrix}.$$

We note that $J_2 \in K_{n_r}(m)$ and $J_1 \in K_{n-n_r}(m)$ hence $J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C)$ belongs to $\varpi_F^{m+1}M_{(n-n_r)\times n_r}(\mathcal{O}_F)$. This shows that

$$\{j^{-1}u^-j(u^-)^{-1}\} \in K_I(m)\cap P_I(m+1)$$

Now the element $(u^-)u^+(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_{n-n_r} - \varpi_F^m BC & B \\ -\varpi_F^{2m}CBC & 1_{n_r} + \varpi_F^m CB \end{pmatrix}. \tag{3.1}$$

Since $2m \geq m+1$ the matrix in (3.1) is contained in the group $K_I(m)\cap P_I(1,m+1)$. ☐

We now observe that $K_I(m)P_I(1,m+1) = P_I(1,m)$. From Mackey decomposition we get that

$$\mathrm{res}_{K_I(m)}\,\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(m)\cap P_I(1,m+1)}^{K_I(m)}(\mathrm{id}).$$

Hence the above restriction decomposes into a direct sum of representations of the group

$$\frac{K_I(m)}{K_I(m)\cap P_I(1,m+1)}. \tag{3.2}$$

The inclusion map of $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ induces the natural homomorphism

$$\tilde{\theta}_I : \frac{K_I(m)\cap\bar{U}_{(n-n_r,n_r)}}{P_I(1,m+1)\cap\bar{U}_{(n-n_r,n_r)}} \to \frac{K_I(m)}{K_I(m)\cap P_I(1,m+1)}.$$

Lemma 3.0.13. *The map $\tilde{\theta}_I$ is an $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant isomorphism.*

Proof. The map is clearly injective and surjectivity follows from the Iwahori decomposition of $K_I(m)$ with respect to the Levi-subgroup M_I. The inclusion of $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ is an $M_{n-n_r,n_r} \cap P_I(1,m)$ equivariant map. □

Let u^- be an element of the group $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ and its block form be given by

$$\begin{pmatrix} 1_{(n-n_r,n_r)} & 0 \\ U^- & 1_{n_r} \end{pmatrix}.$$

The map $u^- \mapsto \varpi_F^{-m} U^-$ induces an isomorphism between the groups $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ and $M_{n_r \times (n-n_r)}(\mathcal{O}_F)$. Let $\bar{U^-}$ be the image of U^- in the mod-$\mathfrak{P}_F$ reduction of $M_{n_r \times (n-n_r)}(\mathcal{O}_F)$. The map $u^- \mapsto \overline{\varpi_F^{-m} U^-}$ induces an isomorphism of the quotient (3.2) with the group of matrices $M_{n_r \times (n-n_r)}(k_F)$. The group $M_{(n-n_r,n_r)}(\mathcal{O}_F) = \mathrm{GL}_{n-n_r}(\mathcal{O}_F) \times \mathrm{GL}_{n_r}(\mathcal{O}_F)$ acts on the group $M_{n_r \times (n-n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction $\mathrm{GL}_{n-n_r}(k_F) \times \mathrm{GL}_{n_r}(k_F)$, the action is given by $(g_1, g_2)U = g_2 U g_1^{-1}$ for all g_1 in $\mathrm{GL}_{n-n_r}(k_F)$, g_2 in $\mathrm{GL}_{n_r}(k_F)$ and U in $M_{n_r \times (n-n_r)}(k_F)$. The map $u^- \mapsto \overline{\varpi_F^{-m} U^-}$ is hence a $M_{(n-n_r,n_r)}(\mathcal{O}_F)$-equivariant map between the quotient (3.2) and $M_{n_r \times (n-n_r)}(k_F)$. Moreover the action of $M_{(n-n_r,n_r)}(\mathcal{O}_F)$ factors through its quotient $M_{(n-n_r,n_r}(k_F)$.

In general the group $G := \mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of matrices $M_{n \times m}(\mathbb{F}_q)$ by setting $(g_1, g_2)U = g_2 U g_1^{-1}$. We also have a G action on the set of matrices $M_{m \times n}(\mathbb{F}_q)$ by setting $(g_1, g_2)V = g_1 V g_2^{-1}$.

Lemma 3.0.14. *There exists a G-equivariant isomorphism between the groups $M_{m \times n}(\mathbb{F}_q)$ and $\widehat{M_{n \times m}(\mathbb{F}_q)}$.*

Proof. Let ψ be a non-trivial character of the additive group $\mathbb{F}_q$. We define a pairing B between $M_{m \times n}(\mathbb{F}_q)$ and $M_{n \times m}(\mathbb{F}_q)$ by defining $B(V, U) = \psi \circ \mathrm{tr}(VU)$. The pairing is non-degenerate and hence we obtain a map T between $M_{m \times n}(\mathbb{F}_q)$ and $\widehat{M_{n \times m}(\mathbb{F}_q)}$ defined by the equation

$$T(V)(U) = \psi \circ \mathrm{tr}(VU).$$

The map T is G equivariant since

$$(g_1, g_2)T(V)(U) = \psi \circ \mathrm{tr}(V g_2^{-1} U g_1) = \psi \circ \mathrm{tr}(g_1 V g_2^{-1} U) = T((g_1, g_2)V)(U).$$

□

The above lemma gives a $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant map between the groups $\widehat{M_{n_r \times (n-n_r)}}(k_F)$ and $M_{(n-n_r) \times n_r}(k_F)$. Hence we get an

$$M_{(n-n_r,n_r)} \cap P_I(1,m)$$

equivariant isomorphism say θ_I between the group of characters of

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}$$

and the group of matrices $M_{(n-n_r)\times n_r}(k_F)$.

Since the group $K_I(m)$ is a normal subgroup of $P_I(1, m)$, we have an action of this group $P_I(1, m)$ on the set of characters of the abelian group

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}.$$

If η is one such character we denote by $Z(\eta)$ the $P_I(1, m)$-stabilizer of this character η. Clifford theory now gives the decomposition

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \bigoplus_{\eta_k} \mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})$$

where $\{\eta_k\}$ is a set of representatives for the orbits under the action of $P_I(1, m)$ and U_{η_k} is some irreducible representation of the group $Z(\eta_k)$. We also note that $Z(\mathrm{id}) = P_I(1, m)$ and the identity character occurs with multiplicity one (which follows from Frobenius reciprocity) and hence

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}). \tag{3.3}$$

Observe that

$$Z(\eta_k) = (Z(\eta_k) \cap M_{(n-n_r,n_r)})K_I(m).$$

Since we have a $M_{(n-n_r,n_r)} \cap P_I(1, m)$ equivariant map between the group of characters of (3.2) and $M_{(n-n_r)\times n_r}(k_F)$, note that

$$Z(\eta_k) \cap M_{(n-n_r,n_r)} = Z_{M_{(n-n_r,n_r)} \cap P_I(1,m)}(A)$$

for some matrix A in $M_{(n-n_r)\times n_r}(k_F)$. The group $M_{(n-n_r,n_r)} \cap P_I(1, m)$ acts on the group of matrices $M_{(n-n_r,n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction. The mod-$\mathfrak{P}_F$ reduction of the group $P_I(1, m) \cap M_{(n-n_r,n_r)}$ is equal to the group $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$. In the next lemma we will bound the mod $\mathfrak{P}_F$ reduction of the group $Z(\eta_k) \cap M_I$ for the proof of the main theorem . Let $\mathcal{O}_A$ be an orbit for the action of $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ on the set of matrices $M_{(n-n_r)\times n_r}(k_F)$. Let p_j be the j^{th} projection of the group $M_I(k_F) = \prod_{i=1}^{r} \mathrm{GL}_{n_i}(k_F)$.

Lemma 3.0.15. *Let $\mathcal{O}_A$ be an orbit consisting of non-zero matrices in*

$$M_{(n-n_r)\times n_r}(k_F).$$

We can choose a representative A such that the $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$-stabilizer of A,

$$Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$$

satisfies one of the following conditions.

1. *There exists a positive integer j, $j \leq r$ such that the image of*

$$p_j : Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F) \to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$.

2. *There exists an i with $1 \leq i \leq r-1$ such that $p_i(g) = p_r(g)$ for all $g \in Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F)$.*

Proof. Let $A = [U_1, U_2, \ldots, U_{(r-1)}]^{tr}$ be the block form (U_k is a matrix of size $n_r \times n_k$ for $1 \leq k \leq r-1$) of a representative m for an orbit $\mathcal{O}_m$ consisting of non-zero matrices. If $((A_{ij}), B) \in Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$ then we have

$$(A_{ij})[U_1, U_2, ..., U_{(r-1)}]^{tr} = [U_1, U_2, ..., U_{(r-1)}]^{tr} B. \tag{3.4}$$

Since $(A_{ij}) \in P_{I'}(k_F)$, we have $A_{ij} = 0$ for all $i > j$. Let $l' \leq r-1$ be the least non-negative integer such that $U_{r-1-l'}$ (A_{ii} matrix of size $n_i \times n_i$) is non-zero and such an l' exists since $m \neq 0$. From (3.4) we get that $A_{ll}U_l^{tr}t = U_l^{tr}B$ where $l = r-1-l'$. There exist matrices $P \in \mathrm{GL}_{n_r}(k_F)$ and $Q \in \mathrm{GL}_{n_l}(k_F)$ such that PU_l^tQ is a matrix of the form

$$\begin{pmatrix} 1_t & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5}$$

where t is the rank of the matrix U_l^{tr}. Now we may change the representative A to $A' = [U_1', U_2', \ldots, U_r']^{tr}$ by the action of the element

$$\mathrm{diag}(1_{n_1}, \ldots, P, \ldots, 1_{n_{r-1}}, Q^{-1})$$

in $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ such that $U_l'^{tr}$ is the matrix (3.5). If $t = n_l = n_r$ then condition (2) is satisfied. Consider the maps $T_1 : k_F^{n_l} \to k_F^{n_r}$ and $T_2 : k_F^{n_r} \to k_F^{n_l}$ given by

$$(a_1, a_2, \ldots, a_{n_l}) \mapsto (a_1, a_2, \ldots, a_{n_l})U_l^{tr}$$

and

$$(a_1, a_2, \ldots, a_{n_r}) \mapsto U_l^{tr}(a_1, a_2, \ldots, a_{n_r})^{tr}$$

respectively. If $t = n_l = n_r$ does not hold then either of T_1 or T_2 has a non-trivial proper kernel (since $U_l \neq 0$). If T_1 has a non-trivial proper kernel then A_{ll} preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_r}(k_F)$. If T_2 has a non-trivial proper kernel then B preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_l}(k_F)$. Hence if $t = n_l = n_r$ does not hold true then condition (1) is satisfied. □

The following lemma is due to Paskunas but we give a mild modification for our applications (see [Pas05, Proposition 6.8]).

Lemma 3.0.16. *Let $m > 1$, σ be any irreducible representation of the group $\mathrm{GL}_m(\mathbb{F}_q)$ and H be a subgroup contained in a proper parabolic subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$. For every irreducible representation γ of H contained in $\mathrm{res}_H(\sigma)$, there exists an irreducible non-cuspidal representation σ' of $\mathrm{GL}_m(\mathbb{F}_q)$ such that $\mathrm{Hom}_H(\gamma, \sigma') \neq 0$.*

Proof. Let P be a proper parabolic subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$ containing H and $\bar{U}$ be the unipotent radical of an opposite parabolic subgroup of P. We observe that $\bar{U} \cap H = \mathrm{id}$. Now if the lemma is false, we have $\mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma) \simeq \oplus_{k \in \Lambda} \sigma_k$ such that σ_k is a cuspidal representation. Using Mackey decomposition we get that

$$\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) = \bigoplus_{u \in \bar{U} \setminus \mathrm{GL}_m(\mathbb{F}_q)/H} \mathrm{Hom}_{\bar{U} \cap H^u}(\mathrm{id}, \gamma^u).$$

If $\bar{U} \cap H = \mathrm{id}$ then $\mathrm{Hom}_{H \cap \bar{U}}(\mathrm{id}, \gamma)$ is non-zero and by the above decomposition

$$\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$

This shows that $\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \sigma_k) \neq 0$ for some $k \in \Lambda$ and this is a contradiction to our assumption. □

Lemma 3.0.17. *Let $m \geq 2$, H be the diagonal subgroup of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$ and $\sigma_1 \boxtimes \sigma_2$ be an irreducible representation of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$. For every irreducible representation γ occurring in $\mathrm{res}_H \sigma_1 \boxtimes \sigma_2$ there exists an irreducible non-cuspidal representation $\sigma_1' \boxtimes \sigma_2'$ of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$ containing γ.*

Proof. Let $\bar{U}$ and U be the subgroups of lower unipotent and upper unipotent matrices of $\mathrm{GL}_m(\mathbb{F}_q)$. Consider the unipotent subgroup $V := \bar{U} \times U$ of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$. Suppose the lemma is false then

$$\mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)}(\gamma) \simeq \oplus_{k \in \Lambda} \sigma_1^k \boxtimes \sigma_2^k$$

such that σ_1^k and σ_2^k are cuspidal representations for all $k \in \Lambda$. We observe that $V \cap H = \mathrm{id}$ and by Mackey decomposition we have

$$\mathrm{Hom}_V(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$

Now by our assumption we have $\mathrm{Hom}_V(\mathrm{id}, \sigma_1^k \boxtimes \sigma_2^k) \neq 0$ for some $k \in \Lambda$ and hence a contradiction. □

The following lemma is similar to proposition 2.2.3. The lemma is just a modified version of the proposition 2.2.3 for our present use.

Lemma 3.0.18. *Let Γ be a $\mathrm{GL}_{n-n_r}(\mathcal{O}_F)$-irreducible sub-representation of*

$$\mathrm{ind}_{P_{(n-n_r, n_r)}(m)}^{\mathrm{GL}_n(\mathcal{O}_F)} \{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

If the irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $s = [M_{I'}, \sigma_{I'}]$, then the representation Γ is atypical for the component $s = [M_I, \sigma_I]$.

Proof. Let ρ be an irreducible sub-representation of $U_m(\tau_{I'})$. If ρ is not typical then, there exists another Bernstein component $[M_J, \lambda_J]$ of $\mathrm{GL}_{n-n_r}(F)$ such that

$$[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$$

and ρ is contained in

$$\mathrm{res}_{\mathrm{GL}_{n-n_r}(\mathcal{O}_F)}\, i_{P_J}^{\mathrm{GL}_{(n-n_r)}(F)}(\lambda_J)$$

where $J = (n'_1, n'_2, \ldots, n'_{r'-1})$ and $\lambda_J = \boxtimes_{i=1}^{r'-1}\lambda_i$. The representation

$$\mathrm{ind}_{P_{(n-n_r,n_r)}(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\rho \boxtimes \tau_r\}$$

is contained in

$$\mathrm{ind}_{P_{(n-n_r,n_r)}\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\rho \boxtimes \tau_r\}. \tag{3.6}$$

The representation (3.6) is contained in the representation

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{P_{(n-n_r,n_r)}}^{\mathrm{GL}_n(F)}\{i_{P_J}^{\mathrm{GL}_{n-n_r}(F)}(\lambda_J) \boxtimes \sigma_r\}.$$

Since $[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$ there exist an inertial class $[\mathrm{GL}_p(F), \sigma]$ occurring in the multi-set

$$\{[\mathrm{GL}_{n_1}(F), \sigma_1], [\mathrm{GL}_{n_2}(F), \sigma_2], \ldots, [\mathrm{GL}_{n_{r-1}}(F), \sigma_{r-1}]\}$$

with a multiplicity not equal to its multiplicity in the multi-set

$$\{[\mathrm{GL}_{n'_1}(F), \lambda_1], [\mathrm{GL}_{n'_2}(F), \lambda_2], \ldots, [\mathrm{GL}_{n'_{r'-1}}(F), \lambda_{r'-1}]\}.$$

Hence the classes $[M_I, \sigma_I]$ and $[M_J \times \mathrm{GL}_{n_r}(F), \lambda_J \boxtimes \sigma_r]$ represent two distinct Bernstein components for the group $\mathrm{GL}_n(F)$. □

3.1 Proof of the main theorem

Proof of theorem 3.0.9. We prove the theorem by using induction on the positive integer n, the rank of $\mathrm{GL}_n(F)$. The theorem is true for $n = 1$ since $U_m(\tau_I)$ is zero. We assume that the theorem is true for all positive integers less than $n+1$. We will show the theorem for the positive integer $n+1$. Let $s = [M_I, \sigma_I]$ be a level-zero inertial class. We assume that the partition $I = (n_1, n_2, \ldots, n_r)$ of $n+1$ satisfies the hypothesis $n_i \leq n_j$ for all $1 \leq i \leq j \leq r$. If $r = 1$ we have $U_m(\tau_I) = 0$ and the theorem holds by default. We now assume that $r > 1$ and let $I' = (n_1, n_2, \ldots, n_{r-1})$.

We now break the proof into two cases. The first case is $n_r = 1$ and the second case is $n_r > 1$.

3.1.1 The case where $n_r = 1$

In this case $n_i = 1$ for $1 \leq i \leq r$ and $P_I = B_n$ where B_n is the Borel subgroup of GL_n. We denote by T_n and U_n the maximal torus and the unipotent radical respectively. We also use the notation $B_n(m)$ for the subgroup $P_I(m)$ and χ_{I_n} for τ_I since $I = (1, 1, \ldots, 1)$ is a tuple of length n. The proof is by induction on the integer n, the rank of T_n. The statement is immediate for $n = 1$ and for $n = 2$ we refer to [BM02, A.2.4] for a proof (We will require the proof for later use and we will recall it at that stage). So we prove the theorem for $n \geq 3$. Suppose the theorem is true for some positive integer $n \geq 2$. The rest of this subsection is to prove the main theorem for $n+1$. By definition of $U_m(\chi_{I_{n+1}})$ we have

$$\mathrm{ind}_{B_{n+1}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}) \simeq U_m(\chi_{I_{n+1}}) \oplus \mathrm{ind}_{B_{n+1}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}).$$

We have the isomorphism

$$\mathrm{ind}_{B_{n+1}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}) \simeq \mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1})\}.$$

We also have the decomposition

$$\begin{aligned} &\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \simeq \\ &\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_n\} \oplus \mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}. \end{aligned}$$

By induction hypothesis and lemma 3.0.18 irreducible sub-representations of

$$\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_{n+1}\}$$

are atypical representations. We now consider the irreducible factors of the representation

$$\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}. \tag{3.7}$$

We use induction on the integer m to show that the representation

$$\begin{aligned} &\mathrm{ind}_{P_{(n,1)}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\simeq \mathrm{ind}_{B_{n+1}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}) \end{aligned}$$

has a complement say $U_{1,m}(\chi_{In+1})$ in the representation (3.7) whose irreducible sub-representations are all atypical representations. This shows that irreducible sub-representations of $U_m(\chi_{I_{n+1}})$ are atypical. To reduce the notations we denote by $P(m)$ the subgroup $P_{(n,1)}(m)$. Applying the decomposition (3.3) to the parabolic subgroup $P_{(n,1)}$ we get that

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) = \mathrm{id} \oplus \mathrm{ind}_{Z(\eta)}^{P(m)}(U_\eta)$$

where η (in the present situation we just have one orbit consisting of non-trivial characters) is any non-trivial character of the group $K_{n+1}(m)U_{n,1}(\mathcal{O}_F)$ which is trivial on
$K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \cap P(m+1)$ and $K_{n+1}(m)$ is the principal congruence subgroup of level m. We have shown a $M \cap P(m)$ equivariant map between the group of characters of

$$\frac{K_{n+1}(m)U_{n,1}(\mathcal{O}_F)}{K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \cap P(m+1)}$$

and $M_{n\times 1}(k_F)$. We choose η to be the character corresponding to the matrix $[1, 0 \ldots, 0]$.

For the above choice of a non-trivial character we have

$$\begin{aligned}
&\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\
&\simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\
&\qquad \oplus \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta \otimes \mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}.
\end{aligned}$$

Since the representation $\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}$ is a level-zero representation,

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}$$

is isomorphic to the inflation of the representation

$$\mathrm{res}_{\overline{Z(\eta)\cap M_{(n,1)}}}\{\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}$$

where $\overline{Z(\eta) \cap M_{(n,1)}}$ is the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta) \cap M_{(n,1)}$. The group $\overline{Z(\eta) \cap M_{(n,1)}}$ is contained in the following subgroup

$$\left\{ \begin{pmatrix} A & B & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} | A \in \mathrm{GL}_{n-1}(k_F), B \in M_{(n-1)\times 1}(k_F) \text{ and } d \in k_F^\times \right\}. \quad (3.8)$$

Let Mir_k be the following group

$$\left\{ \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} | A \in \mathrm{GL}_{k-1}(k_F), B \in M_{(k-1)\times 1}(k_F), \right\}$$

Now we have to understand the restriction

$$\mathrm{res}_{P_{(n-1,1)}} \mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n})$$

which is reduced to understanding the restriction

$$\mathrm{res}_{\mathrm{Mir}_{n-1}} \mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}).$$

We use the theory of derivatives (originally for $\mathrm{GL}_n(F)$ due to Bernstein and Zelevinsky (see [BZ76])) to describe this restriction in a way sufficient for our application. We refer to [Zel81, Chapter 3, §13] for details of these constructions.

In the case of finite fields from Clifford theory one can define four exact functors and we recall the formalism here. The precise definitions are not required for our purpose except for one functor Ψ^+ which will be recalled latter:

$$\mathcal{M}(\mathrm{Mir}_{k-1}) \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \mathcal{M}(\mathrm{Mir}_k) \underset{\Psi^+}{\overset{\Psi^-}{\rightleftarrows}} \mathcal{M}(\mathrm{GL}_{k-1}(k_F))$$

The key results we use from Zelevinsky are summarised below (see [Zel81, Chapter 3, §13]).

Theorem 3.1.1 (Zelevinsky). *The functors Ψ^+ and Φ^- are left adjoint to Ψ^- and Φ^+ respectively. The compositions $\Phi^-\Phi^+$ and $\Psi^-\Psi^+$ are naturally equivalent to identity. Moreover $\Phi^+\Psi^-$ and $\Phi^-\Psi^+$ are zero. The diagram*

$$0 \to \Phi^+\Phi^- \to \mathrm{id} \to \Psi^+\Psi^- \to 0$$

obtained from these properties is exact.

Using this theorem and following Bernstein-Zelevinsky one can define a filtration Fil on a finite dimensional representation τ of Mir_n, for all $n > 1$. The filtration Fil is given by

$$0 \subset \tau_n \subset ... \subset \tau_3 \subset \tau_2 \subset \tau_1 = \tau$$

where $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}$ and $\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+\Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k \geq 1$. The representation $\tau^{(k)} := \Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k \geq 0$ of $\mathrm{GL}_{n-k}(k_F)$ is called the k^{th}-derivative of τ and by convention $\tau^{(0)} := \tau$.

Let R_n be the Grothendieck group of $\mathrm{GL}_n(k_F)$ for all $n \geq 1$ and set $R_0 = \mathbb{Z}$. Zelevinsky defined a ring structure on the group $R = \oplus_{n\geq 0} R_n$ by setting parabolic induction as the product rule. Recall that the ring R has a $\mathbb{Z}$-linear map D defined by setting $D(\pi) = \sum_{k\geq 0}(\pi|_{\mathrm{Mir}_n})^{(k)}$ for all π in R_n. It follows from [Zel81, Chapter 3, §13] that

$$D(\mathrm{ind}_P^{\mathrm{GL}_n(k_F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)) = \prod_{i=1}^{r} D(\tau_i)$$

where the product on the right hand side is in the ring R. The map D is hence an endomorphism of the ring R. If π is a supercuspidal representation of $\mathrm{GL}_n(k_F)$ then by Gelfand-Kazhdan theory it follows that $\pi^{(n)} = 1$, $\pi^{(0)} = \pi$

and all other derivatives are zero (see [Zel81, Chapter 3, §13]). Let $1_R \in R_0$ be the identity element of R.

In our present situation we have

$$D(\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})) = \prod_{i=1}^{n} D(\chi_i) = \prod_{i=1}^{n}(\chi_i + 1_R).$$

Let X_{n-k} be the term of degree $(n-k)$ in the expansion of the above product (it is a representation of $\mathrm{GL}_{n-k}(k_F)$ in the Grothendieck group R_{n-k}. Since the coefficients of the above expansion are positive X_{n-k} is actually a representation and not just a virtual representation.) Then we have

$$\mathrm{res}_{\mathrm{Mir}_{n-1}} \mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}) \simeq \bigoplus_{k\geq 1}^{n} (\Phi^+)^{k-1}\Psi^+(X_{n-k}).$$

Observe that $P_{(n-1,1)} = \mathrm{Mir}_{(n-1)} k_F^\times$ (here $k_F^\times$ is the centre of $\mathrm{GL}_n(k_F)$) and $\mathrm{Mir}_{(n-1)} \cap k_F^\times = \mathrm{id}$. The representation

$$\rho := (\Phi^+)^{k-1}\Psi^+(X_{n-k})$$

extends to a representation of $P_{(n-1,1)}$ by setting $\rho(a) = \chi(a)$ for all $a \in k_F^\times$ where χ is the central character of the representation

$$\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\boxtimes_{i=1}^{n}\chi_i).$$

Since the central character will play some role, we denote the extended representation by

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}.$$

By inflation we extend the $P_{(n,1)}(k_F) \times k_F^\times$-representation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

to a representation of $Z(\eta) \cap M_{(n,1)}$. We continue to use the notation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

for the extended representation. We now have

$$\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_n}) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_n}) \oplus \bigoplus_{k\geq 1}^{n} \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}\}.$$

We will show that any irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1})\}$$

is atypical for the component $[T_n, \chi_{I_n}]$.

We first consider the case when $k \geq 2$. The representation X_{n-k} is a direct sum of the representations:

$$\mathrm{ind}_{B_{n-k}(k_F)}^{\mathrm{GL}_{n-k}(k_F)}(\chi_{i_1} \boxtimes \chi_{i_2} \boxtimes \ldots \cdots \boxtimes \chi_{i_{n-k}}).$$

The above term also occurs in the expansion

$$\prod_{j=1}^{n-k}(1_R + \chi_{i_j})(1_R + \lambda)$$

where λ is a cuspidal representation of $\mathrm{GL}_k(k_F)$. To shorten the notation we use the symbol $\times$ for the multiplication in the ring R. We get that the representation

$$(\Phi^+)^{k-1}\Psi^+(\times_{j=1}^{n-k}\chi_{n_j})$$

occurs in the representation

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}(\times_{j=1}^{n-k}\chi_{n_j} \times \lambda).$$

Since the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta) \cap M_{(n,1)}$ is contained in the subgroup of the form (3.8), even if the central characters of $\times_{j=1}^{n}\chi_j$ and $\times_{j=1}^{n-k}(\chi_j) \times \lambda$ are different we may change χ_{n+1} to χ'_{n+1} such that the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{ext((\Phi^+)^{k-1}\Psi^+(\times_{j=1}^{n-k}\chi_{n_j}))\} \boxtimes \chi_{n+1}$$

occurs in the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}(\times_{j=1}^{n-k}(\chi_j) \times \lambda) \boxtimes \chi'_{n+1}.$$

Hence an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}) \otimes U_\eta\} \tag{3.9}$$

occurs as a sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\{(\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1})\} \otimes U_\eta\}. \tag{3.10}$$

The above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_{(1,1,\ldots,k,1)}\cap\mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1}\}. \tag{3.11}$$

Hence the sub-representation of (3.9) are not typical representations.

Now we are left with the term

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}. \tag{3.12}$$

We might as well repeat the same strategy as for $k \geq 2$ and now λ is one dimensional but the representations (3.11) and $\times_{j=1}^{n+1}\chi_j$ may not have distinct inertial support. In order to tackle the terms of the above representation we use a different technique. We now recall the definition of the representation U_η, the functor Ψ^+ and some facts due to Casselman regarding the restriction of an irreducible smooth representation to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$.

The representation U_η is a character on the group $Z(\eta)$. From (3.8) any element of the group $Z(\eta)$ is of the form

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \tag{3.13}$$

where $A \in \mathrm{GL}_{n-1}(\mathcal{O}_F)$; $(X')^t, X^t, B, C^t \in M_{(n-1)\times 1}(\mathcal{O}_F)$; $e, d \in \mathcal{O}_F^\times$; $y, y' \in \mathcal{O}_F$ and $d \equiv e(\mathfrak{P}_F)$. The character U_η is given by

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \mapsto \eta(\varpi_F^m y).$$

The functor

$$\Psi^+ : \mathcal{M}(\mathrm{GL}_{k-1}(k_F)) \to \mathcal{M}(\mathrm{Mir}_k)$$

is the inflation functor via the quotient map of Mir_k modulo the unipotent radical of Mir_k.

Let (π, V_π) be an irreducible smooth representation of $\mathrm{GL}_2(F)$. We denote by $c(\pi)$ and ϖ_π the conductor and central character of the representation π respectively. Let V^N be the space of all vectors fixed by the principal congruence subgroup of level N for all $N \geq 1$. For all $i > c(\varpi_\pi)$ we define the representation $U_i(\chi)$ as the complement of the representation $\mathrm{ind}_{B_2(i-1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$ in $\mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$. For $i = c(\varpi_\pi)$ we set

$$U_i(\varpi_\pi) = \mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\varpi_\pi \boxtimes \mathrm{id}).$$

It follows from [Cas73, Proposition 1] that the representation $U_i(\varpi_\pi)$ is an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_F)$. From the result [Cas73, Proposition 2] we get that $c(\pi) \geq c(\varpi_\pi)$. By [Cas73, Theorem 1] we have

$$\mathrm{res}_{\mathrm{GL}_2(\mathcal{O}_F)} V_\pi = V^{(c(\pi)-1)} \oplus \bigoplus_{i \geq c(\pi)} U_i(\varpi_\pi). \tag{3.14}$$

We now describe the representation $U_i(\varpi_\pi)$ in our language. Let κ be a non-trivial character of the group $K_2(m)U_{(1,1)}(\mathcal{O}_F)$ trivial modulo

$K_2(m)U_{(1,1)}(\mathcal{O}_F) \cap B_2(m+1)$. Let $Z(\kappa)$ be a $B_2(m)$ stabilizer of κ. Any element of the group $Z(\kappa)$ is of the form (for an appropriate choice of a non-trivial character κ)

$$\begin{pmatrix} a & b \\ \varpi_F^m c & d \end{pmatrix}$$

where $a, d \in \mathcal{O}_F^\times$; $b \in \mathcal{O}_F$, $c \in \mathfrak{P}_F^m$ and $d \equiv a$ modulo $\mathfrak{P}_F$. We define a character U_η by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \eta(c).$$

We then have

$$U_m(\varpi) \simeq \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_2(\mathcal{O}_F)}(U_\eta \otimes (\varpi \boxtimes \mathrm{id})).$$

Now let us resume the proof in the general case $n > 2$ the representation

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}$$

is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)}(m)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}(X_{n-1} \boxtimes U_m(\chi)) \tag{3.15}$$

where χ is given by $\prod_{i=1}^n \chi_i$ of $\mathcal{O}_F^\times$. This representation, by the theorem of Casselman (see the decomposition 3.14) is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)} \cap \mathrm{GL}_{(n+1)}(\mathcal{O}_F)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}(X'_{n-1} \boxtimes \sigma)$$

where σ is a supercuspidal representation of level-zero with central character χ (see the remark below for the existence) and X'_{n-1} is the $(n-1)$ derivative of the representation

$$i_{B_n}^{\mathrm{GL}_n(F)}(\chi_{I_n}).$$

Hence irreducible sub-representations of (3.12) are atypical. This completes the proof that irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta \otimes \mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}$$

are atypical. From the decomposition

$$\begin{aligned} &\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\qquad \oplus \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta \otimes \mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}. \end{aligned}$$

we get the theorem for the case where $n_r = 1$.

Remark 3.1.2. *The existence of the cuspidal representation of* $\mathrm{GL}_2(k_F)$ *with a given central character can be deduced from the explicit formula for such representations, we refer to [BH06, Theorem section 6.4]. To be precise we begin with a quadratic extension* k *of* k_F *and* θ *a character of* $k^\times$ *such that* $\theta^q \neq \theta$ *where* $q = \#k_F$*. These characters are called regular characters and for any regular character one can define a supercuspidal representation* π_θ *and conversely all supercuspidal representations are of the form* π_θ *for some regular character* θ*. The central character of* π_θ *is given by* $\mathrm{res}_{k_F^\times}(\theta)$*. Now to get a supercuspidal representation with a central character* χ *we begin with a character* χ *on* $k_F^\times$*, there are* $\#k_F + 1$ *possible extensions to* $k^\times$*. The set of characters* θ *such that* $\theta^q = \theta$ *has cardinality* $\#k_F - 1$*. Hence there exists at least one supercuspidal representation with a given central character* χ*. This shows that irreducible sub-representations of* (3.15) *are not typical and this completes the proof of the theorem in this case.*

3.1.2 The case where $n_r > 1$

By transitivity of induction we have

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\}.$$

We note that $P_I(1,m) \cap U_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap \bar{U}_{(n-n_r+1,n_r)}$ hence lemma 2.2.6 gives the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\}.$$

Splitting the representation $\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})$ as $\tau_{I'} \oplus U_m^0(\tau_{I'})$ we get that

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I).$$

From Frobenius reciprocity the representation τ_I occurs in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $U_{(1,m)}^0(\tau_I)$ be the complement of τ_I in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$. With this we conclude that

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus U_{(1,m)}^0(\tau_I) \oplus \tau_I.$$

By definition $U_m(\tau_I) = \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_m^0(\tau_I))$ which shows that

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U_{(1,m)}^0(\tau_I)).$$

We observe that $P_I(1,m) \cap U_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap \bar{U}_{(n-n_r+1,n_r)}$ hence lemma

2.2.6 applied to the groups $J_2 = P_I(1,m)$ and $J_1 = P_{(n-n_r+1,n_r)}(m)$ and $\lambda = U^0_m(\tau_{I'}) \boxtimes \tau_r$ gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U^0_m(\tau_{I'}) \boxtimes \tau_r\} \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

With this we are in a place to use the induction hypothesis through the isomorphism

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\tau_{I'})\boxtimes\tau_r\}\oplus\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I)). \quad (3.16)$$

By induction hypothesis $\mathrm{GL}_{n-n_r+1}(\mathcal{O}_F)$-irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $[M_{I'}, \sigma_{I'}]$. Now lemma 3.0.18 and the equation (3.16) reduce the proof of the theorem to showing that irreducible sub-representations of $\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I))$ are atypical representations.

Proposition 3.1.3. *The irreducible sub-representations of*

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I))$$

are atypical for $m \geq 1$.

Proof. We observe that

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\tau_I)\}$$

and the decomposition (3.3) gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) = \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\}$$

which gives the equality

$$U^0_{(1,m+1)}(\tau_I) = U^0_{(1,m)}(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}.$$

If we show that the irreducible sub-representations of

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}$$

(for $\eta_k \neq \mathrm{id}$) are atypical for $[M_I, \sigma_I]$ then induction on the positive integer m completes the proof of the proposition in this case. To begin with we note that

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\}$$
$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k)\cap M_I}\,\tau_I)\}.$$

The representation τ_I is a level zero representation. Hence $\mathrm{res}_{Z(\eta_k)\cap M_I}\tau_I$ is isomorphic to the inflation of the representation $\mathrm{res}_{\overline{Z(\eta_k)\cap M_I}}\tau_I$ where $\overline{Z(\eta_k)\cap M_I}$ is mod-$\mathfrak{P}_F$ reduction of $Z(\eta_k)\cap M_I$. Let $A=\theta_I(\eta_k)$ where θ_I is the map defined in the paragraph just after lemma 3.0.14. The mod-$\mathfrak{P}_F$ reduction $\overline{Z(\eta_k)\cap M_I}$ is contained in $Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$. If η_k is a nontrivial character then $A\neq 0$ and we can apply lemma 3.0.15. For convenience we break the proof of this proposition into subsections considering different possibilities in lemma 3.0.15.

3.1.3 Condition (1) of lemma 3.0.15

We first assume that A satisfies the condition (1) in lemma 3.0.15. There exists at least one n_j with $1\leq j\leq r$ such that the image of the projection

$$p_j: Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)\cap M_I\to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$. Here p_j is the projection onto the j-th factor of M_I. In particular n_j is greater than 1. Let γ be an irreducible sub-representation of the restriction $\mathrm{res}_H\tau_j$ where H is the image of $M_I\cap Z(\eta_k)$ under the projection p_j. It follows from lemma 3.0.16 that there exists an irreducible non-cuspidal representation τ' of $\mathrm{GL}_{n_j}(k_F)$ such that $\tau_j\not\simeq\tau'$ and γ is contained in $\mathrm{res}_H\tau'$. Let τ' (as a representation of $\mathrm{GL}_{n_j}(\mathcal{O}_F)$ obtained by inflation) be a sub-representation of

$$\Gamma=i_{P_J(1)}^{\mathrm{GL}_{n_j}(\mathcal{O}_F)}(\kappa_J)$$

where $J=(m_1,m_2,\dots,m_t)$ is an ordered partition of the positive integer n_j and each of κ_l for $1\leq l\leq t$ is a cuspidal representation of $\mathrm{GL}_{m_l}(k_F)$. Define a representation τ_I^1 (a first modification of τ_I) of $Z(\eta_k)\cap M_I$ by setting

$$\tau_I^1:=\tau_1\boxtimes\tau_2\boxtimes\cdots\boxtimes\tau_{j-1}\boxtimes\gamma\boxtimes\tau_{j+1}\boxtimes\cdots\boxtimes\tau_r.$$

The representation

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\tau_I^1)\}\tag{3.17}$$

is contained in the representation

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\mathrm{res}_{Z(\eta_k)\cap M_I}\tau_I^2)\}$$

where τ_I^2 (the second modification) is the representation

$$\tau_I^1:=\tau_1\boxtimes\tau_2\boxtimes\cdots\boxtimes\tau_{j-1}\boxtimes\Gamma\boxtimes\tau_{j+1}\boxtimes\cdots\boxtimes\tau_r.$$

Observe that

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\tau_I^2)\}\simeq\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})\otimes\tau_I^2\}$$

The representation $\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})\otimes\tau_I^2\}$ is a sub-representation of the representation $\mathrm{ind}_{P_I(1,m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2)$ which in turn is contained in the representation $\mathrm{ind}_{P_I\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2)$. We denote by I_1 the refinement of the ordered partition I obtained by replacing n_j with the ordered partition $J=(m_1,m_2,\dots,m_t)$. We define κ_{I_1} a representation of $M_{I_1}(\mathcal{O}_F)$ by setting

$$\kappa_{I_1}:=\tau_1\boxtimes\cdots\boxtimes\tau_{j-1}\boxtimes\kappa_1\boxtimes\cdots\boxtimes\kappa_t\boxtimes\tau_{j+1}\boxtimes\cdots\boxtimes\tau_r.$$

By setting these notations we now note that

$$\mathrm{ind}_{P_I\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2)\subset\mathrm{ind}_{P_{I_1}\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\kappa_{I_1}).$$

Since I_1 is a proper partition of I the Bushnell-Kutzko types $(P_I(1),\tau_I)$ and $(P_{I_1}(1),\kappa_{I_1})$ represent two distinct inertial classes.

3.1.4 Condition (2) of lemma 3.0.15

Let $A=\theta_I(\eta_k)$ satisfy the condition (2) in the lemma 3.0.15. In this case there exists a j with $1\le j<r$ such that the mod $\mathfrak{P}_F$ reduction of $Z(\eta_k)\cap M_I$ is contained in the subgroup of the form

$$\{(A_1,\dots,A_j,\dots,A_r)|\ A_i\in\mathrm{GL}_{n_i}(k_F)\ \forall\ i\in\{1,2,\dots,r\}\text{ and }A_j=A_r\}.$$

Note that $n_j=n_r$ and we assumed that $n_r>1$. Consider the representation $\tau_j\boxtimes\tau_r$ of $\mathrm{GL}_{n_j}(k_F)\times\mathrm{GL}_{n_r}(k_F)$ and $H=\{(A,A)\ |A\in\mathrm{GL}_{n_r}(k_F)\}$. For every irreducible sub-representation γ of $\mathrm{res}_H(\tau_j\boxtimes\tau_r)$ using lemma 3.0.17 we obtain an irreducible non-cuspidal representation $\tau_j^1\boxtimes\tau_r^1$ such that γ is contained in $\mathrm{res}_H(\tau_j^1\boxtimes\tau_r^1)$. Now define a representation τ_I^1 by setting

$$\tau_I^1:=\tau_1\boxtimes\tau_2\boxtimes\cdots\boxtimes\tau_j^1\boxtimes\cdots\boxtimes\tau_r^1.$$

We note here that τ_I^1 may not be independent of γ in the sense that that (τ_j^1,τ_r^1) depends on the irreducible sub-representation γ of $\mathrm{res}_H(\tau_k\boxtimes\tau_r)$. Any irreducible sub-representation Γ of

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\mathrm{res}_{Z(\eta_k)\cap M_I}\tau_I)\}$$

occurs as a sub-representation of some

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\mathrm{res}_{Z(\eta_k)\cap M_I}\tau_I^1)\}.\tag{3.18}$$

The representation in (3.18) is contained as a sub-representation of

$$\mathrm{ind}_{P_I\cap\mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau_I^1).\tag{3.19}$$

Let τ_j^1 and τ_r^1 be sub-representations of $\mathrm{ind}_{P_{J_1}(1)}^{\mathrm{GL}_{n_k}(\mathcal{O}_F)}(\kappa_{J_1})$ and $\mathrm{ind}_{P_{J_2}(1)}^{\mathrm{GL}_{n_r}(\mathcal{O})\ F}(\kappa_{J_2})$ respectively. Let I_1 be the partition of the positive integer n obtained by replacing n_j and n_r by the partitions J_1 and J_2 in $(n_1, n_2 \dots, n_j, \dots, n_r)$. We denote by τ_{I_1} the representation

$$\tau_{n_1} \boxtimes \cdots \boxtimes \tau_{n_{j-1}} \boxtimes \kappa_{J_1} \boxtimes \tau_{n_{j+1}} \boxtimes \cdots \boxtimes \kappa_{J_2}$$

of $M_{I_1}(\mathcal{O}_F)$. The representation (3.19) is contained in the representation

$$\mathrm{ind}_{P_{I_1} \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau_{I_1}).$$

The Bushnell-Kutzko types $(P_I(1), \tau_I)$ and $(P_{I_1}(1), \tau_{I_1})$ represent two distinct inertial classes since I_1 is a proper refinement of I (see lemma 3.0.17). □

This completes the proof of the proposition and also the proof of the theorem. □