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Chapter 3

Level zero inertial classes

Definition 3.0.8. Let \( I = (n_1, n_2, \ldots, n_r) \) be an ordered partition of \( n \). An inertial class \( s = [M_I, \bigotimes_{i=1}^{r} \sigma_i] \) is called a level-zero inertial class if for every \( \sigma_i \) there exists an irreducible representation \( \tau_i \) of \( \text{GL}_{n_i}(O_F) \) such that \( \tau_i \) is the inflation of an irreducible cuspidal representation of \( \text{GL}_{n_i}(k_F) \) and \( \text{Hom}_{\text{GL}_{n_i}(O_F)}(\tau_i, \sigma_i) \neq 0 \).

We fix a level-zero inertial class \( s = [M_I, \sigma_I] \) with the pairs \((\text{GL}_{n_i}(O_F), \tau_i)\) as in the above definition. The pair \((\text{GL}_{n_i}(O_F), \tau_i)\) is the Bushnell-Kutzko type for the inertial class \([\text{GL}_{n_i}(F), \sigma_i]\). Let \( m \) be a positive integer and \( P_I(m) \) be the inverse image of \( P_I(O_F/P_F^m) \) under the mod-\( P_F^m \) reduction map \( \pi_m : \text{GL}_n(O_F) \to \text{GL}_n(O_F/P_F^m) \).

The representation \( \bigotimes_{i=1}^{r} \tau_i \) of \( M_I(k_F) \) extends to a representation of \( P_I(k_F) \) by inflation via the quotient map \( P_I(k_F) \to P_I(k_F)/U_I(k_F) \simeq M_I(k_F) \).

The representation \( \bigotimes_{i=1}^{r} \tau_i \) of \( P_I(k_F) \) extends to a representation of \( P_I(1) \) by inflation via the map \( \pi_1 \). We note that \( P_I(1) \cap U_I \) and \( P_I(1) \cap \bar{U}_I \) are contained in the kernel of this extension. The pair \((P_I(1), \tau_I)\) is the Bushnell-Kutzko type for the component \( s \) (see [BK99][Section 8.3.1]). The irreducible sub-representations of

\[ \text{ind}_{P_I(1)}^{\text{GL}_n(O_F)}(\tau_I) \]

are thus typical for \( s \).

We note that the groups \( P_I(m) \) satisfy Iwahori decomposition with respect to \( P_I \) and \( M_I \). The \( M_I(O_F) \) representation \( \tau_I \) extends to a representation of \( P_I(m) \) such that \( P_I(m) \cap U_I \) and \( P_I(m) \cap \bar{U}_I \) are contained in the kernel of the extension. This shows that the sequence of groups \( \{P_I(m) \mid m \geq 1\} \) and \( \tau_I \) satisfy the hypothesis for the groups \( \{H_m \mid m \geq 1\} \) and \( \tau \) in lemma 2.2.5 hence we have the isomorphism

\[ \bigcup_{m \geq 1} \text{ind}_{P_I(m)}^{\text{GL}_n(O_F)}(\tau_I) \simeq \text{ind}_{P_I \cap \text{GL}_n(O_F)}^{\text{GL}_n(O_F)}(\tau_I) \]

We recall that the lemma 2.2.4 shows that typical representations for the component \( s \) can only occur in the above representation.
Using Frobenius reciprocity we get that the representation $\tau_I$ occurs in $\text{ind}^{P_1(1)}_{P_1(m)}(\tau_I)$ with multiplicity one. Let $m \geq 1$ and $U^0_m(\tau_I)$ be the $P_1(1)$-stable complement of the representation $\tau_I$ in $\text{ind}^{P_1(1)}_{P_1(m)}(\tau_I)$. Let $U_m(\tau_I)$ be the representation $\text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(U^0_m(\tau_I))$.

We note that $\text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(\tau_I) \oplus U_m(\tau_I) \simeq \text{ind}^{\text{GL}_n(O_F)}_{P_1(m)}(\tau_I)$

We will show that irreducible sub-representations of $U_m(\tau_I)$ are atypical.

**Theorem 3.0.9 (Main).** Let $m \geq 1$. The $\text{GL}_n(O_F)$-irreducible subrepresentations of $U_m(\tau_I)$ are atypical.

The classification of typical representations for the inertial class $s$ is given by the following corollary.

**Corollary 3.0.10.** The irreducible sub-representations of $\text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(\tau_I)$ are precisely the typical representations for the level-zero inertial class $[M_I, \sigma_I]$. Moreover if $\Gamma$ is a typical representation then

$$\text{dim}_C \text{Hom}_{\text{GL}_n(O_F)}(\Gamma, \text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(\tau_I)) = \text{dim}_C \text{Hom}_{\text{GL}_n(O_F)}(\Gamma, \iota^{\text{GL}_n(F)}_{P_1}(\sigma_I)).$$

**Proof.** Given a typical representation $\Gamma$ for the inertial class $s$, the theorem shows that $\Gamma$ is a sub-representation of $\text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(\tau_I)$ and the multiplicity formula follows from 2.2.4 and the above theorem. Conversely if $\Gamma$ is a sub-representation of $\text{ind}^{\text{GL}_n(O_F)}_{P_1(1)}(\tau_I)$ then by Frobenius reciprocity we get that $\text{Hom}_{P_1(1)}(\tau_I, \Gamma) \neq 0$. If $\Gamma$ is contained as a $\text{GL}_n(O_F)$-irreducible sub-representation in an irreducible smooth representation $\pi$ of $\text{GL}_n(F)$ then the restriction of $\pi$ to $P_1(1)$ contains the representation $\tau_I$. The pair $(P_1(1), \tau_I)$ is the Bushnell-Kutzko type for the inertial class $s = [M_I, \sigma_I]$ hence the inertial support of $\pi$ is $s$. Hence $\Gamma$ is a typical representation and this proves the corollary. \qed

We will need a few lemmas before the proof of this theorem. Let $I = (n_1, n_2, \ldots, n_r)$ be the ordered partition of the positive integer $n$ as fixed at the beginning of this chapter. **Until the beginning of the section 3.1 we assume that** $r > 1$. We denote by $I'$ the ordered partition $(n_1, n_2, \ldots, n_{r-1})$ of $n - n_r$. Let $m$ be a positive integer and $P_1(1, m)$ be the following set

$$\left\{\left(\begin{array}{cc} A & B \\ \overline{w}_F C & D \end{array}\right) \mid A \in P_1(1); B, C^t \in M_{n_r \times (n-n_r)}(O_F); D \in \text{GL}_{n_r}(O_F)\right\}.$$ 

Note that $P_1(1, 1) = P_1(1)$. 46
Lemma 3.0.11. The set $P_I(1, m)$ is a subgroup of $P_I(1)$.

Proof. The group $GL_n(\mathcal{O}_F)$ acts on the set of lattices of $F^n$ contained in the lattice $\mathcal{O}_F^n$. If $r-1 = 1$ the set $P_I(1, m)$ is the $GL_n(\mathcal{O}_F)$-stabilizer of the lattice $(\mathcal{O}_F)^{n_1} \oplus (\mathcal{O}_F)^{n_2}$. In the case $r-1 > 1$ the set $P_I(1, m)$ is the $GL_n(\mathcal{O}_F)$-stabilizer of lattices $L_k$ for $1 < k \leq r-1$ defined as:

$$L_k = (\mathcal{O}_F)^{n_1} \oplus \cdots \oplus (\mathcal{O}_F)^{n_k-1} \oplus (\mathcal{O}_F)^{n_k} \oplus \cdots \oplus (\mathcal{O}_F)^{n_{r-1}} \oplus (\mathcal{O}_F)^{n_r}.$$  

This shows that $P_I(1, m)$ is a subgroup and is contained in $P_I(1)$ from the definition.

The structure of the representation

$$\text{ind}^{P_I(1,m)}_{P_I(1,m+1)}(\text{id})$$

will be used in the proof of the main theorem. Using Clifford theory we decompose the above representation. Let $K_I(m)$ be the group $K_n(m) U_{n-r,n_r} (\mathcal{O}_F)$.

Lemma 3.0.12. The group $K_I(m)$ is a normal subgroup of $P_I(1, m)$ and $K_I(m) \cap P_I(1, m + 1)$ is a normal subgroup of $K_I(m)$.

Proof. The groups $K_I(m)$ and $P_I(1, m)$ satisfy Iwahori decomposition with respect to $U_{n-r,n_r}$, $\bar{U}_{n-r,n_r}$ and $M_{n-r,n_r}$. We also note that

$$K_I(m) \cap U_{n-r,n_r} = P_I(1, m) \cap U_{n-r,n_r}$$

and

$$K_I(m) \cap \bar{U}_{n-r,n_r} = P_I(1, m) \cap \bar{U}_{n-r,n_r}.$$  

Hence $P_I(1, m) \cap U_{n-r,n_r}$ and $P_I(1, m) \cap \bar{U}_{n-r,n_r}$ normalize $K_I(m)$. Since $K_I(m)$ is a product of the group $K_n(m)$ and $U_{n-r,n_r}(\mathcal{O}_F)$ the group $P_I(1, m) \cap M_{n-r,n_r}$ normalizes the group $K_I(m)$. This shows the first part.

Notice that $K_I(m) \cap U_{n-r,n_r}$ is equal to $K_I(m) \cap P_I(1, m+1) \cap U_{n-r,n_r}$ and $K_I(m) \cap M_{n-r,n_r}$ is equal to $K_I(m) \cap P_I(1, m+1) \cap M_{n-r,n_r}$ hence it is enough to check that $K_I(m) \cap \bar{U}_{n-r,n_r}$ normalizes the group $K_I(m) \cap P_I(1, m+1)$. Since $K_I(m) \cap P_I(1, m+1) \cap \bar{U}_{n-r,n_r}$ is abelian and is contained in $K_I(m) \cap \bar{U}_{n-r,n_r}$ hence we need to check that $u^- j(u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ are contained in $K_I(m) \cap P_I(1, m+1)$ for all $u^-, j$ and $u^+$ in $K_I(m) \cap \bar{U}_{n-r,n_r}$.

$$K_I(m) \cap \bar{U}_{n-r,n_r},$$

$$K_I(m) \cap P_I(1, m+1) \cap M_{n-r,n_r}$$  

and

$$K_I(m) \cap P_I(1, m+1) \cap U_{n-r,n_r} = U_{n-r,n_r}(\mathcal{O}_F).$$

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respectively.

Let \( u^+, u^- \) and \( j \) be three elements from \( U_{n-n_r,n_r}(\mathcal{O}_F) \), \( K_I(m) \cap \tilde{U}_{(n-n_r,n_r)} \) and \( K_I(m) \cap P_I(1,m+1) \cap M_{(n-n_r,n_r)} \) respectively. We write them in their block form as:

\[
u^+ = \begin{pmatrix} 1_{n-n_r} & B \\ 0 & 1_{n_r} \end{pmatrix}
\]

where \( B \in M_{(n-n_r) \times n_r}(\mathcal{O}_F) \),

\[
u^- = \begin{pmatrix} 1_{n-n_r} & 0 \\ \varpi_F^m C & 1_{n_r} \end{pmatrix}
\]

where \( C \in M_{n_r \times (n-n_r)}(\mathcal{O}_F) \) and

\[
j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.
\]

We observe that \( u^- j(u^-)^{-1} = j \{ j^{-1} u^- j(u^-)^{-1} \} \) and the commutator \( \{ j^{-1} u^- j(u^-)^{-1} \} \) in its block form is as follows:

\[
\begin{pmatrix} 1_{n-n_r} \\ J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C) & 0 \\ 0 & 1_{n_r} \end{pmatrix}.
\]

We note that \( J_2 \in K_{n_r}(m) \) and \( J_1 \in K_{n-n_r}(m) \) hence \( J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C) \) belongs to \( \varpi_F^{m+1} M_{(n-n_r) \times n_r}(\mathcal{O}_F) \). This shows that

\[
\{ j^{-1} u^- j(u^-)^{-1} \} \in K_I(m) \cap P_I(m+1)
\]

Now the element \( (u^-) u^+ (u^-)^{-1} \) is of the form

\[
\begin{pmatrix} 1_{n-n_r} - \varpi_F^m BC \\ -\varpi_F^{2m} CB & 1_{n_r} + \varpi_F^m CB \end{pmatrix}, \quad (3.1)
\]

Since \( 2m \geq m+1 \) the matrix in (3.1) is contained in the group \( K_I(m) \cap P_I(1,m+1) \). \( \square \)

We now observe that \( K_I(m) P_I(1,m+1) = P_I(1,m) \). From Mackey decomposition we get that

\[
\text{res}_{K_I(m)} \text{ind}_{P_I(1,m+1)}^{P_I(1,m+1)}(\text{id}) \simeq \text{ind}_{K_I(m) \cap P_I(1,m+1)}^{K_I(m)}(\text{id}).
\]

Hence the above restriction decomposes into a direct sum of representations of the group

\[
\frac{K_I(m)}{K_I(m) \cap P_I(1,m+1)}.
\]

The inclusion map of \( K_I(m) \cap \tilde{U}_{(n-n_r,n_r)} \) in \( K_I(m) \) induces the natural homomorphism

\[
\tilde{\theta}_I : \frac{K_I(m) \cap \tilde{U}_{(n-n_r,n_r)}}{P_I(1,m+1) \cap \tilde{U}_{(n-n_r,n_r)}} \to \frac{K_I(m)}{K_I(m) \cap P_I(1,m+1)}.
\]

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Lemma 3.0.13. The map $\tilde{\theta}_I$ is an $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant isomorphism.

Proof. The map is clearly injective and surjectivity follows from the Iwahori decomposition of $K_I(m)$ with respect to the Levi-subgroup $M_I$. The inclusion of $K_I(m) \cap \tilde{U}_{(n-n_r,n_r)}$ in $K_I(m)$ is an $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant map. \[\square\]

Let $u^-$ be an element of the group $K_I(m) \cap \tilde{U}_{(n-n_r,n_r)}$ and its block form be given by

$$
\begin{pmatrix}
1_{(n-n_r,n_r)} & 0 \\
U^- & 1_{n_r}
\end{pmatrix}.
$$

The map $u^- \mapsto \omega_F^{-m} U^-$ induces an isomorphism between the groups $K_I(m) \cap \tilde{U}_{(n-n_r,n_r)}$ and $M_{n_r \times (n-n_r)}(O_F)$. Let $\bar{U}^-$ be the image of $U^-$ in the mod-$\mathfrak{P}_F$ reduction of $M_{n_r \times (n-n_r)}(O_F)$. The map $u^- \mapsto \omega_F^{-m} \bar{U}^-$ induces an isomorphism of the quotient (3.2) with the group of matrices $M_{n_r \times (n-n_r)}(k_F)$.

The group $M_{(n-n_r,n_r)}(O_F) = \text{GL}_{n-n_r}(O_F) \times \text{GL}_{n_r}(O_F)$ acts on the group $M_{n_r \times (n-n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction $\text{GL}_{n-n_r}(k_F) \times \text{GL}_{n_r}(k_F)$, the action is given by $(g_1,g_2)U = g_2 U g_1^{-1}$ for all $g_1$ in $\text{GL}_{n-n_r}(k_F)$, $g_2$ in $\text{GL}_{n_r}(k_F)$ and $U$ in $M_{n_r \times (n-n_r)}(k_F)$. The map $u^- \mapsto \omega_F^{-m} \bar{U}^-$ is hence a $M_{(n-n_r,n_r)}(O_F)$-equivariant map between the quotient (3.2) and $M_{n_r \times (n-n_r)}(k_F)$. Moreover the action of $M_{(n-n_r,n_r)}(O_F)$ factors through its quotient $M_{(n-n_r,n_r)}(k_F)$.

In general the group $G := \text{GL}_m(F_q) \times \text{GL}_n(F_q)$ acts on the set of matrices $M_{m \times n}(F_q)$ by setting $(g_1,g_2)U = g_2 U g_1^{-1}$. We also have a $G$ action on the set of matrices $M_{m \times n}(F_q)$ by setting $(g_1,g_2)V = g_1 V g_2^{-1}$.

Lemma 3.0.14. There exists a $G$-equivariant isomorphism between the groups $M_{m \times n}(F_q)$ and $\text{M}_{m \times n}(F_q)$.

Proof. Let $\psi$ be a non-trivial character of the additive group $F_q$. We define a pairing $B$ between $M_{m \times n}(F_q)$ and $M_{n \times m}(F_q)$ by defining $B(V,U) = \psi \circ \text{tr}(VU)$. The pairing is non-degenerate and hence we obtain a map $T$ between $M_{m \times n}(F_q)$ and $\text{M}_{n \times m}(F_q)$ defined by the equation

$$
T(V)(U) = \psi \circ \text{tr}(VU).
$$

The map $T$ is $G$ equivariant since

$$
(g_1,g_2)T(V)(U) = \psi \circ \text{tr}(V g_2^{-1} U g_1) = \psi \circ \text{tr}(g_1 V g_2^{-1} U) = T((g_1,g_2)V)(U).
$$

The above lemma gives a $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant map between the groups $M_{n_r \times (n-n_r)}(k_F)$ and $M_{(n-n_r) \times n_r}(k_F)$. Hence we get an $M_{(n-n_r,n_r)} \cap P_I(1,m)$
equivariant isomorphism say $\theta_I$ between the group of characters of

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m + 1)}$$

and the group of matrices $M_{(n-n_r)\times n_r}(k_F)$.

Since the group $K_I(m)$ is a normal subgroup of $P_I(1, m)$, we have an action of this group $P_I(1, m)$ on the set of characters of the abelian group $\eta$.

If $\eta$ is one such character we denote by $Z(\eta)$ the $P_I(1, m)$-stabilizer of this character $\eta$. Clifford theory now gives the decomposition

$$\text{ind}^{P_I(1,m)}_{P_I(1,m+1)}(\text{id}) \simeq \bigoplus_{\eta_k \in \eta} \text{ind}^{P_I(1,m)}_{Z(\eta_k)}(U_{\eta_k})$$

where $\{\eta_k\}$ is a set of representatives for the orbits under the action of $P_I(1, m)$ and $U_{\eta_k}$ is some irreducible representation of the group $Z(\eta_k)$. We also note that $Z(\text{id}) = P_I(1, m)$ and the identity character occurs with multiplicity one (which follows from Frobenius reciprocity) and hence

$$\text{ind}^{P_I(1,m)}_{P_I(1,m+1)}(\text{id}) \simeq \text{id} \oplus \bigoplus_{\eta_k \neq \text{id}} \text{ind}^{P_I(1,m)}_{Z(\eta_k)}(U_{\eta_k}). \quad (3.3)$$

Observe that

$$Z(\eta_k) = (Z(\eta_k) \cap M_{(n-n_r,n_r)}) K_I(m).$$

Since we have a $M_{(n-n_r,n_r)} \cap P_I(1, m)$ equivariant map between the group of characters of (3.2) and $M_{(n-n_r)\times n_r}(k_F)$, note that

$$Z(\eta_k) \cap M_{(n-n_r,n_r)} = Z_{M_{(n-n_r,n_r)} \cap P_I(1,m)}(A)$$

for some matrix $A$ in $M_{(n-n_r)\times n_r}(k_F)$. The group $M_{(n-n_r,n_r)} \cap P_I(1, m)$ acts on the group of matrices $M_{(n-n_r,n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction. The mod-$\mathfrak{P}_F$ reduction of the group $P_I(1, m) \cap M_{(n-n_r,n_r)}$ is equal to the group $P_I(k_F)\times \text{GL}_{n_r}(k_F)$. In the next lemma we will bound the mod $\mathfrak{P}_F$ reduction of the group $Z(\eta_k) \cap M_I$ for the proof of the main theorem. Let $O_A$ be an orbit for the action of $P_I(k_F)\times \text{GL}_{n_r}(k_F)$ on the set of matrices $M_{(n-n_r)\times n_r}(k_F)$.

Let $p_j$ be the $j^{th}$ projection of the group $M_I(k_F) = \prod_{i=1}^r \text{GL}_{n_i}(k_F)$.

**Lemma 3.0.15.** Let $O_A$ be an orbit consisting of non-zero matrices in $M_{(n-n_r)\times n_r}(k_F)$.

We can choose a representative $A$ such that the $P_I(k_F)\times \text{GL}_{n_r}(k_F)$-stabilizer of $A$,

$$Z_{P_I(k_F)\times \text{GL}_{n_r}(k_F)}(A)$$

satisfies one of the following conditions.
1. There exists a positive integer $j$, $j \leq r$ such that the image of $p_j : Z_{P^r(k_F) \times \text{GL}_{n_r}(k_F)}(A) \cap M_I(k_F) \to \text{GL}_{n_j}(k_F)$

is contained in a proper parabolic subgroup of $\text{GL}_{n_j}(k_F)$.

2. There exists an $i$ with $1 \leq i \leq r - 1$ such that $p_i(g) = p_r(g)$ for all $g \in Z_{P^r(k_F) \times \text{GL}_{n_r}(k_F)}(A) \cap M_I(k_F)$.

Proof. Let $A = [U_1, U_2, \ldots, U_{(r-1)}]^t$ be the block form ($U_k$ is a matrix of size $n_r \times n_k$ for $1 \leq k \leq r - 1$) of a representative $m$ for an orbit $O_m$ consisting of non-zero matrices. If $((A_{ij}), B) \in Z_{P^r(k_F) \times \text{GL}_{n_r}(k_F)}(A)$ then we have

$$(A_{ij})[U_1, U_2, \ldots, U_{(r-1)}]^t = [U_1, U_2, \ldots, U_{(r-1)}]^t B.$$  \hspace{1cm} (3.4)

Since $(A_{ij}) \in P^r(k_F)$, we have $A_{ij} = 0$ for all $i > j$. Let $l' \leq r - 1$ be the least non-negative integer such that $U_{r-1-l'}$ (matrix of size $n_i \times n_j$) is non-zero and such an $l'$ exists since $m \neq 0$. From (3.4) we get that $A_{ll'}U_l^t = U_{l'}^t B$ where $l = r - 1 - l'$. There exist matrices $P \in \text{GL}_{n_r}(k_F)$ and $Q \in \text{GL}_{n_l}(k_F)$ such that $PU_l^tQ$ is a matrix of the form

$$
\begin{pmatrix}
1_t & 0 \\
0 & 0
\end{pmatrix}
\hspace{1cm} (3.5)
$$

where $t$ is the rank of the matrix $U_l^t$. Now we may change the representative $A$ to $A' = [U_1', U_2', \ldots, U_r]^t$ by the action of the element $\text{diag}(1_{n_1}, \ldots, P, \ldots, 1_{n_{r-1}}, Q^{-1})$ in $P^r(k_F) \times \text{GL}_{n_r}(k_F)$ such that $U_l'^t$ is the matrix (3.5). If $t = n_l = n_r$ then condition (2) is satisfied. Consider the maps $T_1 : k_{F}^{n_l} \to k_{F}^{n_r}$ and $T_2 : k_{F}^{n_r} \to k_{F}^{n_l}$ given by

$$(a_1, a_2, \ldots, a_{n_l}) \mapsto (a_1, a_2, \ldots, a_{n_l})U_l^t$$

and

$$(a_1, a_2, \ldots, a_{n_r}) \mapsto U_l^t(a_1, a_2, \ldots, a_{n_r})^t$$

respectively. If $t = n_l = n_r$ does not hold then either of $T_1$ or $T_2$ has a non-trivial proper kernel (since $U_l \neq 0$). If $T_1$ has a non-trivial proper kernel then $A_{ll}$ preserves this kernel and hence belongs to a proper parabolic subgroup of $\text{GL}_{n_r}(k_F)$. If $T_2$ has a non-trivial proper kernel then $B$ preserves this kernel and hence belongs to a proper parabolic subgroup of $\text{GL}_{n_l}(k_F)$. Hence if $t = n_l = n_r$ does not hold true then condition (1) is satisfied.

The following lemma is due to Paskunas but we give a mild modification for our applications (see [Pas05 Proposition 6.8]).
Lemma 3.0.16. Let $m > 1$, $\sigma$ be any irreducible representation of the group $\GL_m(\mathbb{F}_q)$ and $H$ be a subgroup contained in a proper parabolic subgroup of $\GL_m(\mathbb{F}_q)$. For every irreducible representation $\gamma$ of $H$ contained in $\res_H(\sigma)$, there exists an irreducible non-cuspidal representation $\sigma'$ of $\GL_m(\mathbb{F}_q)$ such that $\Hom(\gamma, \sigma') \neq 0$.

Proof. Let $P$ be a proper parabolic subgroup of $\GL_m(\mathbb{F}_q)$ containing $H$ and $U$ be the unipotent radical of an opposite parabolic subgroup of $P$. We observe that $\bar{U} \cap H = \text{id}$. Now if the lemma is false, we have $\ind_{H}^{\GL_m(\mathbb{F}_q)}(\gamma) \simeq \bigoplus_{k \in \Lambda} \sigma_k$ such that $\sigma_k$ is a cuspidal representation. Using Mackey decomposition we get that

$$\Hom(\text{id}, \ind_{H}^{\GL_m(\mathbb{F}_q)}(\gamma)) = \bigoplus_{u \in U \backslash \GL_m(\mathbb{F}_q)/H} \Hom_{U \cap H^u}(\text{id}, \gamma^u).$$

If $\bar{U} \cap H = \text{id}$ then $\Hom_{U \cap H}(\text{id}, \gamma)$ is non-zero and by the above decomposition

$$\Hom(\text{id}, \ind_{H}^{\GL_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$ 

This shows that $\Hom(\text{id}, \sigma_k) \neq 0$ for some $k \in \Lambda$ and this is a contradiction to our assumption.

Lemma 3.0.17. Let $m \geq 2$, $H$ be the diagonal subgroup of $\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)$ and $\sigma_1 \boxtimes \sigma_2$ be an irreducible representation of $\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)$. For every irreducible representation $\gamma$ occurring in $\res_H(\sigma_1 \boxtimes \sigma_2)$ there exists an irreducible non-cuspidal representation $\sigma'_1 \boxtimes \sigma'_2$ of $\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)$ containing $\gamma$.

Proof. Let $\bar{U}$ and $U$ be the subgroups of lower unipotent and upper unipotent matrices of $\GL_m(\mathbb{F}_q)$. Consider the unipotent subgroup $V := \bar{U} \times U$ of $\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)$. Suppose the lemma is false then

$$\ind_{H}^{\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)}(\gamma) \simeq \bigoplus_{k \in \Lambda} \sigma_k^1 \boxtimes \sigma_k^2$$

such that $\sigma_k^1$ and $\sigma_k^2$ are cuspidal representations for all $k \in \Lambda$. We observe that $V \cap H = \text{id}$ and by Mackey decomposition we have

$$\Hom_V(\text{id}, \ind_{H}^{\GL_m(\mathbb{F}_q) \times \GL_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$ 

Now by our assumption we have $\Hom_V(\text{id}, \sigma_k^1 \boxtimes \sigma_k^2) \neq 0$ for some $k \in \Lambda$ and hence a contradiction.

The following lemma is similar to proposition 2.2.3. The lemma is just a modified version of the proposition 2.2.3 for our present use.

Lemma 3.0.18. Let $\Gamma$ be a $\GL_n-n_\tau(\mathcal{O}_F)$-irreducible sub-representation of $\ind_{\mathcal{P}_{(n-n_\tau, n_\tau)}}^{\GL_n(\mathcal{O}_F)} \{U_m(\tau_{\Gamma'}) \boxtimes \tau_{\tau'}\}$. 

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If the irreducible sub-representations of \( U_m(\tau_I) \) are atypical for the component \( s = [M_I', \sigma_I'] \), then the representation \( \Gamma \) is atypical for the component \( s = [M_I, \sigma_I] \).

**Proof.** Let \( \rho \) be an irreducible sub-representation of \( U_m(\tau_I) \). If \( \rho \) is not typical then, there exists another Bernstein component \([M_J, \lambda_J]\) of \( \text{GL}_{n-r}(F) \) such that

\[
[M_I', \sigma_I'] \neq [M_J, \lambda_J]
\]

and \( \rho \) is contained in

\[
\text{res}_{\text{GL}_{n-r}(\mathcal{O}_F) / \text{GL}_{n-r}(F)} \left( \text{ind}_{\mathcal{P}_{n-r}(m)} \{ \rho \otimes \tau_I \} \right)
\]

where \( J = (n_1', n_2', \ldots, n_{r'-1}') \) and \( \lambda_J = \otimes_{i=1}^{r'-1} \lambda_i \). The representation

\[
\text{ind}_{\text{GL}_{n}(\mathcal{O}_F) / \text{GL}_{n-r}(\mathcal{O}_F)} \{ \rho \otimes \tau_I \}
\]

is contained in

\[
\text{ind}_{\text{GL}_{n}(\mathcal{O}_F) / \text{GL}_{n-r}(\mathcal{O}_F)} \{ \rho \otimes \tau_I \}.
\]

The representation \([3.6]\) is contained in the representation

\[
\text{res}_{\text{GL}_{n}(\mathcal{O}_F) / \text{GL}_{n-r}(F)} \left( \text{ind}_{\mathcal{P}_{n-r}(m)} \{ \rho \otimes \tau_I \} \otimes \sigma_I \right).
\]

Since \([M_I', \sigma_I'] \neq [M_J, \lambda_J]\) there exist an inertial class \([\text{GL}_{p}(F), \sigma]\) occurring in the multi-set

\[
\{[\text{GL}_{n_1}(F), \sigma_1], [\text{GL}_{n_2}(F), \sigma_2], \ldots, [\text{GL}_{n_{r-1}}(F), \sigma_{r-1}]\}
\]

with a multiplicity not equal to its multiplicity in the multi-set

\[
\{[\text{GL}_{n_1'}(F), \lambda_1], [\text{GL}_{n_2'}(F), \lambda_2], \ldots, [\text{GL}_{n_{r'-1}'}(F), \lambda_{r'-1}]\}.
\]

Hence the classes \([M_I, \sigma_I]\) and \([M_J \times \text{GL}_{n_r}(F), \lambda_J \otimes \sigma_r]\) represent two distinct Bernstein components for the group \( \text{GL}_n(F) \). \( \Box \)

### 3.1 Proof of the main theorem

**Proof of theorem 3.0.9.** We prove the theorem by using induction on the positive integer \( n \), the rank of \( \text{GL}_n(F) \). The theorem is true for \( n = 1 \) since \( U_m(\tau_I) \) is zero. We assume that the theorem is true for all positive integers less than \( n + 1 \). We will show the theorem for the positive integer \( n + 1 \). Let \( s = [M_I, \sigma_I] \) be a level-zero inertial class. We assume that the partition \( I = (n_1, n_2, \ldots, n_r) \) of \( n + 1 \) satisfies the hypothesis \( n_i \leq n_j \) for all \( 1 \leq i \leq j \leq r \). If \( r = 1 \) we have \( U_m(\tau_I) = 0 \) and the theorem holds by default. We now assume that \( r > 1 \) and let \( I' = (n_1, n_2, \ldots, n_{r-1}) \).

We now break the proof into two cases. The first case is \( n_r = 1 \) and the second case is \( n_r > 1 \). 53
3.1.1 The case where \( n_r = 1 \)

In this case \( n_i = 1 \) for \( 1 \leq i \leq r \) and \( P_1 = B_n \) where \( B_n \) is the Borel subgroup of \( \text{GL}_n \). We denote by \( T_n \) and \( U_n \) the maximal torus and the unipotent radical respectively. We also use the notation \( B_n(m) \) for the subgroup \( P_1(m) \) and \( \chi_{I_n} \) for \( \tau_I \) since \( I = (1, 1, \ldots, 1) \) is a tuple of length \( n \). The proof is by induction on the integer \( n \), the rank of \( T_n \). The statement is immediate for \( n = 1 \) and for \( n = 2 \) we refer to [BM02, A.2.4] for a proof (We will require the proof for later use and we will recall it at that stage). So we prove the theorem for \( n \geq 3 \). Suppose the theorem is true for some positive integer \( n \geq 2 \). The rest of this subsection is to prove the main theorem for \( n + 1 \). By definition of \( U_m(\chi_{I_{n+1}}) \) we have

\[
\text{ind}_{B_{n+1}(m)}^{\text{GL}_{n+1}(O_F)}(\chi_{I_{n+1}}) \simeq U_m(\chi_{I_{n+1}}) \oplus \text{ind}_{B_{n+1}(1)}^{\text{GL}_{n+1}(O_F)}(\chi_{I_{n+1}}).
\]

We have the isomorphism

\[
\text{ind}_{B_{n+1}(m)}^{\text{GL}_{n+1}(O_F)}(\chi_{I_{n+1}}) \simeq \text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{\text{ind}_{B_{n}(m)}^{\text{GL}_{n}(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}.
\]

We also have the decomposition

\[
\text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{\text{ind}_{B_{n}(m)}^{\text{GL}_{n}(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \simeq \text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_{n+1}\} \oplus \text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{\text{ind}_{B_{n}(1)}^{\text{GL}_{n}(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}.
\]

By induction hypothesis and lemma [3.0.18] irreducible sub-representations of

\[
\text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_{n+1}\}
\]

are atypical representations. We now consider the irreducible factors of the representation

\[
\text{ind}_{P_{(n,1)}(m)}^{\text{GL}_{n+1}(O_F)}\{\text{ind}_{B_{n}(1)}^{\text{GL}_{n}(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}.
\]

(3.7)

We use induction on the integer \( m \) to show that the representation

\[
\text{ind}_{P_{(n,1)}(1)}^{\text{GL}_{n+1}(O_F)}\{\text{ind}_{B_{n}(1)}^{\text{GL}_{n}(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}
\]

\[
\simeq \text{ind}_{B_{n+1}(1)}^{\text{GL}_{n+1}(O_F)}(\chi_{I_{n+1}})
\]

has a complement say \( U_{1,m}(\chi_{I_{n+1}}) \) in the representation \([3.7]\) whose irreducible sub-representations are all atypical representations. This shows that irreducible sub-representations of \( U_m(\chi_{I_{n+1}}) \) are atypical. To reduce the notations we denote by \( P(m) \) the subgroup \( P_{(n,1)}(m) \). Applying the decomposition \([3.3]\) to the parabolic subgroup \( P_{(n,1)} \) we get that

\[
\text{ind}_{P_{(m,1)}(1)}^{P(m)}(\text{id}) = \text{id} \oplus \text{ind}_{Z(\eta)}^{P(m)}(U_\eta)
\]

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where \( \eta \) (in the present situation we just have one orbit consisting of non-trivial characters) is any non-trivial character of the group \( K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \) which is trivial on \( K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \cap P(m+1) \) and \( K_{n+1}(m) \) is the principal congruence subgroup of level \( m \). We have shown a \( M \cap P(m) \) equivariant map between the group of characters of

\[
\frac{K_{n+1}(m)U_{n,1}(\mathcal{O}_F)}{K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \cap P(m+1)}
\]

and \( M_{n \times 1}(k_F) \). We choose \( \eta \) to be the character corresponding to the matrix \([1,0 \ldots,0]\).

For the above choice of a non-trivial character we have

\[
\text{ind}_{P(m+1)}^{GL_{n+1}(\mathcal{O}_F)} \{ \text{ind}_{B_n(1)}^{GL_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \} \\
\cong \text{ind}_{P(m)}^{GL_{n+1}(\mathcal{O}_F)} \{ \text{ind}_{B_n(1)}^{GL_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \} \\
\oplus \text{ind}_{Z(\eta)}^{GL_{n+1}(\mathcal{O}_F)} \{ U_{\eta} \otimes \text{res}_{Z(\eta) \cap M_{n,1}} \{ \text{ind}_{B_n(1)}^{GL_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \} \}.
\]

Since the representation \( \text{ind}_{B_n(1)}^{GL_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \) is a level-zero representation,

\[
\text{res}_{Z(\eta) \cap M_{n,1}} \{ \text{ind}_{B_n(1)}^{GL_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \}
\]

is isomorphic to the inflation of the representation

\[
\text{res}_{\overline{Z(\eta) \cap M_{n,1}}} \{ \text{ind}_{B_n(k_F)}^{GL_n(k_F)}(\chi_{I_n}) \boxtimes \chi_{n+1} \}
\]

where \( \overline{Z(\eta) \cap M_{n,1}} \) is the mod-\( \mathfrak{q}_F \) reduction of the group \( Z(\eta) \cap M_{n,1} \). The group \( \overline{Z(\eta) \cap M_{n,1}} \) is contained in the following subgroup

\[
\left\{ \begin{pmatrix} A & B & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \bigg| A \in \text{GL}_{n-1}(k_F), B \in M_{(n-1) \times 1}(k_F) \text{ and } d \in k_F^\times \right\}. \quad (3.8)
\]

Let \( \text{Mir}_k \) be the following group

\[
\left\{ \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} \bigg| A \in \text{GL}_{k-1}(k_F), B \in M_{(k-1) \times 1}(k_F) \right\}.
\]

Now we have to understand the restriction

\[
\text{res}_{P_{(n-1,1)}} \text{ind}_{B_n(k_F)}^{GL_n(k_F)}(\chi_{I_n})
\]

which is reduced to understanding the restriction

\[
\text{res}_{\text{Mir}_{n-1}} \text{ind}_{B_n(k_F)}^{GL_n(k_F)}(\chi_{I_n}).
\]
We use the theory of derivatives (originally for \( \text{GL}_n(F) \) due to Bernstein and Zelevinsky (see \cite{BZ76}) to describe this restriction in a way sufficient for our application. We refer to \cite{Zel81} Chapter 3, §13 for details of these constructions.

In the case of finite fields from Clifford theory one can define four exact functors and we recall the formalism here. The precise definitions are not required for our purpose except for one functor \( \Psi^+ \) which will be recalled latter:

\[
\begin{array}{ccc}
\mathcal{M}(\text{Mir}_{k-1}) & \xrightarrow{\Phi^+} & \mathcal{M}(\text{Mir}_k) \\
\Phi^- & \xlongleftarrow{\psi^-} & \mathcal{M}(\text{GL}_{k-1}(k_F)) \\
\Psi^+ & \xlongleftarrow{\psi^+} & \end{array}
\]

The key results we use from Zelevinsky are summarised below (see \cite{Zel81} Chapter 3, §13).

**Theorem 3.1.1** (Zelevinsky). The functors \( \Psi^+ \) and \( \Phi^- \) are left adjoint to \( \Psi^- \) and \( \Phi^+ \) respectively. The compositions \( \Phi^- \Phi^+ \) and \( \Psi^- \Psi^+ \) are naturally equivalent to identity. Moreover \( \Phi^+ \Psi^- \) and \( \Phi^- \Psi^+ \) are zero. The diagram

\[
0 \rightarrow \Phi^+ \Phi^- \rightarrow \text{id} \rightarrow \Psi^+ \Psi^- \rightarrow 0
\]

obtained from these properties is exact.

Using this theorem and following Bernstein-Zelevinsky one can define a filtration \( \text{Fil} \) on a finite dimensional representation \( \tau \) of \( \text{Mir}_n \), for all \( n > 1 \). The filtration \( \text{Fil} \) is given by

\[
0 \subset \tau_n \subset \cdots \subset \tau_3 \subset \tau_2 \subset \tau_1 = \tau
\]

where \( \tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1} \) and \( \tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+\Psi^-(\Phi^-)^{k-1}(\tau) \) for all \( k \geq 1 \). The representation \( \tau^{(k)} := \Psi^-(\Phi^-)^{k-1}(\tau) \) for all \( k \geq 0 \) of \( \text{GL}_{n-k}(k_F) \) is called the \( k \text{th} \)-derivative of \( \tau \) and by convention \( \tau^{(0)} := \tau \).

Let \( R_n \) be the Grothendieck group of \( \text{GL}_n(k_F) \) for all \( n \geq 1 \) and set \( R_0 = \mathbb{Z} \). Zelevinsky defined a ring structure on the group \( R = \oplus_{n \geq 0} R_n \) by setting parabolic induction as the product rule. Recall that the ring \( R \) has a \( \mathbb{Z} \)-linear map \( D \) defined by setting \( D(\pi) = \sum_{k \geq 0} (\pi|_{\text{Mir}_n})^{(k)} \) for all \( \pi \) in \( R_n \). It follows from \cite{Zel81} Chapter 3, §13 that

\[
D(\text{ind}_{F}^{\text{GL}_n(k_F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)) = \prod_{i=1}^{r} D(\tau_i)
\]

where the product on the right hand side is in the ring \( R \). The map \( D \) is hence an endomorphism of the ring \( R \). If \( \pi \) is a supercuspidal representation of \( \text{GL}_n(k_F) \) then by Gelfand-Kazhdan theory it follows that \( \pi^{(n)} = 1, \pi^{(0)} = \pi \).
and all other derivatives are zero (see [Zel81, Chapter 3, §13]). Let $1_R \in R_0$ be the identity element of $R$.

In our present situation we have

$$D(\text{ind}_{B_n(1)}^{GL_n(O_F)}(\chi I_n)) = \prod_{i=1}^{n} D(\chi_i) = \prod_{i=1}^{n} (\chi_i + 1_R).$$

Let $X_{n-k}$ be the term of degree $(n - k)$ in the expansion of the above product (it is a representation of $GL_{n-k}(k_F)$ in the Grothendieck group $R_{n-k}$. Since the coefficients of the above expansion are positive $X_{n-k}$ is actually a representation and not just a virtual representation.) Then we have

$$\text{res}_{\text{Mir}_{n-1}} \text{ind}_{B_n(k_F)}^{GL_n(k_F)}(\chi I_n) \simeq \bigoplus_{k \geq 1} (\Phi^+)^{k-1} \Psi^+(X_{n-k}).$$

Observe that $P_{(n-1,1)} = \text{Mir}_{(n-1)} k_F^\times$ (here $k_F^\times$ is the centre of $GL_n(k_F)$) and $\text{Mir}_{(n-1)} \cap k_F^\times = \text{id}$. The representation

$$\rho := (\Phi^+)^{k-1} \Psi^+(X_{n-k})$$

extends to a representation of $P_{(n-1,1)}$ by setting $\rho(a) = \chi(a)$ for all $a \in k_F^\times$ where $\chi$ is the central character of the representation

$$\text{ind}_{B_n(k_F)}^{GL_n(k_F)}(\bigotimes_{i=1}^{n} \chi_i).$$

Since the central character will play some role, we denote the extended representation by

$$\text{ext}\{(\Phi^+)^{k-1} \Psi^+(X_{n-k})\}.$$  

By inflation we extend the $P_{(n,1)}(k_F) \times k_F^\times$-representation

$$\text{ext}\{(\Phi^+)^{k-1} \Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

to a representation of $Z(\eta) \cap M_{(n,1)}$. We continue to use the notation

$$\text{ext}\{(\Phi^+)^{k-1} \Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

to the extended representation. We now have

$$\text{ind}_{P_{(m+1)}}^{GL_{n+1}(O_F)}(\chi I_n) \simeq \text{ind}_{P_{(m)}}^{GL_{n+1}(O_F)}(\chi I_n) \oplus \bigoplus_{k \geq 1} \text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)}(\text{ext}\{(\Phi^+)^{k-1} \Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}).$$

We will show that any irreducible sub-representation of

$$\text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)}(\text{ext}\{(\Phi^+)^{k-1} \Psi^+(X_{n-k})\} \boxtimes \chi_{n+1})$$

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is atypical for the component $[T_n, \chi_{I_n}]$.

We first consider the case when $k \geq 2$. The representation $X_{n-k}$ is a direct sum of the representations:

$$\text{ind}_{B_{n-k}(k_F)}^{GL_{n-k}(k_F)} (\chi_{i_1} \boxtimes \chi_{i_2} \boxtimes \cdots \boxtimes \chi_{i_{n-k}}).$$

The above term also occurs in the expansion

$$\prod_{j=1}^{n-k} (1_R + \chi_{i_j})(1_R + \lambda)$$

where $\lambda$ is a cuspidal representation of $GL_k(k_F)$. To shorten the notation we use the symbol $\times$ for the multiplication in the ring $R$. We get that the representation

$$(\Phi^+)^{k-1}\Psi^+ (\chi_{j=1}^{n-k} \chi_{n_j})$$

occurs in the representation

$$\text{res}_{M_{n-1}} (\chi_{j=1}^{n-k} \chi_{n_j} \times \lambda).$$

Since the mod-$\Psi_F$ reduction of the group $Z(\eta) \cap M_{(n,1)}$ is contained in the subgroup of the form (3.8), even if the central characters of $\chi_{j=1}^{n-k} \chi_{j}$ and $\chi_{j=1}^{n-k} \chi_{j} \times \lambda$ are different we may change $\chi_{n+1}$ to $\chi'_{n+1}$ such that the representation

$$\text{res}_{Z(\eta) \cap M_{(n,1)}} \{\text{ext}\{(\Phi^+)^{k-1}\Psi^+ (\chi_{j=1}^{n-k} \chi_{n_j})\} \boxtimes \chi_{n+1}$$

occurs in the representation

$$\text{res}_{Z(\eta) \cap M_{(n,1)}} (\chi_{j=1}^{n-k} \chi_{j} \times \lambda) \boxtimes \chi'_{n+1}.$$

Hence an irreducible sub-representation of

$$\text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)} \{(\text{ext}\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}) \otimes U_{\eta}\}$$

occurs as a sub-representation of

$$\text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)} \{\{\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1}\} \otimes U_{\eta}\}. \quad (3.9)$$

The above representation occurs as a sub-representation of

$$\text{ind}_{P_{1,1,\ldots,1,k,1}}^{GL_{n+1}(O_F)} \{\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1}\}. \quad (3.10)$$

Hence the sub-representation of (3.9) are not typical representations.

Now we are left with the term

$$\text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)} \{(\text{ext}\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_{\eta}\}. \quad (3.11)$$
We might as well repeat the same strategy as for \( k \geq 2 \) and now \( \lambda \) is one dimensional but the representations \( (3.11) \) and \( \chi_{j=1}^{n+1} \lambda_j \) may not have distinct inertial support. In order to tackle the terms of the above representation we use a different technique. We now recall the definition of the representation \( U_{\eta} \), the functor \( \Psi^+ \) and some facts due to Casselman regarding the restriction of an irreducible smooth representation to the maximal compact subgroup \( \text{GL}_2(\mathcal{O}_F) \).

The representation \( U_{\eta} \) is a character on the group \( Z(\eta) \). From (3.8) any element of the group \( Z(\eta) \) is of the form

\[
\begin{pmatrix}
A & B & X^t \\
\varpi_F C & d & y \\
\varpi_F^m X & \varpi_F^m y' & e
\end{pmatrix}
\] (3.13)

where \( A \in \text{GL}_{n-1}(\mathcal{O}_F) \); \( (X')^t, X^t, B, C^t \in M_{(n-1) \times 1}(\mathcal{O}_F) \); \( e, d \in \mathcal{O}_F^\times \); \( y, y' \in \mathcal{O}_F \) and \( d \equiv e(\mathfrak{P}_F) \). The character \( U_{\eta} \) is given by

\[
\begin{pmatrix}
A & B & X^t \\
\varpi_F C & d & y \\
\varpi_F^m X & \varpi_F^m y' & e
\end{pmatrix} \mapsto \eta(\varpi_F^m y).
\]

The functor

\[
\Psi^+: \mathcal{M}(\text{GL}_{n-1}(k_F)) \to \mathcal{M}(\text{Mir}_k)
\]

is the inflation functor via the quotient map of \( \text{Mir}_k \) modulo the unipotent radical of \( \text{Mir}_k \).

Let \((\pi, V_\pi)\) be an irreducible smooth representation of \( \text{GL}_2(F) \). We denote by \( c(\pi) \) and \( \varpi_\pi \) the conductor and central character of the representation \( \pi \) respectively. Let \( V^N \) be the space of all vectors fixed by the principal congruence subgroup of level \( N \) for all \( N \geq 1 \). For all \( i > c(\varpi_\pi) \) we define the representation \( U_i(\chi) \) as the complement of the representation \( \text{ind}_{\text{GL}_2(\mathcal{O}_F)}^{\text{B}_2(i-1)}(\chi) \) in \( \text{ind}_{\text{B}_2(i)}^{\text{GL}_2(\mathcal{O}_F)}(\chi) \). For \( i = c(\varpi_\pi) \) we set

\[
U_i(\varpi_\pi) = \text{ind}_{\text{B}_2(i)}^{\text{GL}_2(\mathcal{O}_F)}(\varpi_\pi \boxtimes \text{id}).
\]

It follows from [Cas73] Proposition 1] that the representation \( U_i(\varpi_\pi) \) is an irreducible representation of \( \text{GL}_2(\mathcal{O}_F) \). From the result [Cas73] Proposition 2| we get that \( c(\pi) \geq c(\varpi_\pi) \). By [Cas73] Theorem 1| we have

\[
\text{res}_{\text{GL}_2(\mathcal{O}_F)} V_\pi = V^{(c(\pi) - 1)} \oplus \bigoplus_{i \geq c(\pi)} U_i(\varpi_\pi). \quad (3.14)
\]

We now describe the representation \( U_i(\varpi_\pi) \) in our language. Let \( \kappa \) be a non-trivial character of the group \( K_2(m)U_{(1,1)}(\mathcal{O}_F) \) trivial modulo
$K_2(m)U_{(1,1)}(O_F) \cap B_2(m+1)$. Let $Z(\kappa)$ be a $B_2(m)$ stabilizer of $\kappa$. Any element of the group $Z(\kappa)$ is of the form (for an appropriate choice of a non-trivial character $\kappa$)
\[
\begin{pmatrix}
a & b \\
\varpi^m_F c & d
\end{pmatrix}
\]
where $a, d \in O_F^\times$; $b \in O_F$, $c \in \Psi_F^m$ and $d \equiv a$ modulo $\Psi_F$. We define a character $U_\eta$ by setting
\[
\begin{pmatrix} a & b \\
c & d \end{pmatrix} \mapsto \eta(c).
\]
We then have
\[
U_m(\varpi) \simeq \text{ind}_{Z(\eta)}^{P(n-1,2)}(X_{n-1} \boxtimes U_m(\chi)).
\]
Now let us resume the proof in the general case $n > 2$ the representation
\[
\text{ind}_{Z(\eta)}^{GL(n+1)(O_F)}\langle\{\text{ext}\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}\} \otimes U_\eta\rangle
\]
is contained in the representation
\[
\text{ind}_{P(n-1,2)\cap GL(n+1)(O_F)}^{GL(n+1)(O_F)}(X_{n-1}^1 \boxtimes \sigma)
\]
where $\sigma$ is a supercuspidal representation of level-zero with central character $\chi$ (see the remark below for the existence) and $X_{n-1}'$ is the $(n-1)$ derivative of the representation
\[
i_{B_n}^{GL_n(F)}(\chi_{I_n}).
\]
Hence irreducible sub-representations of (3.12) are atypical. This completes the proof that irreducible sub-representations of
\[
\text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)}\langle\{U_\eta \otimes \text{res}_{Z(\eta) \cap M(n,1)}\{\text{ind}_{B_n(1)}^{GL_n(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\} \otimes \chi_{n+1}\rangle
\]
are atypical. From the decomposition
\[
\begin{align*}
\text{ind}_{P(m+1)}^{GL_{n+1}(O_F)}\langle\{\text{ind}_{B_n(1)}^{GL_n(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\
\simeq \text{ind}_{P(m)}^{GL_{n+1}(O_F)}\langle\{\text{ind}_{B_n(1)}^{GL_n(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\
\oplus \text{ind}_{Z(\eta)}^{GL_{n+1}(O_F)}\langle\{U_\eta \otimes \text{res}_{Z(\eta) \cap M(n,1)}\{\text{ind}_{B_n(1)}^{GL_n(O_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \otimes \chi_{n+1}\} \rangle
\end{align*}
\]
we get the theorem for the case where $n_r = 1$. 

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Remark 3.1.2. The existence of the cuspidal representation of $GL_2(k_F)$ with a given central character can be deduced from the explicit formula for such representations, we refer to [BH06, Theorem section 6.4]. To be precise we begin with a quadratic extension $k$ of $k_F$ and $\theta$ a character of $k^\times$ such that $\theta^q \neq \theta$ where $q = \# k_F$. These characters are called regular characters and for any regular character one can define a supercuspidal representation $\pi_\theta$ and conversely all supercuspidal representations are of the form $\pi_\theta$ for some regular character $\theta$. The central character of $\pi_\theta$ is given by $\text{res}_{k_F}(\theta)$. Now to get a supercuspidal representation with a central character $\chi$ we begin with a character $\chi$ on $k_F^\times$, there are $\# k_F + 1$ possible extensions to $k^\times$. The set of characters $\theta$ such that $\theta^q = \theta$ has cardinality $\# k_F - 1$. Hence there exists at least one supercuspidal representation with a given central character $\chi$. This shows that irreducible sub-representations of (3.15) are not typical and this completes the proof of the theorem in this case.

3.1.2 The case where $n_r > 1$

By transitivity of induction we have

$$\text{ind}_{P_I(1),m}(\tau_I) \simeq \text{ind}_{P_I(1),m}(\text{ind}_{P_I(1),m}(\tau_I)).$$

We note that $P_I(1,m) \cap U_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \tilde{U}_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap \tilde{U}_{(n-n_r+1,n_r)}$ hence lemma 2.2.6 gives the isomorphism

$$\text{ind}_{P_I(1),m}(\text{ind}_{P_I(1),m}(\tau_I)) \simeq \text{ind}_{P_I(1),m}(\text{ind}_{P_I(1),m}(\tau_I)).$$

Splitting the representation $\text{ind}_{P_I(1),m}(\tau_I)$ as $\tau_I \oplus U^0_m(\tau_I)$ we get that

$$\text{ind}_{P_I(1),m}(\text{ind}_{P_I(1),m}(\tau_I)) \simeq \text{ind}_{P_I(1),m}(U^0_m(\tau_I) \boxtimes_\tau_I) \oplus \text{ind}_{P_I(1),m}(\tau_I).$$

From Frobenius reciprocity the representation $\tau_I$ occurs in $\text{ind}_{P_I(1),m}(\tau_I)$ with multiplicity one. Let $U^0_{m}(\tau_I)$ be the complement of $\tau_I$ in $\text{ind}_{P_I(1),m}(\tau_I)$. With this we conclude that

$$\text{ind}_{P_I(1),m}(\tau_I) \simeq \text{ind}_{P_I(1),m}(U^0_m(\tau_I) \boxtimes_\tau_I) \oplus U^0_{m}(\tau_I) \oplus \tau_I.$$

By definition $U_m(\tau_I) = \text{ind}_{P_I(1)}^{GL_n(O_F)}(U^0_m(\tau_I))$ which shows that

$$U_m(\tau_I) \simeq \text{ind}_{P_I(1),m}^{GL_n+1(O_F)}(U^0_m(\tau_I) \boxtimes_\tau_I) \oplus \text{ind}_{P_I(1)}^{GL_n+1(O_F)}(U^0_{m}(\tau_I)).$$

We observe that $P_I(1,m) \cap U_{(n-n_r+1,n_r)} = P_{I(n-n_r+1,n_r)}(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \tilde{U}_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap \tilde{U}_{(n-n_r+1,n_r)}$ hence lemma
Proof. We observe that irreducible sub-representations of equation (3.16) reduce the proof of the theorem to showing that irreducible sub-representations of \( \lambda \) gives us the isomorphism

\[
\text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1,m)} (U_m^0(\tau_I) \boxtimes \tau_r) \simeq \text{ind}^{\text{GL}_{n+1}(O_F)}_{P_{n-n_r+1,n_r}(m)} (U_m(\tau_I) \boxtimes \tau_r).
\]

With this we are in a place to use the induction hypothesis through the isomorphism

\[
U_m(\tau_I) \simeq \text{ind}^{\text{GL}_{n+1}(O_F)}_{P_{n-n_r+1,n_r}(m)} (U_m(\tau_I) \boxtimes \tau_r) \oplus \text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1)} (U_{0}^0(1,m)(\tau_I)). \tag{3.16}
\]

By induction hypothesis \( \text{GL}_{n-n_r+1}(O_F) \)-irreducible sub-representations of \( U_m(\tau_I) \) are atypical for the component \([M_I', \sigma_I']\). Now lemma 3.0.18 and the equation (3.16) reduce the proof of the theorem to showing that irreducible sub-representations of \( \text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1,m)} (U_{0}^0(1,m)(\tau_I)) \) are atypical representations.

**Proposition 3.1.3.** The irreducible sub-representations of

\[
\text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1)} (U_{0}^0(1,m)(\tau_I))
\]

are atypical for \( m \geq 1 \).

**Proof.** We observe that

\[
\text{ind}^{P_l(1)}_{P_l(1,m+1)} (\tau_I) \simeq \text{ind}^{P_l(1)}_{P_l(1,m)} \{ \text{ind}^{P_l(1,m+1)}_{P_l(1,m)} (\tau_I) \}
\]

and the decomposition (3.3) gives us the isomorphism

\[
\text{ind}^{P_l(1)}_{P_l(1,m+1)} (\tau_I) = \text{ind}^{P_l(1)}_{P_l(1,m)} (\tau_I) \oplus \bigoplus_{\eta_k \neq \text{id}} \text{ind}^{P_l(1)}_{P_l(1,m)} \{ \text{ind}^{Z(\eta_k)}_{P_l(1,m)} (U_{\eta_k}) \otimes \tau_I \}
\]

which gives the equality

\[
U_{0}^0(1,m+1)(\tau_I) = U_{0}^0(1,m)(\tau_I) \oplus \bigoplus_{\eta_k \neq \text{id}} \text{ind}^{P_l(1)}_{P_l(1,m)} \{ \text{ind}^{Z(\eta_k)}_{P_l(1,m)} (U_{\eta_k}) \otimes \tau_I \}.
\]

If we show that the irreducible sub-representations of

\[
\text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1,m)} \{ \text{ind}^{Z(\eta_k)}_{P_l(1,m)} (U_{\eta_k}) \otimes \tau_I \}
\]

(for \( \eta_k \neq \text{id} \)) are atypical for \([M_I, \sigma_I]\) then induction on the positive integer \( m \) completes the proof of the proposition in this case. To begin with we note that

\[
\text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1,m)} \{ \text{ind}^{Z(\eta_k)}_{P_l(1,m)} (U_{\eta_k}) \otimes \tau_I \}
\]

\[
\text{ind}^{\text{GL}_{n+1}(O_F)}_{P_l(1,m)} \{ \text{ind}^{Z(\eta_k)}_{P_l(1,m)} (U_{\eta_k} \otimes \text{res} Z(\eta_k) \cap M_I \tau_I) \}.
\]

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The representation $\tau_I$ is a level zero representation. Hence $\text{res}_{Z(\eta_k) \cap M_I} \tau_I$ is isomorphic to the inflation of the representation $\text{res}_{Z(\eta_k) \cap M_I} \tau_I$ where $Z(\eta_k) \cap M_I$ is mod-$\Psi_F$ reduction of $Z(\eta_k) \cap M_I$. Let $A = \theta_I(\eta_k)$ where $\theta_I$ is the map defined in the paragraph just after lemma 3.0.14. The mod-$\Psi_F$ reduction $Z(\eta_k) \cap M_I$ is contained in $Z_{P_I'}(k_F) \times \text{GL}_{n_I'}(k_F)(A)$. If $\eta_k$ is a nontrivial character then $A \neq 0$ and we can apply lemma 3.0.15. For convenience we break the proof of this proposition into subsections considering different possibilities in lemma 3.0.15.

### 3.1.3 Condition (1) of lemma 3.0.15

We first assume that $A$ satisfies the condition (1) in lemma 3.0.15. There exists at least one $n_j$ with $1 \leq j \leq r$ such that the image of the projection

$$p_j : Z_{P_I'}(k_F) \times \text{GL}_{n_I}(k_F)(A) \cap M_I \to \text{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\text{GL}_{n_j}(k_F)$. Here $p_j$ is the projection onto the $j$-th factor of $M_I$. In particular $n_j$ is greater than 1. Let $\gamma$ be an irreducible sub-representation of the restriction $\text{res}_H \tau_j$ where $H$ is the image of $M_I \cap Z(\eta_k)$ under the projection $p_j$. It follows from lemma 3.0.16 that there exists an irreducible non-cuspidal representation $\tau'$ of $\text{GL}_{n_j}(k_F)$ such that $\tau_j \ncong \tau'$ and $\gamma$ is contained in $\text{res}_H \tau'$. Let $\tau'$ (as a representation of $\text{GL}_{n_j}(O_F)$ obtained by inflation) be a sub-representation of

$$\Gamma = i_{P_j(1)}^{\text{GL}_{n_j}(O_F)}(\kappa_j)$$

where $J = (m_1, m_2, \ldots, m_t)$ is an ordered partition of the positive integer $n_j$ and each of $\kappa_l$ for $1 \leq l \leq t$ is a cuspidal representation of $\text{GL}_{m_l}(k_F)$. Define a representation $\tau_I^1$ (a first modification of $\tau_I$) of $Z(\eta_k) \cap M_I$ by setting

$$\tau_I^1 := \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_{j-1} \boxtimes \gamma \boxtimes \tau_{j+1} \boxtimes \cdots \boxtimes \tau_r.$$  

The representation

$$\text{ind}_{P_I(1,m)}^{\text{GL}_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \tau_I^1) \}$$

is contained in the representation

$$\text{ind}_{P_I(1,m)}^{\text{GL}_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \text{res}_{Z(\eta_k) \cap M_I} \tau_I^2) \}$$

where $\tau_I^2$ (the second modification) is the representation

$$\tau_I^2 := \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_{j-1} \boxtimes \Gamma \boxtimes \tau_{j+1} \boxtimes \cdots \boxtimes \tau_r.$$  

Observe that

$$\text{ind}_{P_I(1,m)}^{\text{GL}_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \tau_I^2) \} \simeq \text{ind}_{P_I(1,m)}^{\text{GL}_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \tau_I^2) \}$$

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The representation \( \text{ind}_{P_I(1,m)}^{GL_n(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \tau_I^2) \} \) is a sub-representation of the representation \( \text{ind}_{P_I(1,m+1)}^{GL_n(O_F)} (\tau_I^2) \) which in turn is contained in the representation \( \text{ind}_{P_I \cap GL_n(O_F)}^{GL_n(O_F)} (\tau_I^2) \). We denote by \( I_1 \) the refinement of the ordered partition \( I \) obtained by replacing \( n_j \) with the ordered partition \( J = (m_1, m_2, \ldots, m_t) \). We define \( \kappa_{I_1} \) a representation of \( M_{I_1}(O_F) \) by setting

\[
\kappa_{I_1} := \tau_1 \boxtimes \cdots \boxtimes \tau_{j-1} \boxtimes \kappa_1 \boxtimes \cdots \boxtimes \kappa_t \boxtimes \tau_{j+1} \boxtimes \cdots \boxtimes \tau_r.
\]

By setting these notations we now note that

\[
\text{ind}_{P_I \cap GL_n(O_F)}^{GL_n(O_F)} (\tau_I^2) \subset \text{ind}_{P_{I_1} \cap GL_n(O_F)}^{GL_n(O_F)} (\kappa_{I_1}).
\]

Since \( I_1 \) is a proper partition of \( I \) the Bushnell-Kutzko types \( (P_I(1), \tau_I) \) and \( (P_{I_1}(1), \kappa_{I_1}) \) represent two distinct inertial classes.

### 3.1.4 Condition (2) of lemma 3.0.15

Let \( A = \theta_I(\eta_k) \) satisfy the condition (2) in the lemma 3.0.15. In this case there exists a \( j \) with \( 1 \leq j < r \) such that the mod \( \mathfrak{P}_F \) reduction of \( Z(\eta_k) \cap M_I \) is contained in the subgroup of the form

\[
\{(A_1, \ldots, A_j, \ldots, A_r)| A_i \in GL_{n_i}(k_F) \forall i \in \{1, 2, \ldots, r\} \text{ and } A_j = A_r\}.
\]

Note that \( n_j = n_r \) and we assumed that \( n_r > 1 \). Consider the representation \( \tau_j \boxtimes \tau_r \) of \( GL_{n_j}(k_F) \times GL_{n_r}(k_F) \) and \( H = \{(A, A) | A \in GL_{n_i}(k_F)\} \). For every irreducible sub-representation \( \gamma \) of \( \text{res}_H (\tau_j \boxtimes \tau_r) \) using lemma 3.0.17 we obtain an irreducible non-cuspidal representation \( \tau_j^1 \boxtimes \tau_r^1 \) such that \( \gamma \) is contained in \( \text{res}_H (\tau_j^1 \boxtimes \tau_r^1) \). Now define a representation \( \tau_I^1 \) by setting

\[
\tau_I^1 := \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_j^1 \boxtimes \cdots \boxtimes \tau_r^1.
\]

We note here that \( \tau_I^1 \) may not be independent of \( \gamma \) in the sense that that \( (\tau_j^1, \tau_r^1) \) depends on the irreducible sub-representation \( \gamma \) of \( \text{res}_H (\tau_k \boxtimes \tau_r) \). Any irreducible sub-representation \( \Gamma \) of

\[
\text{ind}_{P_I(1,m)}^{GL_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \text{res}_{Z(\eta_k) \cap M_I} (\tau_I^1)) \}
\]

occurs as a sub-representation of some

\[
\text{ind}_{P_I(1,m)}^{GL_{n+1}(O_F)} \{ \text{ind}_{Z(\eta_k)}^{P_I(1,m)} (U_{\eta_k} \otimes \text{res}_{Z(\eta_k) \cap M_I} (\tau_I^1)) \}.
\]

The representation in (3.18) is contained as a sub-representation of

\[
\text{ind}_{P_I \cap GL_{n+1}(O_F)}^{GL_{n+1}(O_F)} (\tau_I^1).
\]
Let $\tau_j^1$ and $\tau_r^1$ be sub-representations of $\text{ind}_{P_{J_1}(1)}^{\text{GL}_{n_k}(O_F)}(\kappa_{J_1})$ and $\text{ind}_{P_{J_2}(1)}^{\text{GL}_{n_r}(O_F)}(\kappa_{J_2})$ respectively. Let $I_1$ be the partition of the positive integer $n$ obtained by replacing $n_j$ and $n_r$ by the partitions $J_1$ and $J_2$ in $(n_1, n_2, \ldots, n_j, \ldots, n_r)$. We denote by $\tau_{I_1}$ the representation

$$\tau_{n_1} \boxtimes \cdots \boxtimes \tau_{n_{j-1}} \boxtimes \kappa_{J_1} \boxtimes \tau_{n_{j+1}} \boxtimes \cdots \boxtimes \kappa_{J_2}$$

of $M_{I_1}(O_F)$. The representation (3.19) is contained in the representation

$$\text{ind}_{P_{I_1} \cap \text{GL}_{n+1}(O_F)}^{\text{GL}_{n+1}(O_F)}(\tau_{I_1}).$$

The Bushnell-Kutzko types $(P_I(1), \tau_I)$ and $(P_{I_1}(1), \tau_{I_1})$ represent two distinct inertial classes since $I_1$ is a proper refinement of $I$ (see lemma 3.0.17). □

This completes the proof of the proposition and also the proof of the theorem. □