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Chapter 6

Collective emission in a cavity: roles of initial state and pure dephasing

In previous chapters, we experimentally investigated the interaction between Yb^{3+} ions and optical modes of a ring resonator. In the analysis and modelling of the experimental data, we assumed independent ions, i.e., the ions are not coupled to each other directly through dipole-dipole interactions or indirectly through the cavity modes. However, it is interesting to know the boundary of this assumption and to see effects beyond it. The dipole-dipole interaction between rare-earth ions has been intensively studied in the context of gain media for lasers and amplifiers. This interaction becomes appreciable when the doping concentration of rare-earth ions is sufficiently high leading to concentration quenching and cooperative emission. On the other hand, collective effects through the cavity modes have a quantum nature and are therefore much more subtle and fundamentally fascinating. In this chapter, we present theoretical study and numerical simulations of collective effects of atoms in a cavity.

6.1 Introduction

The problem of an ensemble of atoms that are initially in their excited states and coupled with each other through a common electromagnetic field was theoretically studied by Robert H. Dicke in 1954 [78]. He predicted that the atoms would correlate through their common field and emit photons with a rate faster than the spontaneous emission rate of an individual atom alone. He also coined a term "superradiance" for such a phenomenon. The Dicke model of superradiance represents a very important quantum many-body system that can be solved exactly and analytically, and has found its application in a large variety of systems [79, 80, 81]. Recent interests lie in the extension of the Dicke model to a steady state with constant excitation [82] and to more realistic cases in which either resonance frequencies or coupling constants of the atoms are inhomogeneous [83]. It has been demonstrated that inhomogeneous frequency detuning of the atoms with respect to the cavity field can transform the system from a "superradiant" state to a "subradiant" state (collectively slow emission). The influence of pure dephasing was studied in a steady-state case which showed that pure dephasing played a role of reduction of collectivity [84]. Dynamics of an ensemble of free-decaying (no excitation) atoms with pure dephasing has not been investigated yet. An extended Dicke model with different initial states and pure dephasing rates will complement the understanding of the collective emission effects for a more realistic system.

In this chapter we shall theoretically study the dynamics of an ensemble of atoms in a cavity in a collective-bad-cavity regime with different initial states and pure dephasing rates. The collective-bad-cavity regime is defined as $\sum_i^N g_i^2 \ll \kappa^2$, where N is the number of atoms, g_i is the coupling constant of the i-th atom to the cavity, and κ is the cavity decay rate. In such a regime, the cavity field can be adiabatically eliminated from the master equation of the system. We solve the master equation by using the quantum Monte Carlo method with maximally eight atoms in the cavity.

6.2 Description of the system

We consider an ensemble of N two-level atoms in a single-mode cavity. The atoms are located in a volume with all the dimensions much smaller than the wavelength of the cavity field such that the atoms experience the same phase of the cavity field. The distances between any two atoms are much larger than the dimension of the atomic wavefunction such that dipole-dipole interactions can be neglected and the atoms only interact with each other indirectly through the cavity field. The atoms are assumed to be on resonance with the cavity mode. All these requirements above are the same as in the original Dicke model.

The interaction Hamiltonian between the atoms and cavity field is given by

$$H_{\text{int}} = i\hbar \sum_i^N g_i (a^\dagger \sigma_-^i - a\sigma_+^i), \tag{6.1}$$

where g_i is the coupling constant of the i-th atom to the cavity field and is assumed to be a real number, a ($a^\dagger$) is the photon annihilation (creation) operator of the cavity field, and σ_-^i (σ_+^i) is the lowering (rising) operator of the i-th atom. The master equation for the density matrix of the atom-cavity system $\rho_{\rm ac}$ can be written as

$$\dot{\rho_{\rm ac}} = -\frac{i}{\hbar}[H_{\rm int}, \rho_{\rm ac}] + \mathcal{L}_{\rm c}\rho_{\rm ac} + \mathcal{L}_{\rm a}\rho_{\rm ac} + \mathcal{L}^*_{\rm a}\rho_{\rm ac}, \tag{6.2}$$

where $\mathcal{L}_{\rm c}$, $\mathcal{L}_{\rm a}$, and $\mathcal{L}^*_{\rm a}$ are the Lindblad superoperators accounting for the decay of the cavity field, decay of the atoms, and pure dephasing of the atoms, respectively. Their explicit expressions are

$$\begin{aligned}
\mathcal{L}_{\rm c}\rho_{\rm ac} &= \kappa(a\rho_{\rm ac}a^+ - a^+a\rho_{\rm ac}/2 - \rho_{\rm ac}a^+a/2), \\
\mathcal{L}_{\rm a}\rho_{\rm ac} &= \gamma\sum_i^N(\sigma^i_-\rho_{\rm ac}\sigma^i_+ - \sigma^i_+\sigma^i_-\rho_{\rm ac}/2 - \rho_{\rm ac}\sigma^i_+\sigma^i_-/2), \\
\mathcal{L}^*_{\rm a}\rho_{\rm ac} &= \frac{\gamma^*}{2}\sum_i^N(\sigma^i_z\rho_{\rm ac}\sigma^i_z - \sigma^i_z\sigma^i_z\rho_{\rm ac}/2 - \rho_{\rm ac}\sigma^i_z\sigma^i_z/2),
\end{aligned} \tag{6.3}$$

where κ is the cavity decay rate, γ is the atom decay rate, γ^* is the atom pure dephasing rate, and $\sigma^i_z = (\sigma^i_+\sigma^i_- - \sigma^i_-\sigma^i_+)/2$. In the collective-bad-cavity regime $\sum_i^N g_i^2 \ll \kappa^2$, the cavity field can be adiabatically eliminated and Eq. (6.2) is reduced to the master equation for the density matrix of the atoms $\rho_{\rm a}$:

$$\dot{\rho_{\rm a}} = \mathcal{L}_J\rho_{\rm a} + \mathcal{L}_a\rho_{\rm a} + \mathcal{L}^*_a\rho_{\rm a}, \tag{6.4}$$

with

$$\mathcal{L}_J\rho_{\rm a} = J_-\rho_{\rm a}J_+ - J_+J_-\rho_{\rm a}/2 - \rho_{\rm a}J_+J_-/2 \text{ and } J_\pm = \frac{2}{\sqrt{\kappa}}\sum_i^N g_i\sigma^i_\pm. \tag{6.5}$$

Equations (6.4) and (6.5) contain products of atomic lowering and rising operators of different atoms describing the effective interactions among the atoms through the cavity field. For $N = 1$, equation (6.4) is reduced to the master equation for a single atom with

$$\mathcal{L}^{N=1}_J\rho_{\rm a} = \frac{4g^2}{\kappa}(\sigma_-\rho_{\rm a}\sigma_+ - \sigma_+\sigma_-\rho_{\rm a}/2 - \rho_{\rm a}\sigma_+\sigma_-/2), \tag{6.6}$$

where the term $4g^2/\kappa$ is the Purcell-enhanced decay rate in addition to the natural decay rate γ.

6.3 The quantum Monte Carlo method

We solve the master equation by using the quantum Monte Carlo method. In this method a probable instance of the wavefunction $|\psi(t)\rangle$ coined as a quantum trajectory evolves continuously according to a modified Schrödinger equation with a non-Hermitian effective Hamiltonian H_{eff} as

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = H_{\text{eff}}|\psi(t)\rangle, \tag{6.7}$$

until this continuous evolution is interrupted by "quantum jumps". A quantum jump that occurs at time t_1 changes the quantum trajectory abruptly from $|\psi(t_{1-})\rangle$ to $|\psi(t_{1+})\rangle$ as

$$|\psi(t_{1+})\rangle = \frac{C_j(t_1)|\psi(t_{1-})\rangle}{\sqrt{\langle\psi(t_{1-})|C_j(t_1)^\dagger C_j(t_1)|\psi(t_{1-})\rangle}}, \tag{6.8}$$

where C_j is one of the collapse operators of the system. The probability density (probability per unit time) for a quantum jump due to C_j to happen at time t is given by

$$p_j(t) = \frac{\langle\psi(t)|C_j(t)^\dagger C_j(t)|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle}. \tag{6.9}$$

The effective Hamiltonian is given by

$$H_{\text{eff}} = H - \frac{i\hbar}{2}\sum_j C_j^\dagger C_j, \tag{6.10}$$

where H is the Hamiltonian of the system without dissipation, C_j ($C_j^\dagger$) are collapse (creation) operators as defined in the master equation of the system:

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_j (C_j\rho C_j^\dagger - C_j^\dagger C_j\rho/2 - \rho C_j^\dagger C_j/2). \tag{6.11}$$

A quantum trajectory can be simulated starting with an initial wavefunction $|\psi(t_0)\rangle$ evolving in a series of time intervals of $\mathrm{d}t$. In each time interval the probability of the quantum jump is calculated as $P(t) = \sum_j p_j(t)\mathrm{d}t$ and is compared with a random number ϵ in between 0 and 1. If $P(t) \leq \epsilon$, the wavefunction will evolve to $|\psi(t+\mathrm{d}t)\rangle$ according to Eq. (6.7), otherwise a quantum jump at time $t+\mathrm{d}t$ occurs according to Eq. (6.8) under a randomly chosen C_j with a weighted probability of $p_j(t)\mathrm{d}t/P(t)$. The expectation value of an arbitrary operator X for each trajectory can be evaluated as

$$\langle X(t)\rangle = \frac{\langle\psi(t)|X(t)|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle}. \tag{6.12}$$

A large number of quantum trajectories can be simulated according to above procedures and the averaged $\langle X(t)\rangle$ represents a solution of the master equation as given by Eq. (6.11). As to our case for the master equation as given by Eq. (6.4), the operators in the quantum Monte Carlo method explicitly read

$$H = 0,\ C_i = \sqrt{\gamma}\sigma_-^i,\ C_{i+N} = \sqrt{\gamma^*/2}\sigma_z^i,\ \text{and}\ C_{2N+1} = J_-. \tag{6.13}$$

We implement the quantum Monte Carlo method by using a Matlab code within a quantum optics toolbox developed by S. M. Tan [85]. In this chapter we focus on the case in which all the atoms share the same coupling constants, i.e. $g_i \equiv g$. We fix the parameters $g = 10^3\gamma$ and $\kappa = 10^6\gamma$ and vary the initial atomic states and the number of atoms N from 1 to 8 in the simulation. These parameters well satisfy the relation of $\gamma \ll \sqrt{N}g \ll \kappa$ in order to validate the collective-bad-cavity approximation. The Purcell-enhanced decay rate is $4g^2/\kappa = 4\gamma$ and therefore a single atom in the cavity decays with a rate of 5γ. We numerically calculate the expectation values of $J_+(t)J_-(t)$ denoted by $\langle J_+J_-\rangle$. $\langle J_+J_-\rangle$ possesses not only the terms of $\langle\sigma_+^i\sigma_-^i\rangle$ related to the emission intensity from individual atoms, but also the terms of $\langle\sigma_+^i\sigma_-^j\rangle$ $(i \neq j)$ as the correlation among the atoms. According to the input-output formalism of the cavity QED theory [86], $\langle J_+J_-\rangle$ is proportional to the optical power of the cavity field that leaks out of the cavity, a physical quantity that one can directly measure in the experiment.

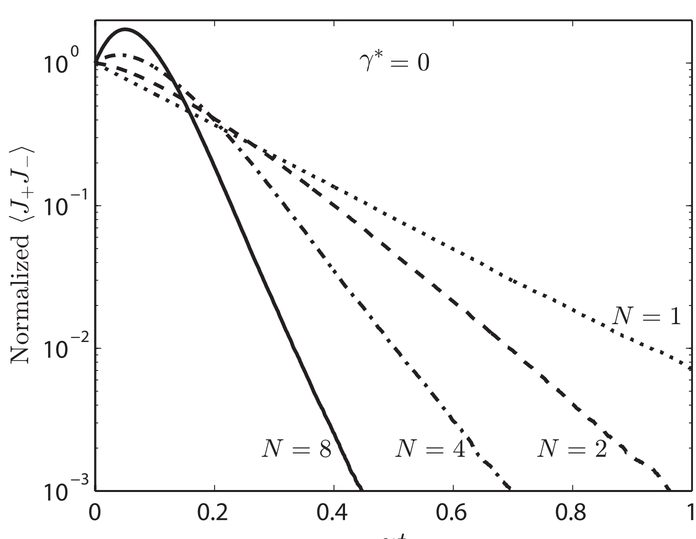

Figure 6.1: Time evolution of $\langle J_+J_-\rangle$ of $N = 1$ (dotted curve), 2 (dashed curve), 4 (dash-dot curve), and 8 (solid curve) atoms in a cavity. The atoms are all in their excited states at zero time and the pure dephasing rate is set to be zero.

6.4 Simulation results

We start with a simple example of N atoms initially in their excited states with $\gamma^* = 0$. The calculated $\langle J_+ J_- \rangle$ as a function of time are shown in Fig. 6.1 for 1, 2, 4, and 8 atoms. $\langle J_+ J_- \rangle$ for a single atom is a single exponential decay with a rate of 5γ as a result of the Purcell effect. With increasing N, $\langle J_+ J_- \rangle$ deviates from a single exponential function and tends to increase after the start of the evolution. The initial pulse of $\langle J_+ J_- \rangle$ is a demonstration of the Dicke's superradiance resulting from the self-formation of the correlations among the atoms through a common cavity field. The duration of the superradiance pulse $\tau_{\rm sr}$ and its rate $\gamma_{\rm sr}$ are approximately given by [78]

$$\tau_{\rm sr} = \frac{\kappa}{2Ng^2} \text{ and } \gamma_{\rm sr} = \frac{1}{\tau_{\rm sr}}. \tag{6.14}$$

We obtain $\tau_{\rm sr}\gamma = 1/8$ for $N = 4$ and $\tau_{\rm sr}\gamma = 1/16$ for $N = 8$. These timescales are approximately characterized in Fig. 6.1. After the superradiance pulse $\langle J_+ J_- \rangle$ decay approximately as single exponential functions with rates of 8.2γ, 12.1γ, and 21.0γ that are extracted from linear fits to the data for $N = 2$, 4, and 8, respectively.

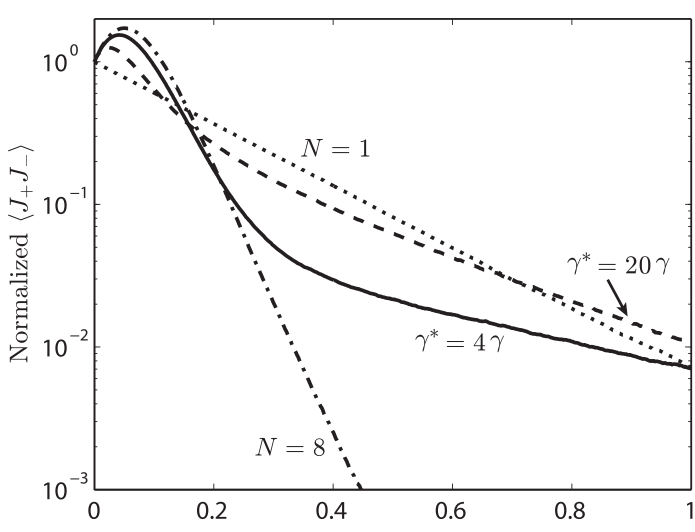

Figure 6.2: Time evolution of $\langle J_+ J_- \rangle$ of $N = 8$ atoms with $\gamma^* = 4\gamma$ (solid curve) and 20γ (dashed curve) in a cavity in contrast to the cases of $N = 1$ (dotted curve) and 8 (dash-dot curve) with $\gamma^* = 0$. The atoms are all in their excited states at zero time.

We add pure dephasing to the atoms and study the dynamics of $\langle J_+ J_- \rangle$. The effect of superradiance strongly depends on the quantum coherence of the atoms and generally requires $\gamma_{\rm sr} > \gamma^*$. We expect the superradiance to be weaker with a larger γ^* and to be significantly suppressed when γ^* is comparable with $\gamma_{\rm sr}$. The calculated $\langle J_+J_-\rangle$ as a function of time for 8 atoms with $\gamma^* = 4\gamma$ and 20γ are shown in Fig. 6.2 in contrast to the cases of $N = 1$ and $N = 8$ with $\gamma^* = 0$. The pure dephasing suppresses the superradiance pulse before the atoms reach the maximum correlation and this suppression is stronger for a larger γ^*. Notably, the evolution of $\langle J_+J_-\rangle$ after the superradiance pulse shows slower decay compared with a single atom. This effect has been studied for an ensemble of atoms with inhomogeneous transition frequencies and was referred to as subradiance [83]. Here we show similar effects caused by pure dephasing of the atoms with the same transition frequency on resonance with the cavity mode. $\langle J_+J_-\rangle$ for 8 atoms with $\gamma^* = 20\gamma$ that is larger than $\gamma_{\rm sr} = 16\gamma$ approaches that for a single atom.

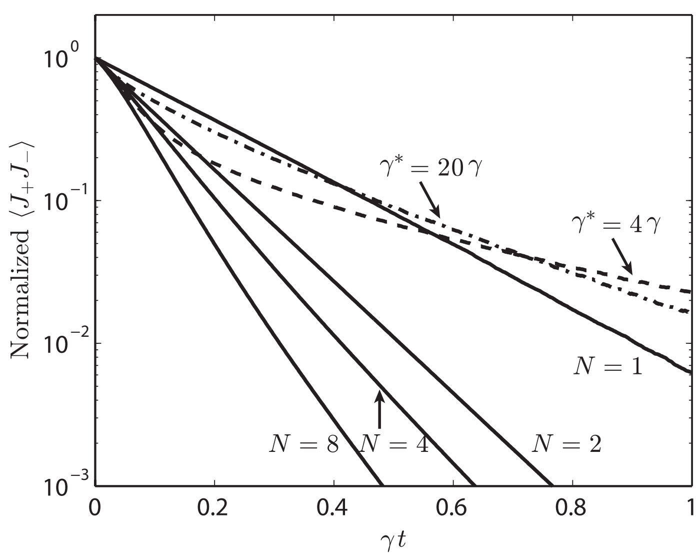

Figure 6.3: Time evolution of $\langle J_+J_-\rangle$ of $N = 1$, 2, 4, and 8 atoms with $\gamma^* = 0$ (solid curves) and $N = 8$ atoms with $\gamma^* = 4\gamma$ (dashed curves) and 20γ (dash-dot curve) in a cavity. For $N > 1$, initially half of the atoms are in their excited states and the other half of the atoms are in their ground states.

We then study the dynamics of initially half-inverted atoms, i.e., half of the atoms are in their excited states and the other half of the atoms are in their ground states. Figure 6.3 shows the calculated $\langle J_+J_-\rangle$ as a function of time for $N = 1$, 2, 4, and 8 atoms. $\langle J_+J_-\rangle$ do not exhibit superrandiance with or without pure dephasing. With a zero pure dephasing rate, $\langle J_+J_-\rangle$ approximately decay as single exponential functions with rates of 9.0γ, 11.0γ, and 14.7γ as extracted from linear fits to the data for $N = 2$, 4, and 8, respectively. With the pure dephasing, the initial decay rates become smaller compared with those without pure dephasing for the same number of atoms, and the atoms evolve into subradiant states similar with the case of fully inverted atoms. With a large value of $\gamma^* = 20\gamma$, $\langle J_+ J_- \rangle$ approaches that for a single atom.

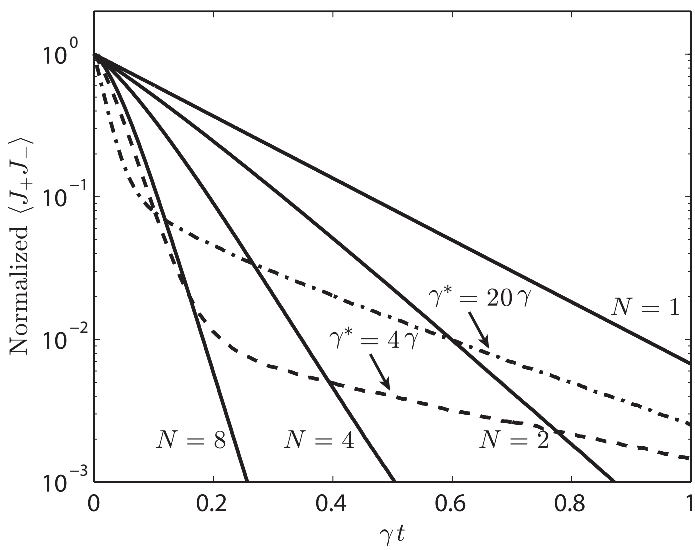

Figure 6.4: Time evolution of $\langle J_+ J_- \rangle$ of $N = 1$, 2, 4, and 8 atoms with $\gamma^* = 0$ (solid curves), and $N = 8$ atoms with $\gamma^* = 4\gamma$ (dashed curves) and 20γ (dash-dot curve) in a cavity. The atoms are initially in the superposition states of $(|g\rangle + |e\rangle)/\sqrt{2}$ with $|g\rangle$ and $|e\rangle$ being the ground and the excited states.

Here we also calculate $\langle J_+ J_- \rangle$ for the atoms that are initially in superposition states. Figure 6.4 shows the calculated $\langle J_+ J_- \rangle$ as a function of time for $N = 1$, 2, 4, and 8 atoms that are initially in states of $(|g\rangle + |e\rangle)/\sqrt{2}$ with $|g\rangle$ and $|e\rangle$ being the ground and the excited states. Without pure dephasing, the initial evolution of $\langle J_+ J_- \rangle$ exhibits a certain nonlinearity, but there is no superradiance effect. Afterwards $\langle J_+ J_- \rangle$ decay approximately as single exponential functions with rates of 8.4γ, 14.8γ, and 31.1γ as extracted from linear fits to the data for $N = 2$, 4, and 8 atoms, respectively. Upon the pure dephasing, the initial decay of $\langle J_+ J_- \rangle$ becomes even faster than that without the pure dephasing for the same number of atoms and this initial decay rate is higher for a larger value of γ^* in stark contrast to the case of half-inverted atoms. The atoms eventually evolve into subradiant states similar with the cases of fully and half-inverted atoms. The calculation for the atoms with the initial states of $(|g\rangle - |e\rangle)/\sqrt{2}$ yields exactly the same results as that with the initial states of $(|g\rangle + |e\rangle)/\sqrt{2}$.

We consider another initial states in which half of the atoms are in the superposition states of $(|g\rangle + |e\rangle)/\sqrt{2}$, while the other half of the atoms are in the superposition states of $(|g\rangle - |e\rangle)/\sqrt{2}$. Figure 6.5 shows the calculated $\langle J_+ J_- \rangle$ as a function of time for $N = 2$, 4, and 8 atoms. Without pure dephasing, the

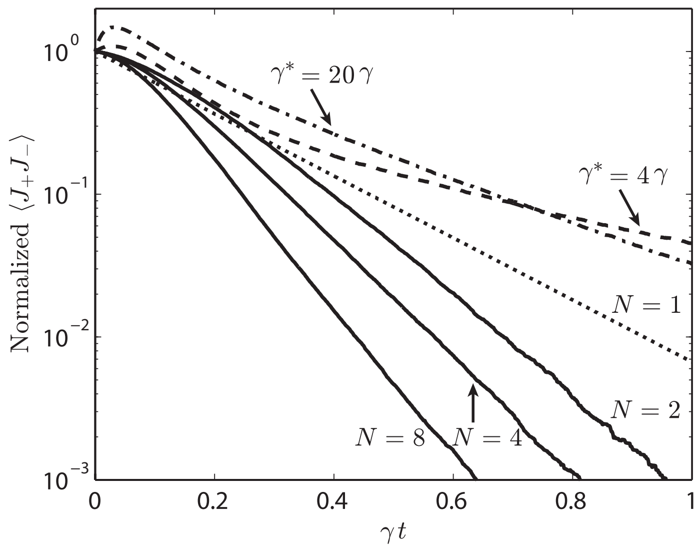

Figure 6.5: Time evolution of $\langle J_+ J_- \rangle$ of $N = 1$ (dotted line), 2, 4, and 8 (solid curves) atoms with $\gamma^* = 0$ and $N = 8$ atoms with $\gamma^* = 4\gamma$ (dashed curves) and 20γ (dash-dot curve) in a cavity. Initially half of the atoms are in the superposition states of $(|g\rangle + |e\rangle)/\sqrt{2}$, while the other half of the atoms are in the superposition states of $(|g\rangle - |e\rangle)/\sqrt{2}$.

atoms build up a certain amount of correlation after the start of the evolution which is however not sufficient to generate a superradiance pulse. $\langle J_+ J_- \rangle$ then decay approximately as single exponential functions with rates of 8.4γ, 9.6γ, and 11.2γ for $N = 2$, 4, and 8 atoms, respectively. Surprisingly, the atoms can build up higher correlation under pure dephasing with a superradiance-like pulse after the start of the evolution as shown in Fig. 6.5 and this effect becomes stronger for a larger value of γ^*. Figure 6.6 shows details of this effect in the initial time of $0.1/\gamma$ for $\gamma^* = 4\gamma$, 20γ, and 100γ. This pure dephasing induced correlation is a new phenomenon and has not been demonstrated before. After the pulse the atoms evolve into subradiant states.

In conclusion, we numerically studied the collective emission of an ensemble of atoms in a cavity in the collective-bad-cavity regime. We solved the master equation of the system with different initial states and different values of pure dephasing rate by using the quantum Monte Carlo method. Without pure dephasing, superradiance occurs for fully inverted atoms but does not occur for half-inverted atoms, neither for the atoms with the same superposition state, nor for the atoms with mixed superposition states. Pure dephasing reduces the correlation among the atoms for fully inverted atoms whereas it enhances the correlation among the atoms with mixed superposition states and generates a superradiance-like pulse. With pure dephasing, the atoms eventually evolve

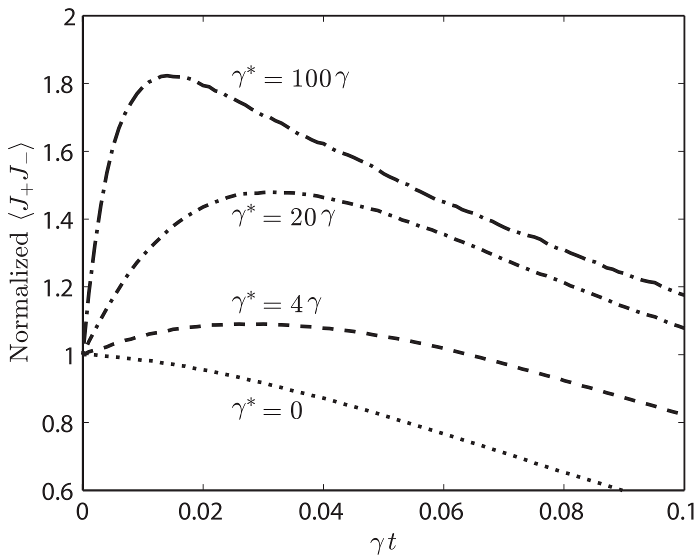

Figure 6.6: Time evolution of $\langle J_+ J_- \rangle$ of $N = 8$ atoms with $\gamma^* = 0$ (dotted curve), 4γ (dashed curve), 20γ (short-dash-dot curve), and 100γ (long-dash-dot curve) in a cavity. Initially half of the atoms are in the superposition states of $(|g\rangle + |e\rangle)/\sqrt{2}$, while the other half of the atoms are in the superposition states of $(|g\rangle - |e\rangle)/\sqrt{2}$.

into subradiant states for all the cases.