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Title: Typical representations for GLn(F)
Issue Date: 2015-06-16

Chapter 2

Preliminaries

The following notation will be used in all chapters of this thesis.

2.1 Basic notation

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{P}_F$ and a finite residue field k_F. We denote by G the F-rational points of an algebraic reductive group and by P a F-rational parabolic subgroup of G. All our representations are on vector spaces over $\mathbb{C}$. Let σ be a smooth representation of a Levi-subgroup M of P. We denote by $i_P^G(\sigma)$ the normalized parabolically-induced representation.

Let H be a closed subgroup of G and (τ, V) be a smooth representation of H, we denote by $\mathrm{ind}_H^G(\tau)$ the space of functions $f : G \to V$ such that $f(hg) = \tau(h)f(g)$ and there exists a compact open subgroup K_f of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K_f$. The group G acts on these functions by right multiplication i.e $(g'f)(g) = f(gg')$ for all $g, g' \in G$ and $f \in \mathrm{ind}_H^G(\tau)$. We denote by c-$\mathrm{ind}_H^G(\tau)$ the sub-space of $\mathrm{ind}_H^G(\tau)$ consisting of functions f such that $\sup(f) \subset HX_f$ where X_f is a compact set. This is the compactly induced representation.

Let H_1 and H_2 be two subgroups such that $H_2 \subset H_1$, σ be a representation of H_1 we denote by $\mathrm{res}_{H_2}(\sigma)$ the restriction of σ to H_2. We use $\boxtimes$ and $\otimes$ for the tensor product of representations of two different groups and the same group respectively. If H_2 is a subgroup of a group H_1, τ is a representation of H_2 and $h \in H_1$ then we denote by τ^h the representation of hH_2h^{-1} given by $h' \mapsto \tau(h^{-1}h'h)$ for all $h' \in hH_2h^{-1}$.

After recalling some general definitions we will restrict ourself to the case $G = \mathrm{GL}_n(F)$ and we will use the following notation: Let $I = (n_1, n_2, n_3, \ldots, n_r)$ be an ordered partition of a positive integer n. Let P_I be the group of invertible block upper triangular matrices of the type $(n_1, n_2, \ldots, n_r)$. We denote by M_I and U_I the group of block diagonal matrices of the type I and the unipotent radical of P_I respectively. We call P_I and M_I the standard parabolic subgroup and standard Levi-subgroup of type I respectively. We denote by $K_n(m)$ the principal congruence subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ of level m.

2.2 Bernstein decomposition and typical representations

Let $B(G)$ be the set of pairs (M, σ) where M is a Levi-subgroup of a F-parabolic subgroup P of G and σ is an irreducible supercuspidal representation of M. We define an equivalence relation on $B(G)$ by setting

$$(M_1, \sigma_1) \sim (M_2, \sigma_2)$$

if and only if there exists an element $g \in G$ and an unramified character χ of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We denote by $\mathcal{B}_G$ the set of such equivalence classes called **inertial classes or Bernstein components**. Any irreducible smooth representation π of G occurs as a sub-representation of a parabolic induction $i_P^G(\sigma)$ where σ is an irreducible supercuspidal representation of a Levi-subgroup M of P. The pair (M, σ) is well determined up to G-conjugation. We call the class $s = [M, \sigma]$ the **inertial support of** π. Let $\mathcal{M}(G)$ be the category of all smooth representations of G. For an inertial class $s = [M, \sigma]$ we denote by $\mathcal{M}_s(G)$ the full sub-category consisting of smooth representations all of whose irreducible sub-quotients appear in the composition series of $i_P^G(\sigma \otimes \chi)$ where χ is an unramified character of M. It is shown by Bernstein (see [Ren10][VI.7.2, Theorem]) that the category $\mathcal{M}(G)$ decomposes as a direct product of $\mathcal{M}_s(G)$ in particular every smooth representation can be written as a direct sum of objects in the categories $\mathcal{M}_s(G)$. We denote by $\mathcal{A}_G(s)$ the set of isomorphism classes of simple objects in the category $\mathcal{M}_s(G)$. If $G = \mathrm{GL}_n(F)$ we use the notation $\mathcal{A}_n(s)$ for $\mathcal{A}_G(s)$ and $\mathcal{B}_n$ for $\mathcal{B}_{\mathrm{GL}_n(F)}$.

Given an irreducible smooth representation ρ of a maximal compact subgroup K of G the compact induction $\pi := \text{c-ind}_K^G(\rho)$ is a finitely generated smooth representation of G and hence there exists an irreducible G-quotient of π. By Frobenius reciprocity [BH06, Proposition 2.5] we get that ρ occurs in a smooth irreducible representation of G. For a given inertial class, we are interested in the representations ρ of K which only occur in irreducible smooth representations with inertial support s.

Definition 2.2.1. *Let s be an inertial class for G. An irreducible smooth representation τ of a maximal compact subgroup K of G is called K-typical representation for s if for any irreducible smooth representation π of G, $\mathrm{Hom}_K(\tau, \pi) \neq 0$ implies that $\pi \in \mathcal{A}_G(s)$.*

In this thesis we will confine ourself to the case where $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O}_F)$ and $n \geq 2$ and in this case we call a K-typical representation for s a typical representation for s. An irreducible representation τ of $\mathrm{GL}_n(\mathcal{O}_F)$ is called *atypical* if τ occurs in two smooth representations π_1 from $\mathcal{M}_s(\mathrm{GL}_n(F))$ and π_2 from $\mathcal{M}_{s'}(\mathrm{GL}_n(F))$ such that $s \neq s'$.

For any component $s \in \mathcal{B}_n$, the existence of a typical representation can be deduced from the theory of types developed by Bushnell and Kutzko in the articles [BK99] and [BK93]. Bushnell and Kutzko constructed a pair (J_s, λ_s) where J_s is a compact open subgroup of $\mathrm{GL}_n(F)$ and λ_s is an irreducible representation of J_s. Let π be an irreducible smooth representation of $\mathrm{GL}_n(F)$. The pair (J_s, λ_s) satisfies the condition

$$\mathrm{Hom}_{J_s}(\pi, \lambda_s) \neq 0 \;\Leftrightarrow\; \pi \;\in\; \mathcal{A}_n(s).$$

In the case of $\mathrm{GL}_n(F)$, the group J_s can be arranged to be a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ by conjugating with an element of $\mathrm{GL}_n(F)$ and hence we assume that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. It follows from Frobenius reciprocity that any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s) \tag{2.1}$$

is a typical representation. The irreducible sub-representations of (2.1) are classified by Schneider and Zink in [SZ99, Section 6, $T_{K,\lambda}$ functor].

For $s = [\mathrm{GL}_n(F), \sigma]$, Paskunas in the article [Pas05][Theorem 8.1] showed that up to isomorphism there exists a unique typical representation for s. More precisely,

Theorem 2.2.2 (Paskunas). *Let n be a positive integer greater than one and σ be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. Let (J_s, λ_s) be a Bushnell-Kutzko type for the component $s = [\mathrm{GL}_n(F), \sigma]$ with $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. The representation*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

is the unique typical representation for the component $[\mathrm{GL}_n(F), \sigma]$ and occurs with multiplicity one in $\sigma \otimes \chi$ for all unramified characters χ of $\mathrm{GL}_n(F)$.

We will consider the classification of typical representations for components $[M, \sigma]$ where M is a Levi-subgroup of a proper parabolic subgroup of $\mathrm{GL}_n(F)$.

Let $s = [M, \sigma]$ be an inertial class of $\mathrm{GL}_n(F)$. We will choose a representative for s. Let P be a parabolic subgroup with M a its Levi-factor. There exists a $g \in \mathrm{GL}_n(F)$ such that $gPg^{-1} = P_I$ for some ordered partition $I = (n_1, n_2, \dots, n_r)$ of n. The groups gMg^{-1} and M_I are two Levi-factors of P_I hence we get an $u \in \mathrm{Rad}\, P_I$ such that $ugM(ug)^{-1} = M_I$. This shows that there exists an element $g' \in \mathrm{GL}_n(F)$ such that $g'Mg'^{-1} = M_I$. Let J be a permutation of the ordered partition $(n_1, n_2, \dots, n_r)$. We can choose a $g'' \in \mathrm{GL}_n(F)$ such that M_I and M_J are conjugate, the two pairs (M, σ) and $(M_J, \sigma^{g'g''})$ are inertially equivalent. In certain cases it is convenient to choose a particular permutation. For example in the proof of the main theorem of chapter 3 we choose $J = (n'_1, n'_2, \dots n'_r)$ such that $n'_i \leq n'_j$ for all $i \leq j$. We denote by σ_I and σ_J the representations $\sigma^{g'}$ and $\sigma^{g'g''}$ respectively and hence

$s = [M_I, \sigma_I] = [M_J, \sigma_J]$.

Let τ be a typical representation for the component s. The representation τ occurs as a $\mathrm{GL}_n(\mathcal{O}_F)$ sub-representation of a $\mathrm{GL}_n(F)$-irreducible smooth representation π (see the reasoning given in the paragraph above Definition 2.2.1). From the above paragraph π occurs in the composition series of $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ where σ_I is a supercuspidal representation of M_I. Hence to classify typical representations we fix a pair $(M_I, \sigma_I) \sim (M, \sigma)$ and examine the $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}(i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)),$$

looking for possible typical representations for s.

By the Iwasawa decomposition $\mathrm{GL}_n(F) = \mathrm{GL}_n(\mathcal{O}_F)P_I$ we get that

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}(i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)) \simeq \mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma_I).$$

We write σ_I as $\boxtimes_{i=1}^r \sigma_i$ where σ_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ for $1 \leq i \leq r$. We denote by τ_i the unique typical representation for the component $[\mathrm{GL}_{n_i}(F), \sigma_i]$ for $1 \leq i \leq r$ and let τ_I be the $M_I(\mathcal{O}_F)$-representation $\boxtimes_{i=1}^r \tau_i$. Will Conley observed in his thesis that the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

admits a complement in $\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma_I)$ whose irreducible sub-representations are atypical for s. We prove a mild generalization which will be used later in proofs by induction.

Let $t_i = [M_i, \lambda_i]$ be a Bernstein component of $\mathrm{GL}_{n_i}(F)$ for $1 \leq i \leq r$. Let σ_i be a smooth representation from $\mathcal{M}_{t_i}(\mathrm{GL}_{n_i}(F))$. We suppose $\mathrm{res}_{\mathrm{GL}_{n_i}(\mathcal{O}_F)}\, \sigma_i = \tau_i^0 \oplus \tau_i^1$ for $1 \leq i \leq r$ such that irreducible sub-representations of τ_i^1 are atypical. We denote by t the Bernstein component

$$[M_1 \times M_2 \times \cdots \times M_r, \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_r]$$

of $\mathrm{GL}_n(F)$. The component t is independent of the choice of representatives (M_i, λ_i). Let $\tau_I^0 = \boxtimes_{i=1}^r \tau_i^0$ and $\sigma_I = \boxtimes_{i=1}^r (\sigma_i)$.

Proposition 2.2.3. *The representation*

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^0)$$

admits a complement in $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ *with all its irreducible sub-representations atypical for* t.

Proof. Any $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representation of $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ occurs as a sub-representation of

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \gamma_i) \tag{2.2}$$

where γ_i is a $\mathrm{GL}_{n_i}(\mathcal{O}_F)$-irreducible sub-representation of σ_i. If γ_i occurs in τ_i^1 for some $i = N$ with $N \leq r$ then there exists a component $t'_N \in \mathcal{B}_N$ such that $t'_N = [M'_N, \lambda'_N] \neq t_N$ and γ_N occurs in the restriction $\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N)$. Hence the representation (2.2) occurs as a $\mathrm{GL}_n(\mathcal{O}_F)$-sub-representation of

$$i_P^{\mathrm{GL}_n(F)}\{i_{P_1}^{\mathrm{GL}_{n_1}(F)}(\lambda_1) \boxtimes \cdots \boxtimes i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N) \boxtimes \cdots \boxtimes i_{P_r}^{\mathrm{GL}_{n_r}(F)}(\lambda_r)\}$$

The inertial support t' of the above representation is

$$[M_1 \times \cdots \times M'_N \times \cdots \times M_r, \lambda_1 \boxtimes \cdots \boxtimes \lambda'_N \boxtimes \cdots \boxtimes \lambda_r].$$

We may assume that M_i is a standard Levi-subgroup for $1 \leq i \leq r$. Now

$$[M_N = \prod_{j=1}^{p} \mathrm{GL}_{m_j}(F), \lambda_N = \boxtimes_{j=1}^{p} \zeta_j] \neq [M'_N = \prod_{j=1}^{p'} \mathrm{GL}_{m'_j}(F), \lambda'_N = \boxtimes_{j=1}^{p'} \zeta'_j]$$

implies that there exists a cuspidal component $[\mathrm{GL}_{m_k}(F), \zeta_k]$ occurring in the multi-set

$$\{[\mathrm{GL}_{m_1}(F), \zeta_1], [\mathrm{GL}_{m_2}(F), \zeta_2], \ldots, [\mathrm{GL}_{m_p}(F), \zeta_p]\}$$

which has a different multiplicity in

$$\{[\mathrm{GL}_{m'_1}(F), \zeta'_1], [\mathrm{GL}_{m'_2}(F), \zeta'_2], \ldots, [\mathrm{GL}_{m_{p'}}(F), \zeta'_{p'}]\}.$$

Adding cuspidal components with the same multiplicity to the above two multi-sets cannot make the multiplicities of the component $[\mathrm{GL}_k(F), \zeta_k]$ the same. This shows that $t' \neq t$ and hence the desired complement is the direct sum of the representations as in (2.2) such that γ_i occur in τ_i^1 for some $i \in \{1, 2, \ldots, r\}$. □

Lemma 2.2.4. *Let $t_i = [\mathrm{GL}_{n_i}(F), \sigma_i]$ be a Bernstein component for $\mathrm{GL}_{n_i}(F)$ and τ_i be a typical representation for t_i. The representation*

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

admits a complement in $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ whose irreducible sub-representations are atypical.

Proof. We use the uniqueness of typical representations for supercuspidal representations (see [Pas05]) to decompose $\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, \sigma_i$ as $\tau_i \oplus \tau_i^1$ such that irreducible sub-representations of τ_i^1 are atypical. The lemma follows as a consequence of proposition 2.2.3. □

Given a component $s = [M_I, \sigma_I]$ of $\mathrm{GL}_n(F)$ the above lemma shows that typical representations only occur as sub-representations of

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

The above representation is still an infinite dimensional representation of the compact group $\mathrm{GL}_n(\mathcal{O}_F)$. We write the above representation as an increasing union of finite dimensional representations.

Let $\{H_i\}_{i\geq 1}$ be a decreasing sequence of compact open subgroups of the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. Let $\bar{U}_I$ be the unipotent radical of the opposite parabolic subgroup $\bar{P}_I$ of P_I with respect to the Levi-subgroup M_I. We assume that H_i satisfies Iwahori decomposition with respect to the parabolic subgroup P_I and Levi-subgroup M_I for all $i \geq 1$ i.e. the product map

$$(H_i \cap \bar{U}_I) \times (H_i \cap M_I) \times (H_i \cap U_I) \to H_i$$

is a homeomorphism for any ordering of the factors on the left hand side and that $\bigcap_{i\geq 1} H_i = \mathrm{GL}_n(\mathcal{O}_F) \cap P_I$. Let τ be a finite dimensional smooth representation of the group $M_I(\mathcal{O}_F)$. We assume that τ extends to a representation of H_i for all $i \geq 1$ such that $H_i \cap U_I$ and $H_i \cap \bar{U}_I$ are contained in the kernel of τ. By definition the representation $\mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$ is contained in $\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$.

Lemma 2.2.5. *The union of the representations*

$$\mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$$

for all $i \geq 1$ is equal to the representation

$$\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau).$$

Proof. Let W be the underlying space for the representations τ. Any element f on in the space

$$\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$$

is a function $f : \mathrm{GL}_n(\mathcal{O}_F) \to W$ such that

1. $f(pk) = \tau(p)f(k)$ for all $p \in \mathrm{GL}_n(\mathcal{O}_F) \cap P_I$ and $k \in \mathrm{GL}_n(\mathcal{O}_F)$,
2. There exists a positive integer m (depending on f) such that $f(gk) = f(g)$ for all $k \in K_n(m)$ and $g \in \mathrm{GL}_n(\mathcal{O}_F)$.

Now there exists a positive integer i such that $H_i \cap \bar{U} \subset K_n(m)$. For such a choice of i and $h \in H_i$ write $h = h^- h^+$ where $h^+ \in \mathrm{GL}_n(\mathcal{O}_F) \cap P$, $h^- \in H_i \cap \bar{U}$ and we can do so by Iwahori decomposition of H_i. We observe that $f(hk) = f(h^- h^+ k) = f(h^+ k (h^+ k)^{-1} h^- (h^+ k)) = f(h^+ k) = \tau(h^+) f(k)$ (since $(h^+ k)^{-1} h^- (h^+ k) \in K_n(m)$). Hence $f \in \mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$. □

We shall need the following technical lemma for frequent reference. Let P be any parabolic subgroup of $\mathrm{GL}_n(F)$ with a Levi-subgroup M and U be the unipotent radical of P. Let J_1 and J_2 be two compact open sub-groups of $\mathrm{GL}_n(\mathcal{O}_F)$ such that J_1 contains J_2. Suppose J_1 and J_2 both satisfy Iwahori decomposition with respect to the Levi-subgroup M, $J_1 \cap U = J_2 \cap U$ and $J_1 \cap \bar{U} = J_2 \cap \bar{U}$. Let λ is an irreducible smooth representation of J_2 which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of λ.

Lemma 2.2.6. *The representation* $\mathrm{ind}_{J_2}^{J_1}(\lambda)$ *is the extension of the representation* $\mathrm{ind}_{J_2 \cap M}^{J_1 \cap M}(\lambda)$ *such that* $J_1 \cap U$ *and* $J_1 \cap \bar{U}$ *are contained in the kernel of the extension.*

Proof. From Iwahori decomposition we get that $(J_1 \cap M)J_2 = J_1$ and Mackey decomposition we get that

$$\mathrm{res}_{J_1 \cap M} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2 \cap M}^{J_1 \cap M}(\lambda).$$

We now verify that $J_1 \cap U$ and $J_1 \cap \bar{U}$ act trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Observe that

$$\mathrm{res}_{J_1 \cap P} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2 \cap P}^{J_1 \cap P}(\lambda).$$

Since the double coset representatives for

$$\frac{J_1 \cap P}{J_2 \cap P}$$

can be chosen from $M \cap J_1$ the group $J_1 \cap U$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Similarly $J_1 \cap \bar{U}$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. This concludes the lemma. □

Lemma 2.2.7. *Let* G *be the* F*-rational points of an algebraic reductive group and* χ *be a character of* G*. Let* τ *be a* K*-typical representation for the component* $s = [M, \sigma]$*. The representation* $\tau \otimes \chi$ *is a typical representation for the component* $[M, \sigma \otimes \chi]$*.*

Proof. Let $\mathrm{Hom}_K(\tau \otimes \chi, \pi) \neq 0$ for some irreducible smooth representation π of G. We now have $\mathrm{Hom}_K(\tau, \pi \otimes \chi^{-1}) \neq 0$. This implies that $\pi \otimes \chi^{-1}$ occurs in the composition series of

$$i_P^G(\sigma \otimes \eta)$$

for some parabolic subgroup P containing M is a Levi-factor and η an unramified character of M. Now π occurs in the composition series for the representation

$$i_P^G(\sigma \otimes \chi \otimes \eta)$$

hence $\tau \otimes \chi$ is a K-typical representation for the component $[M, \sigma \otimes \chi]$. □

We now sketch the general strategy to classify typical representations. We choose the sequence $\{H_i \mid i \geq 1\}$ depending on certain class of components s and then we will construct a representation say $U_i(\tau_I)$ such that

$$U_i(\tau_I) \oplus \mathrm{ind}_{H_1}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) = \mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

for $i \geq 2$. We will show by induction on the integer i that the irreducible sub-representations of $U_i(\tau_I)$ are atypical for s. This shows that typical representations occur as sub-representations of

$$\mathrm{ind}_{H_1}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I). \tag{2.3}$$

It is indeed possible that all irreducible sub-representations of (2.3) are typical for s. This will be the case for many Bernstein components for example level-zero (to be defined in the next chapter), principal series components and $s = [\mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$ for $n \geq 2$. The choice of H_i, construction of $U_i(\tau_I)$ and showing irreducible sub-representations of $U_i(\tau_I)$ are atypical representations for various classes of components will occupy the next three chapters.