Stellingen

behorende bij het proefschrift
The mixed Ax-Lindemann theorem and its applications to the Zilber-Pink conjecture
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Let S be a connected mixed Shimura variety. A good, but still special example for S is the universal family $\mathfrak{A}_{g,M}$ over $\mathcal{A}_{g,M}$, the fine moduli space of principally polarized abelian varieties of dimension g with a level-M-structure ($M \geqslant 3$). Associated with S there is a pair $(P, \mathcal{X}^+)$, called a *connected mixed Shimura datum*, consisting of a $\mathbb{Q}$-group P and a topological space $\mathcal{X}^+$ together with a uniformization $\mathrm{unif}\colon \mathcal{X}^+ \to S$. It is known that S has a canonical structure of algebraic variety over $\overline{\mathbb{Q}}$. It is also known that there exists an embedding $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$ which realizes $\mathcal{X}^+$ as an open $\mathbb{R}$-semi-algebraic subset of a complex algebraic variety $\mathcal{X}^\vee$. A *special subvariety* of S is $[i]T$ for some Shimura morphism $[i]\colon T \to S$. A *weakly special subvariety* of S is an irreducible component of $[i]([\varphi]^{-1}(t'))$ for some Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ and $t' \in T'$.

1. Let B be an irreducible subvariety of $\mathcal{A}_{g,M}$ and let $X := [\pi]^{-1}(B)$, where $[\pi]\colon \mathfrak{A}_{g,M} \to \mathcal{A}_{g,M}$ denotes the universal family. Let $\mathcal{C}$ be the isotrivial part of $X \to B$, i.e. the largest isotrivial abelian subscheme of X over B. Assume that $\mathcal{C} \to B$ is a constant family. Then

   $$\{\text{translates of abelian subschemes of } X \to B \text{ by a torsion section and then by a constant section of } \mathcal{C} \to B\} = \{X \cap E \mid E \text{ weakly special in } S\}.$$

   (Proposition 1.2.14)

2. A subset F of S is weakly special iff F is a closed irreducible subvariety of S and one (and hence every) complex analytic irreducible component of $\mathrm{unif}^{-1}(F)$ is $\mathbb{R}$-semi-algebraic. (Remark 1.3.7 and Corollary 2.3.3)

3. Let Y be a closed irreducible subvariety of S and let $\widetilde{Y}$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$. Then there is a unique complex analytic irreducible component $\widetilde{F}$ of $\widetilde{Y}^{\mathrm{Zar}} \cap \mathcal{X}^+$ containing $\widetilde{Y}$, where $\widetilde{Y}^{\mathrm{Zar}}$ denotes the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$. Moreover, $\mathrm{unif}(\widetilde{F})$ is a weakly special subvariety of S. (Theorem 2.3.1)

4. For any $\mathbb{R}$-semi-algebraic subset $\widetilde{Z}$ of $\mathcal{X}^+$, every irreducible component Y of $\overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$ is weakly special. (Theorem 3.1.4)

5. Assume that S is of abelian type and let Σ be the set of special points of S (a good, but still special example is $S = \mathfrak{A}_{g,M}$, in which case the points of Σ

are precisely the points corresponding to the torsion points on the CM abelian varieties). Let Y be a closed irreducible subvariety of S such that $Y \cap \Sigma$ is Zariski dense in Y. Then Y is a special subvariety of S under each of the two conditions: (1) the pure part of S is a subvariety of $\mathcal{A}_{6,M}^N$, e.g. $S = \mathfrak{A}_{6,M}^N$; (2) assume the generalized Riemann hypothesis. (Theorem 4.3.1)

6. Let $S = \mathfrak{A}_{g,M}$ and let Σ be the generalized Hecke orbit of $s \in S$. Let Y be a closed irreducible subvariety of S such that $Y \cap \Sigma$ is Zariski dense in Y. Then Y is weakly special under each of the three conditions: (1) s is a special point; (2) s is a torsion point on its fiber and Y is contained in an abelian scheme over a curve; (3) $s \in \mathfrak{A}_{g,M}(\overline{\mathbb{Q}})$ and Y is a curve. (Theorem 4.3.2, Theorem 5.1.4 and Theorem 5.1.5)

7. For A an abelian variety over a number field $k \subset \mathbb{C}$ and t a torsion point of $A(\mathbb{C})$, denote by $N(t)$ its order and $k(t)$ the field of definition of t over k. Let $g, d \in \mathbb{N}_+$ and let $\varepsilon \in (0,1)$. There exists $c > 0$ s.t. for all number fields $k \subset \mathbb{C}$ of degree d over $\mathbb{Q}$, all g-dimensional CM abelian varieties A over k and all torsion points t in $A(\mathbb{C})$,
$$[k(t) : k] \geqslant cN(t)^{1-\varepsilon}.$$
(A result of Silverberg in *Torsion points on abelian varieties of CM-type*, Compositio Mathematica, 68:241-249, 1988. Reproved by me in the dissertation as Corollary 4.2.4)

8. Consider $\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm) := \varprojlim_K \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K$ where K runs over the open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$. We have
$$\pi_0(\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm)) = \mathrm{GL}_2(\mathbb{A}_f)/\overline{\mathrm{GL}_2(\mathbb{Q})^+} = \widehat{\mathbb{Z}}^*$$
but $\pi_0^{\mathrm{path}}(\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm)) = \mathrm{GL}_2(\mathbb{A}_f)/\,\mathrm{GL}_2(\mathbb{Q})^+$. They do not coincide since $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{A}_f)$.

9. Groups are useful to study mathematical objects. The world of Shimura varieties is ruled by groups.

10. Coffee and math are closely related. In any mathematical institute there is at least one kind of device concerning coffee, even if there is no coffee machine.