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**Author:** Gao, Ziyang

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# The mixed Ax-Lindemann theorem and its applications to the Zilber-Pink conjecture

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door

**Ziyang GAO**  
geboren te Dandong, Liaoning, China  
in 1988

Samenstelling van de promotiecommissie:

**Promotor:** Prof. dr. S.J.Edixhoven

**Promotor:** Prof. dr. E.Ullmo (IHÉS, Université Paris-Sud)

**Overige leden:**

Prof. dr. Y.André (CNRS, Université Paris-Diderot)

Prof. dr. B.Klingler (Université Paris-Diderot)

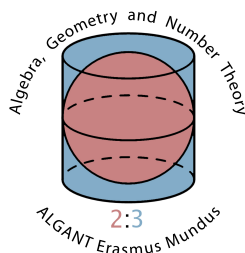
Prof. dr. B.Moonen (Radboud Universiteit Nijmegen)

Prof. dr. P.Stevenhagen

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*par*

Ziyang GAO

## Le théorème d'Ax-Lindemann mixte et ses applications à la conjecture de Zilber-Pink

Soutenue le 24 novembre 2014 devant la Commission d'examen :

|             |             |                              |             |
|-------------|-------------|------------------------------|-------------|
| M. Yves     | ANDRÉ       | CNRS et IMJ                  | Rapporteur  |
| M. Bas      | EDIXHOVEN   | Leiden University            | Directeur   |
| M. Bruno    | KLINGLER    | Université Paris-Diderot     | Rapporteur  |
| M. Ben      | MOONEN      | Radboud University Nijmegen  | Examinateur |
| M. Peter    | STEVENHAGEN | Leiden University            | Examinateur |
| M. Emmanuel | ULLMO       | IHÉS et Université Paris-Sud | Directeur   |



Thèse préparée au  
**Département de Mathématiques d'Orsay**  
Laboratoire de Mathématiques (UMR 8628), Bât. 425  
Université Paris-Sud  
91405 Orsay CEDEX

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