Author: Nadimpalli, Santosh
Title: Typical representations for GLn(F)
Issue Date: 2015-06-16

Chapter 1

Introduction

Let F be a non-discrete non-Archimedean locally compact field. This thesis is concerned with the study of the restriction of an irreducible smooth representation of $\mathrm{GL}_n(F)$ to a **maximal compact subgroup** K where $n \geq 2$. In particular we are interested in those irreducible representations (called **K-typical representations**) of K which determine the inertial support of the given irreducible smooth representation. In the context of local Langlands correspondence, such representations have seen significant arithmetic applications. In this thesis we try and achieve in many cases the classification of such irreducible smooth representations of K for a given inertial support s.

1.1 Motivation

Local Langlands Correspondence

Let $\bar{F}$ be a separable algebraic closure of F. Let F^{un} be the maximal unramified sub-extension of $\bar{F}$. We have the canonical quotient map

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(F^{un}/F).$$

The group $\mathrm{Gal}(F^{un}/F)$ is canonically isomorphic to the Galois group of the residue field $k_{F^{un}}$ of F^{un} over k_F. Since $k_{F^{un}}$ is the algebraic closure of the finite field k_F we get the map

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(k_{F^{un}}/k_F) \simeq \hat{\mathbb{Z}}. \tag{1.1}$$

Let q be the cardinality of the residue field k_F. We denote by Φ_F the automorphism of $k_{F^{un}}$ which sends an element x to x^q. Let W_F be the group consisting of those elements of $\mathrm{Gal}(\bar{F}/F)$ which map to a power of Φ_F under the map (1.1). The group W_F is called the Weil group of F. The group W_F can be made a locally compact group by declaring the open subgroups of $\mathrm{Gal}(\bar{F}/F^{un})$ (under its pro-finite topology) as open subgroups of W_F. Hence we obtain an exact sequence of topological groups

$$0 \to \mathrm{Gal}(\bar{F}/F^{un}) \to W_F \to \Phi_F^{\mathbb{Z}} \to 0$$

where $\Phi_F^{\mathbb{Z}}$ is given discrete topology.

Local class field theory gives us a canonical topological isomorphism

$$W_F^{ab} \simeq F^{\times}.$$

where W_F^{ab} is the quotient of W_F by the closure of the derived group of W_F. This gives us a one-to-one correspondence between the set of continuous characters of W_F and $F^\times$. The local Langlands correspondence establishes a higher dimensional analogue of the correspondence between the characters obtained via local class field theory. Such a correspondence can be stated with a certain algebraic object called Weil-Deligne representation. To begin with we introduce a norm $\|\|$ on the Weil group W_F. Let x be an element of W_F and the image of x under the map (1.1) be Φ_F^r then $\|x\| = q^{-r}$. An n-dimensional Weil Deligne representation is a triple (r, V, N) where V is an n-dimensional complex vector space, r is a homomorphism of W_F into $\mathrm{GL}(V)$ with open kernel, $N \in \mathrm{End}_{\mathbb{C}}(V)$ such that

$$r(x)Nr(x)^{-1} = \|x\|N$$

for all $x \in W_F$. We call the triple (r, V, N) Frobenius semi-simple if the representation (r, V) is semi-simple. The local Langlands conjecture (see [LRS93] for the case where characteristic of F is greater than zero and [HT01] and [Hen00] for the case where characteristic of F is zero) is a natural correspondence between the set of isomorphism classes of n-dimensional Frobenius semi-simple Weil-Deligne representations and the set of isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_n(F)$ (A representation (π, V) is called smooth if and only if the stabiliser of a vector $v \in V$ contains an open subgroup of $\mathrm{GL}_n(F)$ for the topology induced from F).

Let B_n be the set of pairs (M, σ) where M is a Levi-subgroup of a parabolic subgroup of $\mathrm{GL}_n(F)$ and σ is a supercuspidal representation of M. We recall that **inertial equivalence** is an equivalence relation on the set B_n defined by setting $(M_1, \sigma_1) \sim (M_2, \sigma_2)$ if and only if there exists an element $g \in G$ and an unramified character χ of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We use the notation $[M, \sigma]$ for the equivalence class containing the pair (M, σ) the equivalence classes are also called as **inertial classes**. Every irreducible smooth representation π of $\mathrm{GL}_n(F)$ occurs as a sub-representation of a parabolically induced representation $i_P^{\mathrm{GL}_n(F)}(\sigma)$ where σ is a supercuspidal representation of a Levi-subgroup M of P. The pair (M, σ) is well determined upto $\mathrm{GL}_n(F)$-conjugacy (see [BZ77][Theorem 2.5 Theorem 2.9(a)(i)]). The class $[M, \sigma]$ is called the **inertial support** of π. (Inertial equivalence is defined for any reductive group G over F but we need it only for $\mathrm{GL}_n(F)$ in this thesis).

Given two triples (r_1, V_1, N_1) and (r_2, V_2, N_2), it turns out that the restrictions of r_1 and r_2 to the group $\mathrm{Gal}(\bar{F}/F^{un})$ are isomorphic if and only if the smooth representations π_1 and π_2 associated by the local Langlands correspondence to (r_1, V_1, N_1) and (r_2, V_2, N_2) respectively have the same inertial support. In several arithmetic applications (see [BM02] and [EG14] for instance) it is desirable to associate with a given inertial support say s an irreducible smooth representation τ of $\mathrm{GL}_n(\mathcal{O}_F)$ which has the property that

$\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \pi) \neq 0$ implies that the inertial support of π is s and such a representation is called a $\mathrm{GL}_n(\mathcal{O}_F)$**-typical representation or typical representation**. We bring to the attention of the reader that we can expect at best an implication in one direction. One can produce easy examples for s such that there exists no irreducible smooth representation τ_s of $\mathrm{GL}_n(\mathcal{O}_F)$ such that $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_s, \pi) \neq 0$ if and only if the inertial support of π is s.

Theory of Types

Let G be the group of F-rational points of an algebraic reductive group (examples being $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$ and $\mathrm{SO}(V, q)$ for some finite dimensional quadratic space (V, q) over F, etc). It was shown by Bernstein that the category of smooth representations $\mathcal{M}(G)$ admits a decomposition

$$\mathcal{M}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{M}_s(G)$$

where $\mathcal{M}_s(G)$ is the full sub-category consisting of smooth representations with all their irreducible sub-quotients having inertial support s. The theory of types developed initially by Bushnell-Kutzko (see [BK98] for a general discussion on smooth representations via types) gives a construction of pairs (J_s, λ_s) where J_s is a compact open subgroup of G and λ_s is a smooth irreducible representation of J_s such that $\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0$ if and only if $\pi \in \mathcal{M}_s(G)$ for all irreducible smooth representations π of G and such a pair (J_s, λ_s) is called a **type** for s. Such a type (J_s, λ_s) gives a natural equivalence of categories $\mathcal{M}_s(G)$ and the category of modules over the spherical Hecke algebra $\mathcal{H}(J_s, \lambda_s)$.

Let K be a maximal compact subgroup of G and s be an inertial class of G. If we know the existence of a type (J_s, λ_s) such that $J_s \subset K$ then by Frobenius reciprocity any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{K}(\lambda_s) \tag{1.2}$$

if contained in a smooth irreducible representation π of G then π contains the representation λ_s on restriction to the group J_s and hence the inertial support of π is s. This shows that irreducible sub-representations of (1.2) are K-typical representations. As types are not known to exist in every case the following natural questions appear :

1. Does there exist a K-typical representation?
2. For a given inertial class s is the cardinality of K-typical representations finite?
3. What are all typical representations?

For $G = \mathrm{GL}_n(F)$ types (J_s, λ_s) are explicitly constructed by Bushnell-Kutzko in the articles [BK93] and [BK99]. For $\mathrm{GL}_n(F)$ we may and do choose "the Bushnell-Kutzko type" (J_s, λ_s) such that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$ for all $s \in \mathcal{B}_n$. In this thesis we will see that the above questions for $G = \mathrm{GL}_n(F)$ can be answered in terms of and by the use of theory of types. For $\#k_F > 3$ we show in many cases that the irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

are precisely the typical representations for the component s. In this sense we classify typical representations for the component s. We bring to attention of the reader that the types constructed by Bushnell-Kutzko may not be unique even up to conjugation. We use the terminology "the Bushnell-Kutzko type" for the pair (J_s, λ_s) a type for $s = [M, \sigma]$ constructed by the inductive procedure in the article [BK99] after fixing a type for the inertial class $[M, \sigma]$ of M.

There can be various constructions of types (J, λ) for a given component s (in the sense that the pair (J, λ) has the property $\mathrm{Hom}_J(\lambda, \pi) \neq 0$ if and only if the inertial support of π is s for any irreducible smooth representation π of G). For any such construction and K a maximal contact subgroup containing J, the irreducible sub-representations of

$$\mathrm{ind}_J^K(\lambda)$$

are a K-typical representation. Hence the theory of typical representation, at least for the case of GL_n, aims to give a uniform approach. It could be interesting to prove at least the finiteness of typical representations in general case.

1.2 Known results

The case of $\mathrm{GL}_2(F)$ is treated by Henniart in the appendix to the article [BM02]. He completely classified typical representations for all possible inertial classes. Henniart predicted that his results can be extended to $\mathrm{GL}_n(F)$ by similar techniques he used at least in those cases where the underlying Levi-subgroup of the inertial class s is $\mathrm{GL}_n(F)$. Paskunas has classified the typical representations for the inertial classes $[\mathrm{GL}_n(F), \sigma]$. We now describe the results of Henniart and Paskunas. Before going any further we note that J_s can be conjugated to a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ and we assume that indeed J_s is a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$.

Theorem 1.2.1 (Henniart)**.** *Let s be an inertial class for $GL_2(F)$. Let (J_s, λ_s) be the Bushnell-Kutzko type for the inertial class s. If $\#k_F > 2$ or $s = [GL_2(F), \sigma]$ then the typical representations for s occur as sub-representations* *of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s).$$

Let T be the maximal torus of $\mathrm{GL}_2(F)$ consisting of diagonal matrices and $s = [T, \chi]$ be an inertial class for $\mathrm{GL}_2(F)$. Let us identify T with $F^\times \times F^\times$ and the character χ be $\chi((a, b)) = \chi_1(a)\chi_2(b)$ for two characters χ_1 and χ_2 of $F^\times$. Let B be a Borel subgroup containing T. Let $B(m)$ be the group of matrices of $\mathrm{GL}_2(\mathcal{O}_F)$ which under the mod $\mathfrak{P}_F^m$ reduction lie in the group $B(\mathcal{O}_F/\mathfrak{P}_F^m)$. If $\chi_1\chi_2^{-1} \neq \mathrm{id}$, N is the level of $\chi_1\chi_2^{-1}$ and $\#k_F = 2$ then Henniart showed that

$$\mathrm{ind}_{B(N)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

and

$$\mathrm{ind}_{B(N+1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

are typical for the inertial class s. The Bushnell-Kutzko type for the inertial class $s = [T, \chi]$ is given by $(B(N), \chi)$. So this shows that in the present case there are indeed additional representations other than those irreducible sub-representations of (1.2) which are typical for s. For all other inertial classes $[T, \chi]$ the typical representations are shown to be sub-representations of (1.2).

For the inertial class $s = [\mathrm{GL}_n(F), \sigma]$ it follows easily that

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

is an irreducible representation and Paskunas in the article [Pas05] showed that

Theorem 1.2.2 (Paskunas)**.** *For any positive integer $n > 1$ and for any inertial class $s = [\mathrm{GL}_n(F), \sigma]$ there exists a unique typical representation.*

1.3 Results of this thesis

In this thesis we are interested in the classification of typical representations for the inertial classes $[M, \sigma]$ where M is a proper Levi-subgroup of $\mathrm{GL}_n(F)$ and $n \geq 3$. Here we assume the Bushnell- Kutzko type (J_s, λ_s) is conjugated such that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. We give a detailed description of the results from each chapter. Our goal is to describe the results and method of proof briefly.

Results of Chapter 2

If τ is a typical representation for an inertial class s then we show that τ occurs in a smooth irreducible representation say $\pi \in \mathcal{M}_s(G)$ of $\mathrm{GL}_n(F)$. We choose a representative (M, σ) for s such that M is the Levi-subgroup consisting of block diagonal matrices and σ naturally a supercuspidal representation of M. Now the representation π occurs in some representation

$$i_P^{\mathrm{GL}_n(F)}(\sigma \otimes \chi)$$

where P is any parabolic subgroup containing M as a Levi-subgroup and $i_P^{\mathrm{GL}_n(F)}$ denotes the parabolic induction and χ is an unramified character of M. Hence for the classification of typical representations we have to look for the typical representations occurring in the representation

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\{i_P^{\mathrm{GL}_n(F)}(\sigma)\} \simeq \mathrm{ind}_{P\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma). \tag{1.3}$$

Now we identify M with the product $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \dots \mathrm{GL}_{n_r}(F)$ for some ordered partition $(n_1, n_2, \dots, n_r)$ of n and σ as a tensor product $\sigma_1 \boxtimes \sigma_2 \boxtimes \cdots \boxtimes \sigma_r$ where σ_i is a cuspidal representation of $\mathrm{GL}_{n_i}(F)$. Let τ_i be the unique typical representation occurring in σ_i. It was observed by Will Conley that the representation

$$\mathrm{ind}_{P\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i)$$

admits a complement in (1.3) whose $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations are non-typical for the component s. We extend Conley's result, to be used for proofs by induction on n. But for sake of brevity we cannot go into details here.

Now any typical representation occurs among the $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{ind}_{P\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

This representation is still an infinite dimensional representation. The first idea is to construct compact open subgroups groups H_m for $m \geq 1$ such that $H_{m+1} \subset H_m$, $\cap_{m\geq 1} H_m = P\cap\mathrm{GL}_n(\mathcal{O}_F)$ and $\boxtimes_{i=1}^r \tau_i$ extends to a representation of H_1. With some additional conditions we show that

$$\mathrm{ind}_{P\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i) \simeq \bigcup_{m\geq 1} \mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

Now depending on the inertial support we have to define H_m in each chapter and analyse the representations

$$\mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

In addition to this we show several technical lemmas which are frequently used in the entire thesis.

Results of Chapter 3

This chapter concerns those components (called **level zero inertial classes**)

$$[M = \prod_{i=1}^{r} \mathrm{GL}_{n_i}(F), \boxtimes_{i=1}^{r} \sigma_i]$$

where each σ_i contains a non-zero vector fixed by the principal congruence subgroup of level one. In other words the Bushnell-Kutzko type for σ_i is given by $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ where τ_i is obtained by inflating a cuspidal representation of $\mathrm{GL}_{n_i}(k_F)$ for all $i \leq r$. Now choose P to be the group of block upper diagonal matrices containing M as a Levi-subgroup. For a positive integer m we denote by $P(m)$ the group of matrices in $\mathrm{GL}_n(\mathcal{O}_F)$ which under the mod-$\mathfrak{P}_F^m$ reduction lie inside the group $P(\mathcal{O}_F/\mathfrak{P}_F^m)$. Now the representation $\boxtimes_{i=1}^{r}\tau_i$ extends to a representation of $P(1)$ by inflation. The sequence of groups H_m we described in the earlier sub-section are given by $P(m)$.

To simplify the notation we denote by V_m the representation

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^{r}\tau_i).$$

Using induction on the positive integer m we show the theorem

Theorem 1.3.1. *The $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of V_m/V_1 are not typical for the inertial class $s = [M, \sigma]$.*

We actually construct a complement of V_1 in V_m.

We note that the Bushnell-Kutzko type for the inertial class s is given by the pair $(P(1), \boxtimes_{i=1}^{r}\tau_i)$, from which we can conclude that the above theorem completely classifies the typical representations in this case. Through our analysis we gain some additional information. We conclude the result in the following theorem.

Theorem 1.3.2. *Let $s = [M, \sigma]$ be a level zero inertial class. Let Γ be a typical representation for the inertial class s. The representation Γ is an irreducible sub-representation of V_1 and*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, V_1) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, i_P^{\mathrm{GL}_n(F)}(\sigma)).$$

We now sketch the proof of the above theorems. The essential features of the proof are captured in two cases, the first case: M is isomorphic to $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$ and the the second case is: M is the diagonal torus of $\mathrm{GL}_n(F)$.

To begin with lets consider the case $n_1 = n_2 = 1$. In this case Henniart in the article [BM02][Appendix] uses Casselman's description (see [Cas73]) of the complete decomposition of the restriction of a smooth representation to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$. Casselman shows that two smooth representations with the same central character have isomorphic restriction to $\mathrm{GL}_2(\mathcal{O}_F)$ except for a finite part. He effectively controls this finite part as well. It turns out in most of the cases (at least in the principal series case) that this finite part is

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s).$$

The rest of the restriction depends only on the conductor and the central character. Henniart manages to produce two smooth representations with same central character and conductor but changing the inertial support. This gives the result. Although additional work has to be carried out when $\#k_F = 2$.

Let $n_1 + n_2 > 2$. When $n \geq 3$ we do not in general know the complete decomposition into irreducible summands of the restriction of an irreducible smooth representations to the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. In practice we found that such result is not required for our purpose. We already reduced to check for typical representations occurring in the representations

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

for $m \geq 1$. Notice that we have

$$\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \otimes (\tau_1 \boxtimes \tau_2)\}. \tag{1.4}$$

By Frobenius reciprocity we know that id occurs with multiplicity one in the representation

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}).$$

By means of Clifford theory we achieve the following decomposition,

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{j=1}^{t} \mathrm{ind}_{Z_j}^{P(m)}(U_j)$$

where Z_j is a compact open subgroup which is “small enough”. We will come back to what we mean by “small enough". Now let us return to 1.4. We have

$$\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \oplus \bigoplus_{1 \leq j \leq t} \mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}.$$

For some fixed j we wish to compare the terms $\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}$ for various τ_1 and τ_2. It is exactly in this context that we refer to Z_j being “small enough". We have shown that for every irreducible sub-representation ξ of $\mathrm{res}_{Z_j}(\tau_1 \boxtimes \tau_2)$ we can find an irreducible representation $\tau_1' \boxtimes \tau_2'$ such that

1. $\tau_1' \boxtimes \tau_2'$ is the inflation of a non-cuspidal representation of $M(k_F)$.

2. ξ occurs in the representation $\mathrm{res}_{Z_j}(\tau_1' \boxtimes \tau_2')$.

This is enough to show that the irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}$$

are non-typical. By means of induction on the positive integer m we prove the theorem.

The above attempt does not work if both τ_1 and τ_2 are characters. The reason is that $M(k_F)$ has no non-cuspidal representations in this case. For the case of principal series case i.e. $s = [T, \chi]$ (T is the maximal torus consisting of invertible diagonal matrices of dimension $n > 2$) we use slightly different techniques. We assume that we know the result for $n-1$ and show that typical representations occur only in

$$W_m := \mathrm{ind}_{R(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{(\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i))) \boxtimes \chi_n\}$$

where R is a parabolic subgroup of the type $(n-1, 1)$ and B_{n-1} is the Borel subgroup consisting of upper triangular matrices. We now use induction on m as done earlier to show that the typical representations occur as sub-representations of W_1. The induction step is achieved by comparing the terms

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i)) \boxtimes \chi_n)\}. \tag{1.5}$$

The mod $\mathfrak{P}_F$ reduction of the group $Z_j \cap N$ (N is the Levi-subgroup of R consisting of block diagonal matrices of size $(n-1, 1)$) is contained in the following subgroup

$$\left\{ \begin{pmatrix} A & B & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} | A \in \mathrm{GL}_{n-2}(k_F); B \in M_{n-2\times 1}(k_F); d \in k_F^\times \right\}.$$

We will further decompose the representation 1.5 by first decomposing the restriction of the representation

$$\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i))$$

to the group

$$P_{(n-2,1)}(k_F) := \left\{ \begin{pmatrix} A & B \\ 0 & d \end{pmatrix} | A \in \mathrm{GL}_{n-2}(k_F); B \in M_{n-2\times 1}(k_F); d \in k_F^\times \right\}.$$

We then compare the terms

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\gamma_p \boxtimes \chi_n)\}$$

where γ_p is the inflation of an irreducible representation of

$$\mathrm{res}_{P_{(n-2,1)}(k_F)}\{\mathrm{ind}_{B_{n-1}(k_F)}^{\mathrm{GL}_{n-1}(k_F)}(\boxtimes_{i=1}^{n-1}\chi_i)\}.$$

We combine these ideas to complete the proof of theorem 3.0.9.

Results of Chapter 4

Let T_n be the maximal torus, consisting of invertible diagonal matrices in $\mathrm{GL}_n(F)$. For $\#k_F > 3$ we classify the typical representations for the components $s = [T_n, \chi]$. We will show that the typical representations occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s).$$

First our goal is to define the groups H_m such that $H_{m+1} \subset H_m$ and

$$\cap_{m\geq 1} H_m = B_n(\mathcal{O}_F)$$

where B_n is the Borel subgroup consisting of invertible upper triangular matrices. We note that the Bushnell-Kutzko type (J_s, λ_s) in this case has the property that $J_s \cap B_n = B_n(\mathcal{O}_F)$. We can hence choose $H_1 = J_s$ and the other groups need to be defined more carefully. We sketch the details.

We identify T_n with the group $\times_{i=1}^n F^\times$ and the character χ with $\boxtimes_{i=1}^n \chi_i$ where χ_i is a character of $F^\times$. We denote by $l(\chi)$ the least positive integer k such that $1 + \mathfrak{P}_F^k$ is contained in the kernel of χ. Let $J_\chi(m)$ be the set consisting of matrices (a_{ij}) where $a_{ij} \in \mathfrak{P}_F^{l(\chi_i\chi_j^{-1})+m-1}$ for all $i > j$, $a_{ii} \in \mathcal{O}_F^\times$ and $a_{ij} \in \mathcal{O}_F$ for all $i < j$. We will show that $J_\chi(m)$ is indeed a compact open subgroup and we also establish the following

1. $J_s = J_\chi(1)$
2. $\cap_{m\geq 1} J_\chi(m) = B_n(\mathcal{O}_F)$
3. The character χ of $T(\mathcal{O}_F)$ extends to a character of $J_\chi(1)$.

Hence we define $H_m = J_\chi(m)$ for $m \geq 1$. This shows that

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{B_n}^{\mathrm{GL}_n(F)}(\chi) \simeq \bigcup_{m\geq 1} \mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi)$$

We denote by $V_m(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

By using induction on the positive integers n and m we show the following theorem:

Theorem 1.3.3. *The* $\mathrm{GL}_n(\mathcal{O}_F)$*-irreducible sub-representations of* $V_m(\chi)/V_1(\chi)$ *are not typical for the component* $s = [T_n, \chi]$.

From the induction hypothesis on the positive n we can show that the typical representations occur in the following sub-representation of V_m:

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

The group $J_\chi(1,m)$ contains $J_\chi(m)$ and $\cap_{\geq m} J_\chi(1,m)$ is of the form

$$\left\{ \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} | A \in J_{\boxtimes_{i=1}^{n-1}\chi_i}(1); B \in M_{n-1\times 1}(\mathcal{O}_F); c \in \mathcal{O}_F^\times \right\}.$$

Now as seen earlier we decompose the representation

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})$$

as follows

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{1\leq j\leq p} \mathrm{ind}_{Z_j}^{J_\chi(1,m)}(U_j)$$

and show that $\mathrm{res}_{Z_j\cap T_n}(\chi) = \mathrm{res}_{Z_j\cap T_n}(\chi')$ for some character χ' such that $[T,\chi'] \neq [T,\chi]$ and $J_\chi(1) = J_{\chi'}(1)$. This shows that the irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_j \otimes \chi)$$

are not typical. This proves the main result by induction on m. We bring to the attention of the reader that the decomposition of the group

$$\mathrm{ind}_{J_\chi(m+1)}^{J_\chi(m)}(\mathrm{id})$$

is much more involved than the decomposition of the corresponding representation obtained by replacing $J_\chi(m)$ and $J_\chi(m+1)$ by $J_\chi(1,m)$ and $J_\chi(1,m+1)$ respectively. This is the reason why we adapt to do induction on both the variables n and m.

Review of chapter 5

In this chapter we are interested in classification of typical representations for the inertial classes $s = [\mathrm{GL}_n(F) \times GL_1(F), \sigma \boxtimes \chi]$. In this chapter and the next we will use the apparatus of [BK93] like lattice chains, hereditary orders, stata, simple characters, β-extensions etc with precise references. Let P be a parabolic subgroup consisting of block upper triangular matrices of the type $(n,1)$. We denote by M the Levi-subgroup of P consisting of block diagonal matrices. We recall that $P(m)$ has the usual meaning. We denote by τ the unique typical representation occurring in the cuspidal representation σ. We can as well assume that χ is trivial.

Typical representations occur as sub-representations of the representation

$$V_m := \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

for some positive integer $m \geq N$ (N will be explicitly defined later). Let $[\mathfrak{A}, l, 0, \beta]$ be a simple stratum defining the Bushnell-Kutzko type (J^0, λ) for the component $[\mathrm{GL}_n(F), \sigma]$. The representation τ is isomorphic to

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda).$$

As we did in the previous situations we will need the decomposition of the representation

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}).$$

The above representation is decomposed in chapter 3 as

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{j} \mathrm{ind}_{Z'_j}^{P^0(m)}(U_j).$$

Now recall that the groups $Z'_j \cap M$ might be smaller than $M(\mathcal{O}_F)$. Now $\mathrm{res}_{Z'_j \cap M}(\tau \boxtimes \mathrm{id})$ would involve complicated Mackey decompositions and moreover the structure of $(J^0 \times {\mathcal{O}_F}^{\times})^u \cap (J^0 \times {\mathcal{O}_F}^{\times})$ can be complicated for a given $u \in \mathrm{GL}_n(\mathcal{O}_F) \times {\mathcal{O}_F}^{\times}$. To overcome this problem we define a compact open subgroup $P^0(m)$ of $P(m)$ such that $P^0(m) \cap M = J^0 \times \mathcal{O}_F^{\times}$, $P^0(m) \cap U = P(m) \cap U$, $P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$ and $\lambda \boxtimes \mathrm{id}$ extends to a representation of $P^0(m)$. We also show that

$$\mathrm{ind}_{P^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

Now we show that

$$\mathrm{ind}_{P^0(m)}^{P^0(m+1)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{1 \le j \le p} \mathrm{ind}_{Z_j}^{P^0(m)}(U_j)$$

where $Z_j \cap M$ contained a priori in $J^0 \times \mathcal{O}_F^{\times}$ is "small enough" in this group.

To explain the meaning of "small enough" we need to recall certain aspects of the type (J^0, λ). Let B be the algebra $\mathrm{End}_{F[\beta]}(\mathfrak{A} \otimes F)$. Let $\mathfrak{B}$ be the order $\mathrm{End}_{F[\beta]}(\mathfrak{A} \otimes F) \cap \mathfrak{A}$. The group J^0 has a normal subgroup J^1 such that $J^0/J^1 \simeq U^0(\mathfrak{B})/U^1(\mathfrak{B})$. The group $U^0(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_{n'}(k_{F[\beta]})$. The representation λ is isomorphic to $\kappa \otimes \lambda'$ where λ' is a cuspidal representation of J^0/J^1 and κ is a certain representation called β-extension. We refer to $Z_j \cap M$ as "small enough" in the sense that $Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})$ satisfies the important property: when $n' > 1$ for every irreducible sub-representation ξ of

$$\mathrm{res}_{Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})}(\kappa \otimes \lambda')$$

there exists a non-cuspidal representation λ'' of J^0/J^1 such that ξ occurs in

$$\mathrm{res}_{Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})}(\kappa \otimes \lambda'').$$

With this we conclude that any irreducible sub-representation Γ of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \tag{1.6}$$

occurs as an irreducible representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id})$$

(τ' may depend on Γ) where we show that irreducible sub-representations of τ' occur in the restriction to $\mathrm{GL}_n(\mathcal{O}_F)$ of an irreducible non-cuspidal representation of $\mathrm{GL}_n(F)$. For showing that τ' occurs in the restriction to $\mathrm{GL}_{n+1}(\mathcal{O}_F)$ of an irreducible non-cuspidal representation we use the novel feature of simple characters and their compatibility with change of rings ($\mathfrak{A}$) due to Bushnell-Kutzko (see [BK93][Section 3.6, Proposition 8.3.5]). This shows that irreducible sub-representations of 1.6 are not typical representations. Hence by using induction on integer m, $m \geq N$, we can show that $\mathrm{GL}_{n+1}(\mathcal{O}_F)$ sub-representations of V_m/V_1 are non-typical.

The Bushnell-Kutzko type J_s is almost close enough to $P^0(N)$. In the sense that $P^0(N) \cap P = J_s \cap P$ and moreover

$$\bar{U}(\varpi_F^N) \subset P^0(N) \cap \bar{U} \subset J_s \cap \bar{U} \subset \bar{U}(\varpi^{N-1}\,\mathcal{O}_F).$$

We decompose the representation

$$\mathrm{ind}_{P^0(N)}^{J_s}(\mathrm{id}) = \mathrm{id} \oplus \bigoplus_j \mathrm{ind}_{Z_j}^{J_s}(U_j)$$

for "small enough" groups Z_j and show that irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_j \otimes (\lambda \boxtimes \mathrm{id})\}$$

are not typical representations. This gives the result:

Theorem 1.3.4. *Let Γ be a typical representation for the inertial class*

$$s = [\mathrm{GL}_n(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$$

and $\#k_F > 2$. The representation Γ is unique and occurs with a multiplicity one in the representation

$$i_P^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi).$$

We show that typical representations should occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \chi).$$

We know from the Bushnell-Kutzko theory that the above representation is irreducible. This gives our uniqueness of typical representation. The multiplicity one result is not known to the author without the results of this thesis. We note that typical representations for the inertial classes of $\mathrm{GL}_3(F)$ where $\#k_F > 3$ are precisely the sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_3(\mathcal{O}_F)}(\lambda_s).$$

Review of chapter 6

In this chapter we will classify typical representations for certain inertial classes

$$s = [\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2].$$

Let P be the parabolic subgroup consisting of block upper triangular matrices of the type $(2,2)$. Let M be the Levi-subgroup of P and U be the unipotent radical of P. We denote by $\bar{P}$ the opposite parabolic subgroup of P with respect to M. Let $\bar{U}$ be the unipotent radical of $\bar{P}$. Let (J_1^0, λ_1) and (J_2^0, λ_2) be Bushnell-Kutzko's types for the inertial classes $[\mathrm{GL}_2(F), \sigma_1]$ and $[\mathrm{GL}_2(F), \sigma_2]$ respectively. The groups $P(m)$ will have the usual meaning for $m \geq 1$. We also define a group $P^0(m)$ for $m \geq N$ where N will be defined explicitly in the main text of the chapter 6. The groups $P^0(m)$ has Iwahori decomposition with respect to P and M; $P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$, $P^0(m) \cap U = P(m) \cap U$ and $P^0(m) \cap M = J_1^0 \times J_2^0$; and

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P^0(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

for all $m \geq N$.

By methods similar to those of chapters 2,3,4,5 we will reduce the problem of classifying typical representations for s to finding typical representations occurring as sub-representations of

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2). \tag{1.7}$$

It turns out that the Bushnell-Kutzko type for s is not of the form $(P^0(N+1), \lambda_1 \boxtimes \lambda_2)$ for almost all cases. This means that we cannot directly conclude that typical representations for the inertial class s are precisely the sub-representations of (1.7).

We expect (at least when $\#k_F > 2$) that typical representations must occur as sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

We compared the dimensions of the above representation with that of (1.7) and observed that they are indeed different and this gave us the first heuristic to expect that there are non-typical irreducible sub-representations occurring in (1.7). We tried and indeed succeed in many cases to classify typical representations occurring in (1.7). This is the a new feature we show in this chapter. We can explain the main result by first recalling some aspects of the construction of Bushnell-Kutzko (semi-simple) type for s.

Let $[\mathfrak{A}_i, n_i, 0, \beta_i]$ be a simple strata defining the Bushnell-Kutzko type (J_i^0, λ_i) for $i \in 1,2$. We denote by e_1 and e_2 the ramification indices of the orders $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. We denote by ϕ_i the irreducible factor of the characteristic polynomial associated to the simple stratas $[\mathfrak{A}_i, n_i, 0, \beta_i]$ for $i \in \{1,2\}$ (see [BK93][Section 2.3]). We have broadly two cases.

1. $n_1/e_1 \neq n_2/e_2$; $n_1/e_1 = n_2/e_2$ but $\phi_1 \neq \phi_2$.

2. $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$

The representations σ_1 and σ_2 are called **completely distinct** if they satisfy condition (1). Otherwise they are said to have **common approximation**. The case of common approximation can be divided into 2 cases. The first is called homogeneous case i.e common approximation to level zero. The homogeneous case in our present situation (i.e both σ_1 and σ_2 are representations of $\mathrm{GL}_2(F)$) means that $\mathfrak{A}_1 = \mathfrak{A}_2 := \mathfrak{A}$, $n_1 = n_2 := n$ and $\beta_1 = \beta_2 := \beta$. And the simple character defining the β extension κ is also the same for σ_1 and σ_2. The second case is common approximation to level $l > 0$. Due to lack of time we do not treat the case where σ_1 and σ_2 admit common approximation to level $l > 0$. Our main theorem is:

Theorem 1.3.5. *Let $\#k_F > 3$ and s be the inertial class*

$$[\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2]$$

where σ_1 and σ_2 are completely distinct or homogenous. The typical representations for the inertial class s are precisely the irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

We will first consider the classification of typical representations for the homogeneous case. We sketch the proof in some detail:

The Bushnell-Kutzko's type (J_s, λ_s) for the inertial class $s = [M, \sigma_1 \boxtimes \sigma_2]$ with each σ_i containing type (J^0, λ_i) (which is defined by the simple strata $[\mathfrak{A}, n, 0, \beta]$) is given by $\lambda_s = \lambda_1 \boxtimes \lambda_2$ and

$$J_s := \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{(n-t)} \\ \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}$$

for $t = [n/2]$ and $E = F[\beta]$. One important observation is that $J_s \cap U \neq U(\mathcal{O}_F)$. This gives us the main problem. To tackle this situation we first try to modify the induced representation (1.7) at the level of subgroups close to the $P^0(n+1)$ and J_s. We will explain this in the unramified ($e(E|F) = 1$) case since most of notations are already defined.

The first step is to split the representation

$$\rho_1 := \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s).$$

The idea is to replace the group $P^0(n+1)$ with the group J_s and see the intertwining between ρ_1 and

$$\rho_2 := \mathrm{ind}_{J_s}^{P^0(t+1)}(\lambda_s).$$

But we cannot do this for the very basic reason that $P^0(t)$ does not contain the group J_s. Now we use a smaller group J_s' such that J_s' is contained in the group $P^0(t)$. The only change between the J_s' and J_s is their intersection with the lower unipotent group. This group J_s' has the property that $P^0(n+1)J_s' = P^0(t)$ and the induction

$$\rho_3 := \mathrm{ind}_{J_s'}^{P^0(t)}(\lambda_s)$$

is irreducible. We are in good situation since by Mackey decomposition the space of intertwining operators between ρ_1 and ρ_3 are one dimensional and any non-zero intertwining operator is surjective from the irreducibility of ρ_3. The rest of the proof is showing that the kernel of this non-trivial intertwining operator has irreducible sub-representations which also occur in ρ_1 for some suitably modified λ_1 and λ_2. The proof of this will now include the action of the group $U(\mathcal{O}_F)$. Which so far acts trivially on such inductions.

The non-trivial intertwining operator I between ρ_1 and ρ_3 is given by the following integral:

$$I(f)(p) = \int_{u^- \in\ P(s,t)\cap \bar{U}} f(u^- p) du^-.$$

If a function f is in the kernel of I then we have

$$\int_{u^- \in\ P^0(t)\cap \bar{U}} f(u^- u^+ (u^-)^{-1} u^-) du^- = 0$$

for all $u^+ \in P^0(t)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} \mathrm{id} & 0 \\ U^- & \mathrm{id} \end{pmatrix},\ u^+ = \begin{pmatrix} \mathrm{id} & U^+ \\ 0 & \mathrm{id} \end{pmatrix}$$

respectively. The above integral-equation can be written as

$$\int_{u^- \in\ P^0(t)\cap \bar{U}} \psi_{(\beta U^+ - U^+ \beta)}(1+U^-) f(u^-) du^- = 0. \tag{1.8}$$

We first note that the group of characters of the group

$$P^0(t+1)/P^0(n+1) \simeq (P^0(t+1)\cap \bar{U})/(P^0(n+1)\cap \bar{U}) \simeq \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$$

is isomorphic to $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. The kernel of I is spanned by the characters which are not in the image of $[\beta,\cdot]$ (the commutator bracket with β on $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$).

Let W be a sub-representation of $\ker(I)$. We will use the fact in lemma 6.2.3 show that irreducible sub-representations of

$$\mathrm{ind}_{P^0(t)}^{\mathrm{GL}_4(\mathcal{O}_F)}(W)$$

are non-typical with the help of co-restriction map.

Now we are left to understand the representation

$$\mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

The group J'_s is contained in J_s and by familiar transitivity and pull-back push-forward technique we write

$$\mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id}) \otimes (\lambda) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \oplus \Gamma$$

and we show that irreducible sub-representations of Γ are non-typical. Which shows that typical representations are precisely the irreducible sub-representations of the first summand in the above decomposition.

We point out that the proof of the facts about the kernel of the intertwining operator we introduced earlier depends on the exact-sequence machinery of Bushnell-Kutzko. The surjectivity of the operator also follows from the calculation of the sets $\mathfrak{N}(\mathfrak{A}, \beta)$. For the ramified case $(e(E|F) > 1)$ we will meet a situation to calculate $\mathfrak{N}(\mathfrak{A}, \beta)$ when $E = F[\beta]$ does not normalize the order $\mathfrak{A}$. This requires some attention otherwise we can use the Bushnell-Kutzko machinery to complete the classification.

Now we have to treat the case where σ_1 and σ_2 are completely distinct. A careful modification for the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_s)$$

gives the theorem in the case where σ_1 and σ_2 are completely distinct. We will use techniques similar to those developed to treat the homogenous case. This is the reason we choose to put this rather simpler situation at the end.