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Chapter 5

From André-Oort to André-Pink-Zannier

5.1 Main results

5.1.1 Background

In the last chapter we have studied the André-Oort conjecture, which is a subconjecture of the Zilber-Pink conjecture. In particular we have proved a weaker version of the André-Oort conjecture (Theorem 4.3.2). This weaker version corresponds to another important case of the Zilber-Pink conjecture, which we call the André-Pink-Zannier conjecture. The goal of this chapter is to study this André-Pink-Zannier conjecture.

In the whole chapter, we restrict to the case $\mathfrak{A}_g \xrightarrow{[\pi]} \mathcal{A}_g$.

Conjecture 5.1.1. *Let Y be a subvariety of $\mathfrak{A}_g$. Let $s \in \mathfrak{A}_g$ and Σ be the generalized Hecke orbit of s. If $\overline{Y \cap \Sigma} = Y$, then Y is weakly special.*

Several cases of this conjecture had been studied by André before its final form was made by Pink [54, Conjecture 1.6]. It is also closely related to a problem (Conjecture 5.1.3) proposed by Zannier. Pink has also proved [54, Theorem 5.4] that Conjecture 5.1.1 implies that Mordell-Lang conjecture.

Conjecture 5.1.1 for $\mathcal{A}_g$, the pure part of $\mathfrak{A}_g$, has been intensively studied by Orr [43, 42], generalizing the previous work of Habegger-Pila [24, Theorem 3] with the Pila-Zannier method.

The set Σ has good moduli interpretation: by Corollary 5.2.5,

$$\begin{aligned}\Sigma &= \text{division points of the polarized isogeny orbit of } s \\ &= \{t \in \mathfrak{A}_g \mid \exists n \in \mathbb{N} \text{ and a polarized isogeny} \\ &\qquad f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t}) \text{ such that } nt = f(s)\}.\end{aligned} \tag{5.1.1}$$

There are authors who consider isogenies instead of polarized isogenies. However this does not essentially improve the result because of Zarhin's trick (see [42, Proposition 4.4]): for any isogeny $f\colon A \to A'$ between polarized abelian varieties, there exists $u \in \mathrm{End}(A^4)$ such that $f^4 \circ u\colon A^4 \to A'^4$ is a polarized isogeny. See §5.5 for more details.

Although Conjecture 5.1.1 and the André-Oort conjecture do not imply each other, they do have some overlap, which for $\mathfrak{A}_g$ is precisely Theorem 4.3.2 when $S = \mathfrak{A}_g$.

We shall divide Conjecture 5.1.1 into two cases: when s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$ and when s is not a torsion point of $\mathfrak{A}_{g,[\pi]s}$. The diophantine estimates for both cases are not quite the same.

5.1.2 The torsion case

When s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$, this conjecture is related to a special-point problem proposed by Zannier. We define the following "special topology" proposed by Zannier:

Definition 5.1.2. *Fix a point $a \in \mathcal{A}_g$. Then a corresponds to a principally polarized abelian variety (A_a, λ_a) of dimension g.*

1. *We say that a point $t \in \mathfrak{A}_g$ is A_a-**special** (or a-**special**) if there exists an isogeny $A_a \to \mathfrak{A}_{g,[\pi]t}$ and that t is a torsion point on the abelian variety $\mathfrak{A}_{g,[\pi]t}$. We shall denote by Σ'_a (or Σ' when there is no confusion) the set of a-special points.*

2. *We say that a point $t \in \mathfrak{A}_g$ is (A_a, λ_a)-**special** if there exists a polarized isogeny $(A_a, \lambda_a) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and that t is a torsion point on the abelian variety $\mathfrak{A}_{g,[\pi]t}$. We shall denote by Σ_a (or Σ when there is no confusion) the set of (A_a, λ_a)-special points.*

3. *We say that a subvariety Z of $\mathfrak{A}_g$ is a-**special** if Z contains an a-special point, $[\pi]Z$ is a totally geodesic subvariety of $\mathcal{A}_g$ and Z is an irreducible component of a subgroup of $[\pi]^{-1}([\pi]Z)$.*

In view of Proposition 1.2.15, every a-special subvariety is weakly special. The following conjecture is proposed by Zannier.

Conjecture 5.1.3. *Let Y be a subvariety of $\mathfrak{A}_g$ and let $a \in \mathcal{A}_g$. If $\overline{Y \cap \Sigma'_a} = Y$, then Y is a-special.*

By (5.1.1), Conjecture 5.1.1 when s is a torsion point of $\mathfrak{A}_{g,[\pi]s}$ is equivalently to a weaker version of Conjecture 5.1.3, i.e. replace Σ'_a by Σ_a in Conjecture 5.1.3. However by [42, Proposition 4.4], Conjecture 5.1.1 for $\mathfrak{A}_{4g}$ also implies Conjecture 5.1.3 for $\mathfrak{A}_g$. Our first main result is:

Theorem 5.1.4. *Conjecture 5.1.3 holds if* $\dim([\pi](Y)) \leqslant 1$.

The proof of this theorem will be presented in §5.3. Remark that by Corollary 5.2.6, the case where $\dim([\pi]Y) = 0$ (i.e. $[\pi](Y)$ is a point) is nothing but the Manin-Mumford conjecture, which has been proved by many people (the first proof was given by Raynaud).

5.1.3 The non-torsion case

The situation becomes more complicated when s is not a torsion point of $\mathfrak{A}_{g,[\pi]s}$. In this case we prove:

Theorem 5.1.5. *Conjecture 5.1.1 holds if $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ and Y is a curve.*

As we have seen in Theorem 1.1.34, $\mathfrak{A}_g$ is defined over $\overline{\mathbb{Q}}$. Hence it is reasonable to talk about its $\overline{\mathbb{Q}}$-points. Moreover, if $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$, then its generalized Hecke orbit Σ is also contained in $\mathfrak{A}_g(\overline{\mathbb{Q}})$ by Corollary 5.2.6. Hence if $\overline{Y \cap \Sigma} = Y$, then Y itself is defined over $\overline{\mathbb{Q}}$. The proof of this theorem will be presented in §5.4.

5.2 Generalized Hecke orbits in $\mathfrak{A}_g$

In this section, we discuss the matrix expression of a polarized isogeny and then compute the generalized Hecke orbit of a point of $\mathfrak{A}_g$.

5.2.1 Polarized isogenies and their matrix expressions

Let $b \in \mathcal{A}_g$. Denote by $A_b = \mathfrak{A}_{g,b}$ and denote by $\lambda_b \colon A_b \xrightarrow{\sim} A_b^\vee$ the principal polarization induced by $\mathfrak{L}_{g,b}$. Then the point b corresponds to the polarized abelian variety (A_b, λ_b). Let $\mathcal{B}$ be a symplectic basis of $H_1(A_b, \mathbb{Z})$ w.r.t. the polarization λ_b. Let $\widetilde{b} \in \mathbb{H}_g^+$ be the period matrix of A_b w.r.t. the basis $\mathcal{B}$. In this subsection, we fix $\mathcal{B}$ to be the $\mathbb{Q}$-basis of V_{2g}.

Consider all points $b' \in \mathcal{A}_g$ such that there exists a polarized isogeny

$$f \colon (A_b, \lambda_b) \to (A_{b'}, \lambda_{b'})$$

where $(A_{b'}, \lambda_{b'}) = (\mathfrak{A}_{g,b'}, A_{b'} \xrightarrow{\sim} A_{b'}^\vee \text{ induced by } \mathfrak{L}_{g,b'})$. Let $\mathcal{B}'$ be a symplectic basis of $H_1(A_{b'}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{b'}$ and let $\widetilde{b}' \in \mathbb{H}_g^+$ be the period matrix of $A_{b'}$ w.r.t. the basis $\mathcal{B}'$.

Definition 5.2.1. *The matrix $\alpha \in \mathrm{GSp}_{2g}(\mathbb{Q})^+ \cap \mathrm{M}_{2g\times 2g}(\mathbb{Z})$ associated to*

$$f_* \colon H_1(A_b, \mathbb{Z}) \to H_1(A_{b'}, \mathbb{Z})$$

*in terms of $\mathcal{B}$ and $\mathcal{B}'$ is called the **rational representation of** f w.r.t. $\mathcal{B}$ and $\mathcal{B}'$.*

The periods $\widetilde{b}$ and $\widetilde{b}'$ are related by α in the following way:

$$\widetilde{b} = \alpha^t \cdot \widetilde{b}' = (A\widetilde{b}' + B)(C\widetilde{b}' + D)^{-1}, \text{ where } \alpha^t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \widetilde{b}, \widetilde{b}' \in \mathbb{H}_g^+ \subset M_{g\times g}(\mathbb{C}).$$

Under the $\mathbb{Q}$-basis $\mathcal{B}$ of V_{2g}, the matrix α^t corresponds to the dual isogeny of f, i.e. the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}'} & \xrightarrow{\alpha^t \cdot} & (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}}, \quad (v, \widetilde{b}') \mapsto \left(\alpha^t v, \alpha^t \widetilde{b}'\right) = \left(\alpha^t v, \widetilde{b}\right) \\ \downarrow \scriptstyle{\mathrm{unif}} & & \downarrow \scriptstyle{\mathrm{unif}} \\ A_{b'} & & A_b \\ \downarrow \scriptstyle{\lambda_b} \wr & & \downarrow \scriptstyle{\lambda_{b'}} \wr \\ A_{b'}^\vee & \xrightarrow{f^\vee} & A_b^\vee \end{array} \quad . \tag{5.2.1}$$

However, since f is a polarized isogeny, $f^*\mathfrak{L}_{g,b'} = \mathfrak{L}_{g,b}^{\otimes(\deg f)^{1/g}}$. So the following diagram commutes:

$$\begin{array}{ccc} A_b & \xrightarrow{\ f\ } & A_{b'} \\ {\scriptstyle [(\deg f)^{1/g}]\circ\lambda_b}\downarrow & & \downarrow{\scriptstyle \lambda_{b'}}\ \wr \\ A_b^\vee & \xleftarrow{\ f^\vee\ } & A_{b'}^\vee \end{array}. \tag{5.2.2}$$

Therefore by (5.2.1) and (5.2.2), we get the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b}} & \xrightarrow{(\deg f)^{1/g}(\alpha^t)^{-1}\cdot} & (\mathcal{X}_{2g,\mathrm{a}}^+)_{\widetilde{b'}} \\ {\scriptstyle \mathrm{unif}}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}} \\ A_b & \xrightarrow{\ f\ } & A_{b'} \end{array}. \tag{5.2.3}$$

Definition 5.2.2. *The matrix* $(\deg f)^{1/g}(\alpha^t)^{-1}$ *is called the* ***matrix expression of*** f *in coordinates* $\mathcal{B}$ *w.r.t.* $\mathcal{B}'$.

Remark 5.2.3.

1. *The two bases* $\mathcal{B}$ *and* $\mathcal{B}'$ *play different roles for the matrix expression of* f*: the matrix expression of* f *depends on both bases because it depends on the period matrices determined by these bases, but its dependence on* $\mathcal{B}$ *is more important because we fix* $\mathcal{B}$ *to be the* $\mathbb{Q}$*-basis for* V_{2g} *when writing the matrix expression.*

2. *It is good to give the matrix* $(\deg f)^{1/g}(\alpha^t)^{-1}$ *a name because we will use it several times in the proof of Theorem 5.1.5. The name "matrix expression" is given by the author. Remark that this definition only works for polarized isogenies because* (5.2.2) *fails for general non-polarized isogenies.*

5.2.2 Generalized Hecke orbits in $\mathfrak{A}_g$

Lemma 5.2.4. *Let* $\varphi \in \mathrm{Aut}\big((P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)\big)$. *Then there exist* $g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ *and* $v_0 \in V_{2g}(\mathbb{Q})$ *such that the action of* φ *on* $\mathcal{X}_{2g,\mathrm{a}}^+$ *is given by*

$$\varphi\left((v,x)\right) = (g'v + v_0, g'x).$$

Proof. Notice that $\varphi(V_{2g}) = \varphi(\mathcal{R}_u(P_{2g,\mathrm{a}})) \subset \mathcal{R}_u(P_{2g,\mathrm{a}}) = V_{2g}$. Since every two Levi decompositions of $P_{2g,\mathrm{a}}$ differ by the conjugation by an element $v_0 \in V_{2g}(\mathbb{Q})$, there exists a $v_0 \in V_{2g}(\mathbb{Q})$ such that $\psi := \mathrm{Int}(v_0)^{-1} \circ \varphi$ maps $(\{0\} \times \mathrm{GSp}_{2g}, \{0\} \times \mathbb{H}_g^+)$ to itself. Now ψ maps V_{2g} and $(\mathrm{GSp}_{2g}, H_g^+)$ to themselves. So ψ can be written as (A, B), where $A \in \mathrm{GL}_{2g}(\mathbb{Q})$ and $B \in \mathrm{Aut}\big((\mathrm{GSp}_{2g}, \mathbb{H}_g^+)\big) = \mathrm{GSp}_{2g}(\mathbb{Q})^+$. Remark that $\psi \in \mathrm{Aut}(P_{2g,\mathrm{a}})$, so we can do the following computation:

For any $v \in V_{2g}(\mathbb{Q})$ and $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$,

$$\begin{aligned}(Ahv, BhB^{-1}) &= \psi((hv,h)) = \psi((0,h)(v,1)) = \psi(0,h)\psi(v,1) \\ &= (0, BhB^{-1})(Av,1) = (BhB^{-1}Av, BhB^{-1}).\end{aligned}$$

Because v is an arbitrary element of $V_{2g}(\mathbb{Q})$, this implies that $Ah = BhB^{-1}A$ for any $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$. But this tells us that $A^{-1}B$ commutes with any element of $\mathrm{GSp}_{2g}(\mathbb{Q})^+$, and hence $A^{-1}B \in \mathbb{G}_m(\mathbb{Q})$. So ψ acts on the group $P_{2g,\mathrm{a}}$ as $\psi((v,h)) = (cBv, BhB^{-1})$ where $c \in \mathbb{Q}^*$ and $B \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$. Therefore ψ acts on $\mathcal{X}^+_{2g,\mathrm{a}}$ as $\psi((v,x)) = (cBv, Bx) = (cBv, cBx)$. Denote by $g' := cB \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$, then the action of φ on $\mathcal{X}^+_{2g,\mathrm{a}}$ is given by

$$\varphi\left((v,x)\right) = (g'v + v_0, g'x).$$

□

Let $s \in \mathfrak{A}_g$, then $[\pi]s \in \mathcal{A}_g$ corresponds to the polarized abelian variety $(\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})$.

Corollary 5.2.5. *Let $s \in \mathfrak{A}_g$. Then a point t is in the generalized Hecke orbit of s iff there exist a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $n' \in \mathbb{N}$ such that $f(s) = n't$.*

Proof. Let $(v,x) \in \mathcal{X}^+_{2g,\mathrm{a}}$ (resp. $(v_t,x_t) \in \mathcal{X}^+_{2g,\mathrm{a}}$) be such that $s = \mathrm{unif}\left((v,x)\right)$ (resp. $t = \mathrm{unif}\left((v_t,x_t)\right)$). Then by Proposition 1.1.31 and Lemma 5.2.4, t is in the generalized Hecke orbit of s iff

$$(v_t, x_t) = (g'v + v_0, g'x) \tag{5.2.4}$$

for some $g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and $v_0 \in V_{2g}(\mathbb{Q})$.

If (5.2.4) is satisfied, then there exists $c \in \mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^*$ s.t $h := c^{-1}g' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ is a $\mathbb{Z}$-coefficient matrix. Hence h corresponds to a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$. We have $t = \mathrm{unif}\left((chv + v_0, x_t)\right)$ by (5.2.4), and therefore

$$n't = m'f(s) + \mathrm{unif}\left((v_0, x_t)\right)$$

where $c = m'/n'$. But $\mathrm{unif}\left((v_0,x_t)\right)$ is a torsion point of $\mathfrak{A}_{g,[\pi]t}$ since $v_0 \in V_{2g}(\mathbb{Q})$, and therefore can be removed by replacing m' and n' by sufficient large multiples. On the other hand $m'f$ is still a polarized isogeny, and hence replacing f by $m'f$, we may assume $m' = 1$. Finally we may assume $n' \in \mathbb{N}$ by possibly replacing f by $-f$.

Conversly, suppose that there exist a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $n' \in \mathbb{N}$ such that $f(s) = n't$. Let $\mathcal{B}_s$ (resp. $\mathcal{B}_t$) be a symplectic basis of $H_1(\mathfrak{A}_{g,[\pi]s}, \mathbb{Z})$ (resp. $H_1(\mathfrak{A}_{g,[\pi]t}, \mathbb{Z})$) and let h be the matrix expression of f in coordiante $\mathcal{B}_s$ w.r.t. $\mathcal{B}_t$. Then $h \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and there exists $(\gamma_V, \gamma_G) \in \Gamma$ such that

$$(n'v_t, x_t) = (\gamma_V, \gamma_G)(hv, hx) = (\gamma_V + \gamma_G hv, \gamma_G hx).$$

Now $g' := \gamma_G h/n' \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ and $v_0 := \gamma_V/n' \in V_{2g}(\mathbb{Q})$ satisfy (5.2.4). □

Corollary 5.2.6. *Let $s \in \mathfrak{A}_g$ and t be a point in the generalized Hecke orbit of s. Let $f_t\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ be a polarized isogeny of minimal degree. Then there exist*

- *a point $s_0 \in \mathfrak{A}_{g,[\pi]s}$;*
- *$\varphi \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$;*
- *$n_0 \in \mathbb{N}$*

such that $s = n_0 s_0$ and

$$f_t(\varphi(s_0) + p) = t$$

for some torsion point $p \in \mathfrak{A}_{g,[\pi]s}$.

Proof. By Corollary 5.2.5, there exist a polarized isogeny $f\colon (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s}) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ and $m', n' \in \mathbb{N}$ such that $p_1 := m'f(s) - n't$ is a torsion point of $\mathfrak{A}_{g,[\pi]t}$. Now $f_t^{-1} \circ f \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big) \otimes \mathbb{Q}$, i.e. there exist $\varphi' \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$ and $n_0' \in \mathbb{N}$ such that $f_t^{-1} \circ f = \varphi' \otimes (1/n_0)$. So $n_0' \circ f = f_t \circ \varphi'$ and hence

$$m' f_t(\varphi'(s)) = m'n_0' f(s) = n_0'(n't + p_1) = n_0'n't + n_0 p_1.$$

Let $\varphi := m' \circ \varphi' \in \mathrm{End}\big((\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})\big)$ and $n_0 := n_0'n' \in \mathbb{N}$, then there exists a torsion point $p_2 \in \mathfrak{A}_{g,[\pi]t}$ such that

$$f_t(\varphi(s)) = n_0 t + p_2.$$

Hence the conclusion follows. ☐

5.3 Proof for the torsion case

5.3.1 Preliminary

In this subsection, we fix some definitions and notation for the proof of Theorem 5.1.4.

Let $a \in \mathcal{A}_g$. The point $a \in \mathcal{A}_g$ corresponds to the polarized abelian variety $(A_a, \lambda_a) := (\mathfrak{A}_{g,a}, \lambda_a)$. We use Σ instead of Σ_a to denote the set of all (A_a, λ_a)-special points of $\mathfrak{A}_g$. Let $\mathrm{unif}\colon \mathcal{X}_{2g,\mathrm{a}}^+ \to \mathfrak{A}_g$ be the uniformization map and let $\mathcal{F}$ be the fundamental set in $\mathcal{X}_{2g,\mathrm{a}}^+$ defined as in Theorem 1.1.34.(3). Let

$$\widetilde{Y} := \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ and } \widetilde{\Sigma} := \mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}.$$

Let $\mathcal{B}$ be a symplectic basis for $H_1(A_a, \mathbb{Z})$ w.r.t. the polarization λ_a. Let $\widetilde{a}$ be the period matrix of A_a w.r.t. the chosen basis $\mathcal{B}$. In the rest of the paper, we shall sometimes identify $\widetilde{a} \in \mathbb{H}_g^+$ and $(0, \widetilde{a}) \in \{0\} \times \mathbb{H}_g^+ \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$.

For any $t \in \Sigma$, there exists by definition of Σ_a a polarized isogeny $(A_a, \lambda_a) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$. Besides, t is a torsion point of $A_{[\pi]t} := \mathfrak{A}_{g,[\pi]t}$, whose order we denote by $N(t)$.

Definition 5.3.1. *For any $t \in \Sigma$, define its* ***complexity*** *to be*

$$\max\left(\textit{minimum degree of polarized isogenies } (A_a, \lambda_a) \to (A_{[\pi]t}, \lambda_{[\pi]t}), N(t)\right).$$

Besides, define the ***complexity*** *of any point of $\widetilde{\Sigma}$ to be the complexity of its image in Σ.*

5.3.2 Application of Pila-Wilkie

The goal of this subsection is to prove the following proposition:

Proposition 5.3.2. *Let Y, $\widetilde{a}$ be as in the last subsection. Let $\varepsilon > 0$. There exists a constant $c = c(Y, \widetilde{a}, \varepsilon) > 0$ with the following property:*

For every $n \geqslant 1$, there exist at most cn^ε definable blocks $B_i \subset \widetilde{Y}$ such that $\cup B_i$ contains all points of complexity at most n in $\widetilde{Y} \cap \widetilde{\Sigma}$.

Lemma 5.3.3. *There exist constants c', κ depending only on g and $\widetilde{a}$ such that*

For any $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists $(v, h) \in P_{2g}(\mathbb{Q})^+$ such that $(v, h)\widetilde{a} = \widetilde{t}$ and $H((v, h)) \leqslant c'n^\kappa$.

Proof. Let $t = \mathrm{unif}(\widetilde{t})$. By [43, Proposition 4.1], there exist

- a polarized isogeny $f\colon \mathfrak{A}_{g,[\pi]t} \to A_a$;
- a symplectic basis $\mathcal{B}'$ for $H_1(\mathfrak{A}_{g,[\pi]t}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{[\pi]t}$

such that the rational representation h_1 of f w.r.t. the chosen bases satisfies that $H(h_1)$ is polynomially bounded by $\deg(f)$.

But $\mathrm{unif}_G(h_1^t\widetilde{a}) = [\pi]t$ by (5.2.3). Hence there exists a $h_2 \in \Gamma_G$ such that $h_2h_1^t\widetilde{a} = \pi(\widetilde{t}) \in \mathcal{F}_G$. By [49, Lemma 3.2], $H(h_2)$ is polynomially bounded by the norm of $h_1^t \cdot \widetilde{a}$.

Now define $h := h_2h_1^t$. We have then $h\widetilde{a} = \pi(\widetilde{t})$ and

$$H(h) \leqslant c_0 \deg(f)^{\kappa_0}$$

where $c_0 > 0$ and $\kappa_0 > 0$ depend only on g and $\widetilde{a}$.

Next write $\widetilde{t} = (\widetilde{t}_V, \pi(\widetilde{t})) \in \mathcal{F}$. Let $v := \widetilde{t}_V$, then $v \in V_{2g}(\mathbb{Q})$ since t is a torsion point of $\mathfrak{A}_{g,[\pi]t}$. Besides, the denominator of v is precisely the order of the torsion point t. But by choice, $\mathcal{F} \simeq [0, N)^{2g} \times \mathcal{F}_G \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ (see Theorem 1.1.34.(3)). Therefore up to a constant depending on nothing, $H(v)$ is bounded by its denominator, i.e. the order of the torsion point t of $\mathfrak{A}_{g,[\pi]t}$.

To sum it up, (v, h) is the element of $P_{2g}(\mathbb{Q})^+$ which we desire. □

Now we can prove Proposition 5.3.2 with the help of Lemma 5.3.3.

Proof of Proposition 5.3.2. Let

$$\sigma\colon P_{2g}(\mathbb{R})^{+} \to \mathcal{X}_{2g,\mathrm{a}}^{+}$$
$$(v,h) \mapsto (v,h)\widetilde{a}$$

The set $R := \sigma^{-1}(\widetilde{Y}) = \sigma^{-1}(\mathrm{unif}^{-1}(Y) \cap \mathcal{F})$ is definable because σ is semi-algebraic and $\mathrm{unif}|_{\mathcal{F}}$ is definable. Hence we can apply the family version of the Pila-Wilkie theorem ([48, 3.6]) to the definable set R: for every $\varepsilon > 0$, there are only finitely many definable block families $B^{(j)}(\varepsilon) \subset R \times \mathbb{R}^m$ and a constant $C_1(R,\varepsilon)$ such that for every $T \geqslant 1$, the rational points of R of height at most T are contained in the union of at most $C_1 T^{\varepsilon}$ definable blocks $B_i(T,\varepsilon)$, taken (as fibers) from the families $B^{(j)}(\varepsilon)$. Since σ is semi-algebraic, the image under σ of a definable block in R is a finite union of definable blocks in $\widetilde{Y}$. Furthermore the number of blocks in the image is uniformly bounded in each definable block family $B^{(j)}(\varepsilon)$. Hence $\sigma(B_i(T,\varepsilon))$ is the union of at most $C_2 T^{\varepsilon}$ blocks in $\widetilde{Y}$, for some new constant $C_2(Y,\widetilde{a},\varepsilon) > 0$.

By Lemma 5.3.3, for any point $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists a rational element $\gamma \in R$ such that $\sigma(\gamma) = \widetilde{t}$ and $H(\gamma) \leqslant c' n^{\kappa}$. By the discussion in the last paragraph, all such γ's are contained in the union of at most $C_1(c'n^{\kappa})^{\varepsilon}$ definable blocks. Therefore all points of $\widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n are contained in the union of at most $C_1 C_2 c'^{\varepsilon} n^{\kappa\varepsilon}$ blocks in $\widetilde{Y}$. □

5.3.3 Galois orbit

In this section we shall deal with the Galois orbit. We handle the case of $\overline{\mathbb{Q}}$-points at first and then use the standard specialization argument to prove the result for general points of $\Sigma \cap Y$.

Proposition 5.3.4. *Suppose $a \in \mathcal{A}_g(\overline{\mathbb{Q}})$. There exist positive constants $c'_1 = c'_1(g)$, $c'_2 = c'_2(g,k(a))$ and $c'_3 = c'_3(g)$ satisfying the following property:*

For any point $t \in \Sigma \cap Y \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$ of complexity n,

$$[k(t):\mathbb{Q}] \geqslant c'_1 \frac{n^{c'_2}}{\max\left(1, h_F(A_a)\right)^{c'_3}}$$

where $k(t)$ is the definition field of t.

Proof. Define (as Gaudron-Rémond [21])

$$\kappa(\mathfrak{A}_{g,[\pi]t}) := ((14g)^{64g^2}[k([\pi]t):\mathbb{Q}]\max(h_F(\mathfrak{A}_{g,[\pi]t}), \log[k([\pi]t):\mathbb{Q}], 1)^2)^{1024g^3}.$$

Take a point $t \in \Sigma \cap Y \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$ of complexity n. Denote by $k([\pi]t)$ the definition field of $[\pi]t$. Denote by $N(t)$ the order of t as a torsion point of $A_{[\pi]t} := \mathfrak{A}_{g,[\pi]t}$. There are two cases.

Case i $n =$ minimum degree of polarized isogenies $(A_a, \lambda_a) \to (A_{[\pi]t}, \lambda_{[\pi]t})$. Then by [21, Théorème 1.4] and [42, Theorem 5.6],

$$n \leqslant \kappa(\mathfrak{A}_{g,[\pi]t}).$$

On the other hand, by a result of Faltings [16, Chapter II, §4, Lemma 5],

$$h_F(\mathfrak{A}_{g,[\pi]t}) \leqslant h_F(A_a) + (1/2)\log n.$$

Now the conclusion for this case follows from the two inequalities above and the easy fact $[k(t) : \mathbb{Q}] \geqslant [k([\pi]t) : \mathbb{Q}]$.

Case ii $n = N(t)$. By [21, Théorème 1.2], there exist positive natural numbers l, simple abelian varieties A_1,...,A_l over a finite extension k' of $k([\pi]t)$ (A_i and A_j can be isogenious to each other over $\overline{\mathbb{Q}}$ for $i \neq j$) and an isogeny

$$\varphi \colon \mathfrak{A}_{g,[\pi]t} \to \prod_{i=1}^{l} A_i \tag{5.3.1}$$

such that φ is defined over k', $\deg\varphi \leqslant \kappa(\mathfrak{A}_{g,[\pi]t})$ and $[k' : k([\pi]t)] \leqslant \kappa(\mathfrak{A}_{g,[\pi]t})^g$. Call $p_i \colon A \to A_i$ the composite of φ and the i-th projection $\prod_{i=1}^{l} A_i \to A_i$ ($\forall i = 1, ..., l$).

Now $t \in A$ is a torsion point of order $N(t)$. Without any loss of generality we may assume

$$N(p_1(t)) \geqslant N(p_i(t))$$

where $N(p_i(t))$ is the order of $p_i(t)$ as a torsion point of A_i.

Lemma 5.3.5.

$$N(t) \leqslant \kappa(\mathfrak{A}_{g,[\pi]t})N(p_1(t))^g \text{ and } [k(t) : \mathbb{Q}] \geqslant [k(p_1(t)) : \mathbb{Q}]/\kappa(\mathfrak{A}_{g,[\pi]t})^{2g}.$$

where $k(p_1(t))$ is the definition field of $p_1(t)$.

Proof. Denote by $N(\varphi(t))$ the order of $\varphi(t)$ as a torsion point of $\prod_{i=1}^{l} A_i$. We have

$$N(\varphi(t)) \geqslant N(t)/\deg\varphi \geqslant N(t)/\kappa(\mathfrak{A}_{g,[\pi]t}).$$

On the other hand, $N(\varphi(t)) = \mathrm{lcd}(N(p_1(t)), ..., N(p_l(t))) \leqslant N(p_1(t))^g$. Now the first inequality follows.

For the second inequality, first of all since φ and $\prod_{i=1}^{l} A_i$ are both defined over k', we have

$$[k(\varphi(t)) : \mathbb{Q}] \leqslant [k(t)k' : \mathbb{Q}] = [k(t) : \mathbb{Q}][k(t)k' : k(t)] \leqslant [k(t) : \mathbb{Q}][k' : k] \leqslant [k(t) : \mathbb{Q}]\kappa(\mathfrak{A}_{g,[\pi]t})^g.$$

Next since all abelian varieties A_1,...,A_l are defined over k', we have then

$$[k(\varphi(t))k' : \mathbb{Q}] \geqslant [k(p_1(t)) : \mathbb{Q}].$$

But

$$\begin{aligned}[k(\varphi(t))k' : \mathbb{Q}] &= [k(\varphi(t))k' : k'][k' : k][k : \mathbb{Q}] \\ &\leqslant [k(\varphi(t)) : k][k' : k][k : \mathbb{Q}] \\ &= [k(\varphi(t)) : \mathbb{Q}][k' : k] \\ &\leqslant [k(\varphi(t)) : \mathbb{Q}]\kappa(\mathfrak{A}_{g,[\pi]t})^g.\end{aligned}$$

Now the second inequality follows from the three inequalities above. □

By [17, Corollaire 1.5],

$$[k(p_1(t)) : \mathbb{Q}] \geqslant c_0'(g)\frac{N(p_1(t))^{1/(2g)}}{\log N(p_1(t))(h_F(A_1) + \log N(p_1(t)))}. \tag{5.3.2}$$

By the comment below [21, Corollaire 1.5], we have

$$h_F(A_1) \leqslant h_F(\mathfrak{A}_{g,[\pi]t}) + \frac{1}{2}\log\kappa(\mathfrak{A}_{g,[\pi]t}). \tag{5.3.3}$$

By assumption, there exists an isogeny $A_a \to \mathfrak{A}_{g,[\pi]t}$ of degree $\leqslant n$. So by Faltings [16, Chapter II, §4, Lemma 5],

$$h_F(\mathfrak{A}_{g,[\pi]t}) \leqslant h_F(A_a) + (1/2)\log n. \tag{5.3.4}$$

Now because $[k(t) : \mathbb{Q}] \geqslant [k([\pi]t) : \mathbb{Q}]$, the conclusion of *Case ii* now follows from Lemma 5.3.5, (5.3.2), (5.3.3) and (5.3.4). □

Corollary 5.3.6. *Suppose that a is defined over a finitely generated field k. There exist positive constants $c_1 = c_1(A_a, k)$ and $c_2 = c_2(A_a, k)$ satisfying the following property:*

For any point $t \in \Sigma \cap Y$ of complexity n defined over a finitely extension $k(t)$ of k,

$$[k(t) : k] \geqslant c_1 n^{c_2}.$$

Proof. This follows from Proposition 5.3.4 and a specialization argument. The case where $n =$ minimum degree of polarized isogenies $(A_a, \lambda_a) \to (A_{[\pi]t}, \lambda_{[\pi]t})$ is proved by Orr [43, Theorem 5.1] (possibly combined with [42, Theorem 5.6]). The case where $n = N(t)$, the order of t as a torsion point of $\mathfrak{A}_{g,[\pi]t}$, follows from the standard specialization argument introduced by Raynaud (see [43, Section 5] and [56, Section 7]). □

5.3.4 End of the proof for the torsion case

In this section, Y is always an irreducible subvariety of $\mathfrak{A}_g$, $a \in \mathcal{A}_g$ and Σ is the set of all a-strongly special points of $\mathfrak{A}_g$.

Theorem 5.3.7. *If $\overline{Y \cap \Sigma} = Y$, then the union of all positive-dimensional weakly special subvarieties contained in Y is Zariski dense in Y.*

Proof. Let Σ_1 be the set of points $t \in Y \cap \Sigma$ such that there is a positive-dimensional block $B \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B)$. Let Y_1 be the Zariski closure of Σ_1. Let k be the finitely generated field $k(a)$. Enlarge k if necessary such that both Y and Y_1 are defined over k.

Let t be a point in $Y \cap \Sigma$ of complexity n. By Corollary 5.3.6, there exist positive constants c_1 and c_2 depending only on g, A_a and k such that

$$[k(t) : k] \geqslant c_1 n^{c_2/2}.$$

But all $\mathrm{Gal}(\overline{k}/k)$-conjugates of t are contained in $Y \cap \Sigma$ and have complexity n. By Proposition 5.3.2, the preimages in $\mathcal{F}$ of these points are contained in the union of $c(Y, \widetilde{a}, c_2/4)n^{c_2/4}$ definable blocks, each of these blocks being contained in $\widetilde{Y}$.

For n large enough, $c_1 n^{c_2/2} > cn^{c_2/4}$. Hence for $n \gg 0$, there exists a definable block $B \subset \widetilde{Y}$ such that $\mathrm{unif}(B)$ contains at least two Galois conjugates of t, and therefore $\dim B > 0$ since blocks are connected. So being in $\mathrm{unif}(B)$, those conjugates of t are in Σ_1. But Y_1 is defined over k, so $t \in Y_1$.

In summary, all points of $Y \cap \Sigma$ of large enough complexity are in Σ_1. This excludes only finitely many points of $Y \cap \Sigma$. So $Y_1 = Y$.

Let Σ_2 be the set of points $t \in Y \cap \Sigma$ such that there is a connected positive-dimensional semi-algebraic set $B' \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B')$. Let Y_2 be the Zariski closure of Σ_2. By definition of blocks, $\Sigma_2 = \Sigma_1$, and hence $Y_2 = Y_1 = Y$.

But for any connected semi-algebraic set $B' \subset \widetilde{Y}$, the Ax-Lindemann theorem (in the form of Theorem 3.1.4) implies that every irreducible component of $\overline{\mathrm{unif}(B')}$, whose dimension is positive if $\dim(B') > 0$, is weakly special. Now the conclusion follows. □

Proof of Theorem 5.1.4. Let S be the smallest connected mixed Shimura subvariety containing Y. Assume S is associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$. Let $(G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/\mathcal{R}_u(P)$. By Theorem 4.1.3 and Theorem 5.3.7, such a non-trivial group N exists: N is the maximal normal subgroup of P such that the followings hold:

- there exists a diagram of Shimura morphisms

$$\begin{array}{ccccc} (P,\mathcal{X}^+) & \xrightarrow{\rho} & (P',\mathcal{X}'^+) := (P,\mathcal{X}^+)/N & \xrightarrow{\pi'} & (G',\mathcal{X}'^+_G) := (P',\mathcal{X}'^+)/\mathcal{R}_u(P') \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}'} & & \downarrow{\scriptstyle \mathrm{unif}'_G} \\ S & \xrightarrow{[\rho]} & S' & \xrightarrow{[\pi']} & S'_G \end{array}$$

  (then S' is by definition a connected Shimura variety of Kuga type)

- the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is not Zariski dense in Y';

- $Y = [\rho]^{-1}(Y')$.

We prove the theorem by induction on g. When $g = 1$, the only non-trivial case is when Y is a curve. But then Y must be weakly special by Theorem 4.1.3 (Or more simply, one can use Theorem 2.3.3 to avoid using the Ax-Lindemann theorem). Remark that this case has also been proved by André [3, Lecture 4] when he proposed the mixed André-Oort conjecture.

When $\dim([\pi](Y)) = 0$, this is the Manin-Mumford conjecture by Corollary 5.2.6. Hence we only have to treat the case $\dim([\pi](Y)) = 1$. Remark that in this case $[\pi](Y)$ is weakly special by the main result of [43], and hence equals $\mathrm{unif}_G\,(G''(\mathbb{R})^+\widetilde{y})$ for some $G'' < \mathrm{GSp}_{2g}$ of positive dimension and $\widetilde{y} \in \mathbb{H}_g^+$. Now there are two cases:

If $\dim([\pi'](Y')) = 0$, then $[\pi'](Y')$ is a point. In this case Y' is a subvariety of an abelian variety. The hypothesis $\overline{Y \cap \Sigma} = Y$ implies that Y' contains a Zariski dense subset of torsion points. Therefore by the result of the Manin-Mumford conjecture, Y' is a special subvariety, i.e. the translate of an abelian subvariety by a torsion point. But the union of positive-dimensional weakly special subvarieties which are contained in Y' is not Zariski dense in Y', so Y' is a point. Therefore Y is weakly special by definition.

If $\dim([\pi'](Y')) = 1$, then $N/\mathcal{R}_u(N)$ is trivial because the dimension of $[\pi](Y) = \mathrm{unif}_G\,(G''(\mathbb{R})^+\widetilde{y})$ is 1. Therefore $V_N := \mathcal{R}_u(N) < V_{2g}$ is non-trivial since N is non-trivial.

Denote for simplicity by $B := [\pi'](Y') = \mathrm{unif}'_G(G''(\mathbb{R})^+\rho(\widetilde{y}))$ and $X := [\pi']^{-1}(B)$. Then $X \to B$ is a family of abelian varieties of dimension g'. We have $g' < g$ since V_N is non-trivial. Besides, $X \to B$ is non-isotrivial because otherwise G'' acts trivially on V_{2g}/V_N, and therefore $G'' \lhd P'$. This contradicts the maximality of N. Hence there exists, up to taking finite covers of $X \to B$, a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & \mathfrak{A}_{g'} \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\;i_B\;} & \mathcal{A}_{g'} \end{array}$$

such that both i and i_B are finite. Apply induction hypothesis to $i(Y') \subset \mathfrak{A}_{g'}$, we get that $i(Y')$ is weakly special. By the geometric interpretation of weakly special subvarieties (Proposition 1.2.15), $i^{-1}(i(Y'))$ is irreducible. Therefore $Y' = i^{-1}(i(Y'))$ since they are of the same dimension. So Y' is a weakly special subvariety of S' (again by Proposition 1.2.15). But then Y' must be a point because the union of the positive-dimensional weakly special subvarieties contained in Y' is not Zariski dense in Y'. Hence Y is weakly special by definition. □

5.4 Proof for the non-torsion case

We prove Theorem 5.1.5 in this section. Let Y be a curve over $\overline{\mathbb{Q}}$ in $\mathfrak{A}_g$, let $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$ and let Σ be the generalized Hecke orbit of s. Then $\Sigma \subset \mathfrak{A}_g(\overline{\mathbb{Q}})$.

For simplicity, we will denote by $(A, \lambda) := (\mathfrak{A}_{g,[\pi]s}, \lambda_{[\pi]s})$ the polarized abelian variety attached to $[\pi](s)$ in this section. Assume that s is not a torsion point of A. Through all this section, we assume that Y is not contained in a fiber of $[\pi]\colon \mathfrak{A}_g \to \mathcal{A}_g$ (otherwise this is a special case of the Mordell-Lang conjecture, which is proved by a series of work of Vojta, Faltings and Hindry).

We fix some notation here. Let $\mathcal{B}$ be a symplectic basis of $H_1(A, \mathbb{Z})$ w.r.t. the polarization λ. Let $\widetilde{s}_G \in \mathbb{H}_g^+$ be the period matrix of (A, λ) w.r.t. the basis $\mathcal{B}$, then $\mathrm{unif}_G(\widetilde{s}_G) = [\pi]s$. Now let $\widetilde{s} = (\widetilde{s}_V, \widetilde{s}_G) \in V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ be a point in $\pi^{-1}(\widetilde{s}_G) \cap \mathrm{unif}^{-1}(s)$. In the whole section, we will fix $\mathcal{B}$ to be the $\mathbb{Q}$-basis of V_{2g} as in §5.2.1.

Denote by k the definition field of s. Then A is defined over the number field k.

5.4.1 Complexity of points in a generalized Hecke orbit

Let $\mathrm{unif}\colon \mathcal{X}_{2g,\mathrm{a}}^+ \to \mathfrak{A}_g$ be the uniformization map and let $\mathcal{F}$ be the fundamental set in $\mathcal{X}_{2g,\mathrm{a}}^+$ defined in Theorem 1.1.34.(3). Let

$$\widetilde{Y} := \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ and } \widetilde{\Sigma} := \mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}.$$

Let $t \in \Sigma$. Let f_t be as in Corollary 5.2.6 (i.e. a polarized isogeny $(A, \lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ of minimum degree). Define

$$n_t := \min\{n \in \mathbb{N} \mid \exists \varphi \in \mathrm{End}\left((A, \lambda)\right) \text{ such that } nt \in f_t\big(\varphi(s) + A(\overline{\mathbb{Q}})_{\mathrm{tor}}\big)\}.$$

The existence of such an n_t is guaranteed by Corollary 5.2.6. Furthermore, let $s_t := \mathrm{unif}\left((\widetilde{s}_V/n_t, \widetilde{s}_G)\right) \in \mathfrak{A}_{g,[\pi]s} = A$. Then there exist by definition of n_t

- $\varphi_t \in \mathrm{End}\left((A, \lambda)\right)$;
- δ_t a torsion point of A

such that

$$f_t\left(\varphi_t(s_t) + \delta_t\right) = t. \tag{5.4.1}$$

The notation n_t, f_t, φ_t, s_t and δ_t will be used through the whole section.

Definition 5.4.1. *Define the* ***complexity*** *of $t \in \Sigma$ to be*

$$\max\left(n_t, N(\delta_t)\right)$$

where $N(\delta_t)$ is the order of δ_t. Besides, define the ***complexity*** *of any point of $\widetilde{\Sigma}$ to be the complexity of its image in Σ.*

The fact that this complexity is a "good enough" parameter will be proved in §5.4.3.

5.4.2 Galois orbit

In contrast to the torsion case, we deal with the Galois orbit at first for the non-torsion case. Keep the notation of the beginning of this section and §5.4.1.

Proposition 5.4.2. *Let $t \in \Sigma$ be of complexity n, then*

$$[k(t):\mathbb{Q}] \geqslant c_3 n^{c_4}$$

where $c_3 = c_3(A,\lambda,s)$ and $c_4 = c_4(A,\lambda,s)$ are two positive constants.

Proof. By [21, Théorème 1.2] and [42, Theorem 5.6], there exist positive constants $c_5 = c_5(A,\lambda)$ and $c_6 = c_6(A,\lambda)$ such that

$$\deg(f_t) \leqslant c_5[k(t):\mathbb{Q}]^{c_6} \tag{5.4.2}$$

The abelian variety A is defined over k. By the main result of [34], there exist two positive constants c_9 and c_{10} depending only on A and k such that for any torsion point $q \in A$ of order $N(q)$, we have

$$[k(q):\mathbb{Q}] \geqslant c_9 N(q)^{c_{10}}. \tag{5.4.3}$$

Case i $N(\delta_t)^{c_{10}/2} \geqslant n_t^{2g^2+4g+1}$. By [26, Proposition 1] or [36, Theorem 2.1.2], there exists a positive constant $c_{11} = c_{11}(A,s,k)$ such that

$$\mathrm{Gal}\left(k(\varphi_t(s_t), A[n_t])/k(A[n_t])\right) \leqslant c_{11} n_t^{2g}.$$

Hence

$$[k(\varphi_t(s_t)):\mathbb{Q}] \leqslant c'_{11} n_t^{2g^2+4g+1} \tag{5.4.4}$$

for another positive constant c'_{11} depending only on A, s and k. Now by (5.4.4), (5.4.3) and the assumption for this case,

$$[k(\varphi_t(s_t),\delta_t):k(\varphi_t(s_t))] \geqslant c_{12}\frac{N(\delta_t)^{c_{10}}}{n_t^{2g^2+4g+1}} \geqslant c_{12} N(\delta_t)^{c_{10}/2} \tag{5.4.5}$$

for a positive constant $c_{12} = c_{12}(A,s,k)$.

Since A is defined over the number field k, every element of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ induces a homomorphism $A(\overline{\mathbb{Q}}) \to A(\overline{\mathbb{Q}})$, and hence a homomorphism $A \to A$. It is not hard to prove the following claim:

Claim 5.4.3. *For any $\sigma_1, \sigma_2 \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/k(\varphi_t(s_t))\right)$, $\sigma_1(\varphi_t(s_t)+\delta_t) = \sigma_2(\varphi_t(s_t)+\delta_t)$ iff $\sigma_2^{-1}\sigma_1 \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/k(\varphi_t(s_t),\delta_t)\right)$.*

This claim implies $[k(\varphi_t(s_t)+\delta_t):\mathbb{Q}] \geqslant [k(\varphi_t(s_t),\delta_t):k(\varphi_t(s_t))]$. Hence by (5.4.5),

$$[k(\varphi_t(s_t)+\delta_t):\mathbb{Q}] \geqslant c_{12} N(\delta_t)^{c_{10}/2}.$$

Since $t = f_t(\varphi_t(s_t)+\delta_t)$, we have therefore

$$[k(t):\mathbb{Q}] \geqslant c_{12}\frac{N(\delta_t)^{c_{10}/2}}{\deg(f_t)}. \tag{5.4.6}$$

Now the conclusion for this case follows from (5.4.2), (5.4.6) and the definition of complexity (recall that k is the definition field of s, and therefore depends only on s).

Case ii $N(\delta_t)^{c_{10}/2} \leqslant n_t^{2g^2+4g+1}$. Roughly speaking, this case follows from the Kummer theory [26, Appendix 2]. Here are the details of the proof:

Let $\Delta := \mathrm{End}\left((A,\lambda)\right) s$ and let $\overline{\Delta} := \mathrm{End}(A)s \subset A$. Then $\overline{\Delta}$ is a finitely generated subgroup of A. Let k' be the smallest number field over which all points of $\overline{\Delta}$ are defined, then k' depends only on A and s. Then by the Mordell-Weil theorem, $A(k')$ is a finitely generated subgroup of A. By definition of k', $\overline{\Delta} \subset A(k')$. Let $\Delta' := \mathbb{Q}\Delta \cap A(k')$ and let $\overline{\Delta}' := \mathbb{Q}\overline{\Delta} \cap A(k')$. Then $\overline{\Delta}'$ is again a finitely generated subgroup of A. It contains $\overline{\Delta}$ and $\mathrm{rank}\,\overline{\Delta}' = \mathrm{rank}\,\overline{\Delta}$. Therefore $[\overline{\Delta}' : \overline{\Delta}]$ is a finite number depending only on k', and hence only on A and s. On the other hand, $\Delta \subset \overline{\Delta} \cap \Delta' \subset \Delta + A(k')_{\mathrm{tor}}$. So $[\overline{\Delta} \cap \Delta' : \Delta]$ is a finite number depending only on k', and hence only on A and s. Therefore by

$$[\Delta' : \Delta] = [\Delta' : \overline{\Delta} \cap \Delta'][\overline{\Delta} \cap \Delta' : \Delta] \leqslant [\overline{\Delta}' : \overline{\Delta}][\overline{\Delta} \cap \Delta' : \Delta],$$

there exists $c_{13} > 0$ depending only on A and s such that $[\Delta' : \Delta] = c_{13}$.

For each $t \in \Sigma$, define another number $n_t' := \min\{n \in \mathbb{N} |\ nt \in f_t\big(A(k') + A(\overline{\mathbb{Q}})_{\mathrm{tor}}\big)\}$. Let $s' \in A(k')$ be such that $n_t' t = f_t(s' + A(\overline{\mathbb{Q}})_{\mathrm{tor}})$. Then because $t = f_t(\varphi_t(s_t) + \delta_t)$, we have

$$s^\dagger := s' - n_t'\varphi_t(s_t) \in A(\overline{\mathbb{Q}})_{\mathrm{tor}}.$$

So $s' \in n_t'\varphi_t(s_t) + A(\overline{\mathbb{Q}})_{\mathrm{tor}} \subset \mathbb{Q}\Delta$, and therefore $n_t'\varphi_t(s_t) + s^\dagger = s' \in \Delta'$. So

$$n_t' = \min\{n \in \mathbb{N} |\ nt \in f_t(\Delta' + A(\overline{\mathbb{Q}})_{\mathrm{tor}})\}. \tag{5.4.7}$$

However by definition,

$$n_t = \min\{n \in \mathbb{N} |\ nt \in f_t(\Delta + A(\overline{\mathbb{Q}})_{\mathrm{tor}}). \tag{5.4.8}$$

Compare (5.4.7) and (5.4.8), we get

$$n_t/n_t' \leqslant [\Delta' : \Delta] = c_{13}. \tag{5.4.9}$$

By [26, Lemma 14] or [36, Corollary 2.1.5], there exists a positive constant $c_{14} = c_{14}(A, k') = c_{14}(A, s)$ such that

$$\mathrm{Gal}\Big(k'\big(\varphi_t(s_t), A[n_t' N(\delta_t)]\big)/k'\big(A[n_t' N(\delta_t)]\big)\Big) \geqslant c_{14} n_t'.$$

Hence

$$[k(t) : \mathbb{Q}] \geqslant \frac{[k'(\varphi_t(s_t) + \delta_t) : \mathbb{Q}]}{\deg(f_t)[k' : k]} \geqslant \frac{c_{14} n_t'}{\deg(f_t)[k' : k]}. \tag{5.4.10}$$

Now the conclusion follows from (5.4.2), (5.4.9) and (5.4.10) (remark that $[k' : k]$ is a constant depending only on A and s). $\square$

5.4.3 Néron-Tate height in family

Next we prove that the complexity defined in Definition 5.4.1 is a good parameter. More explicitly we have the following proposition:

Proposition 5.4.4. *Let Y be as in the beginning of this section. Let $t \in Y(\overline{\mathbb{Q}}) \cap \Sigma$. Let f_t, n_t, s_t, φ_t and δ_t be as in §5.4.1. Then*

$$\deg(\varphi_t) \leqslant c_7 n_t^{c_8} \quad \text{and} \quad \deg(f_t) \leqslant c_7' n_t^{c_8'}$$

for some positive constants $c_7 = c_7(g, Y, s)$, $c_7' = c_7'(g, Y, s)$ and $c_8 = c_8(g, Y, s)$, $c_8' = c_8'(g, Y, s)$.

We shall prove this proposition with help of a well-chosen family of Néron-Tate heights, i.e. the one related to the $\mathbb{G}_m$-torsor $\mathfrak{L}_g$ defined in Theorem 1.1.34. Then we shall use a theorem of Silverman-Tate [60, Theorem A].

By Theorem 1.1.34(2), $\mathfrak{L}_g \to \mathfrak{A}_g$ is a symmetric and relatively ample $\mathbb{G}_m$-torsor w.r.t. $\mathfrak{A}_g \to \mathcal{A}_g$. Now consider the Néron-Tate height $\widehat{h}_{\mathfrak{L}_{g,b}}$ on A_b for each $b \in \mathcal{A}_g(\overline{\mathbb{Q}})$. For any $s \in \mathfrak{A}_g(\overline{\mathbb{Q}})$, we shall denote by

$$\widehat{h}_{\mathfrak{L}_g}(s) := \widehat{h}_{\mathfrak{L}_{g,[\pi]s}}(s).$$

Lemma 5.4.5. *Let s_1 and s_2 be two points of $\mathfrak{A}_g(\overline{\mathbb{Q}})$. Assume that there exists a polarized isogeny*

$$f\colon (\mathfrak{A}_{g,[\pi]s_1}, \lambda_{[\pi]s_1}) \to (\mathfrak{A}_{g,[\pi]s_2}, \lambda_{[\pi]s_2})$$

such that $s_1 = f(s_2)$. Then $\widehat{h}_{\mathfrak{L}_g}(s_2) = (\deg f)^{1/g}\widehat{h}_{\mathfrak{L}_g}(s_1)$.

Proof. By the moduli interpretation of $\mathfrak{L}_g$ (Theorem 1.1.34(3)), $f^*\mathfrak{L}_{g,[\pi]s_2} = \mathfrak{L}_{g,[\pi]s_1}^{\otimes(\deg f)^{1/g}}$. So we have

$$\begin{aligned}
\widehat{h}_{\mathfrak{L}_g}(s_2) &= \widehat{h}_{\mathfrak{L}_{g,[\pi]s_2}}(f(s_1)) \\
&= \widehat{h}_{\mathfrak{L}_{g,[\pi]s_1}^{\otimes(\deg f)^{1/g}}}(s_1) \\
&= (\deg f)^{1/g}\widehat{h}_{\mathfrak{L}_{g,[\pi]s_1}}(s_1) \\
&= (\deg f)^{1/g}\widehat{h}_{\mathfrak{L}_g}(s_1).
\end{aligned}$$

□

Now we begin the proof of Proposition 5.4.4.

Proof of Proposition 5.4.4. Denote by $\varepsilon\colon \mathcal{A}_g \to \mathfrak{A}_g$ the zero section.

By Theorem 1.1.34(6), we can apply [60, Theorem A]: there exist constants $c_{15} = c_{15}(g) > 0$ and $c_{16} = c_{16}(g)$ such that

$$|\widehat{h}_{\mathfrak{L}_g}(t) - h_{\mathfrak{A}_g,\mathfrak{L}_g}(t)| < c_{15} h_{\mathcal{A}_g,\varepsilon^*\mathfrak{L}_g}([\pi]t) + c_{16} \tag{5.4.11}$$

for any $t \in \mathfrak{A}_g(\overline{\mathbb{Q}})$.

We need the following lemma, which uses the fact that Y is a curve in an essential way:

Lemma 5.4.6. *There exist two constants $c_{17} > 0$ and c_{18} depending only on Y such that*

$$h_{\mathfrak{A}_g,\mathfrak{L}_g}(t) \leqslant c_{17} h_{\mathcal{A}_g,\varepsilon^*\mathfrak{L}_g}([\pi]t) + c_{18}$$

Proof. The idea is due to Lin-Wang [32, Proof of Proposition 2.1]. The following notation will be used only in this proof: denote by $B = [\pi](Y)$ and $X = [\pi]^{-1}(B)$. By abuse of notation, we will not distinguish $[\pi]$ and $[\pi]|_X$. Remark that $X \to B$ is a non-isotrivial family of abelian varieties.

Let Y' be a smooth resolution of $Y \subset \mathfrak{A}_g$, then $X \times_B Y' \to Y'$ is also a non-isotrivial family of abelian varieties of dimension g and we write $\varepsilon_{Y'} \colon Y' \to X \times_B Y'$ to be the zero-section. Let $f \colon Y' \to \mathfrak{A}_g$ be the natural morphism. Consider the following commutative diagram

$$\begin{array}{ccc} X \times_B Y' & \underset{p_2}{\overset{\varepsilon_{Y'}}{\rightleftarrows}} & Y' \\ \downarrow{\scriptstyle p_1} & \lrcorner & \downarrow{\scriptstyle [\pi]\circ f} \\ X & \xrightarrow{[\pi]} & B \end{array} .$$

Now let $t' \in Y'(\overline{\mathbb{Q}})$ be such that $f(t') = t$. Then up to bounded functions,

$$\begin{aligned} h_{\mathfrak{A}_g,\mathfrak{L}_g}(t) &= h_{X,\mathfrak{L}_g|_X}(t) \\ &= h_{X,\mathfrak{L}_g|_X}(f(t')) \\ &= h_{Y',f^*\mathfrak{L}_g|_X}(t') \end{aligned} \qquad \begin{aligned} h_{\mathcal{A}_g,\varepsilon^*\mathfrak{L}_g}([\pi]t) &= h_{B,\varepsilon^*\mathfrak{L}_g|_X}([\pi]t) \\ &= h_{B,\varepsilon^*\mathfrak{L}_g|_X}(f \circ [\pi](t')) \\ &= h_{Y',(f\circ[\pi])^*\varepsilon^*\mathfrak{L}_g|_X}(t') \\ &= h_{Y',\varepsilon_{Y'}^* p_1^*\mathfrak{L}_g|_X}(t'). \end{aligned}$$

Since Y is a curve, the morphism $[\pi] \circ f \colon Y' \to B$ is finite. Therefore $p_1^*\mathfrak{L}_g|_X$ is ample. So $\varepsilon_{Y'}^* p_1^*\mathfrak{L}_g|_X$ is ample. Hence there exist two constants $c_{17} > 0$ and c_{18} depending only on Y' (and hence only on Y) such that

$$h_{Y',f*\mathfrak{L}_g|_X}(t') \leqslant c_{17} h_{Y',\varepsilon_{Y'}^* p_1^*\mathfrak{L}_g|_X}(t') + c_{18} \tag{5.4.12}$$

for any $t' \in Y'(\overline{\mathbb{Q}})$. Now the conclusion follows. □

Now for any $t \in Y \cap \Sigma \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$, by (5.4.1) and Lemma 5.4.5,

$$\widehat{h}_{\mathfrak{L}_g}(t) = \frac{\deg(f_t)^{1/g} \deg(\varphi_t)^{1/g}}{n_t^2} \widehat{h}_{\mathfrak{L}_g}(s). \tag{5.4.13}$$

But for any $t \in \Sigma \cap \mathfrak{A}_g(\overline{\mathbb{Q}})$, we have the following result of Faltings [16, Chapter II, §4, Lemma 5]

$$|h_F(A_{[\pi]t}) - h_F(A)| \leqslant \frac{1}{2} \log \deg(f_t). \tag{5.4.14}$$

Besides by [44, Corollary 1.3], there exists a positive constant $c_{19} = c_{19}(g)$ such that

$$|\frac{1}{2}h_F(A_{[\pi]t}) - h_{\mathcal{A}_g,\varepsilon^*\mathfrak{L}_g}([\pi]t)| \leqslant c_{19}\log\Big(\max\big(1, h_F(A_{[\pi]t})\big)+2\Big) \tag{5.4.15}$$

for any $t \in \mathfrak{A}_g(\overline{\mathbb{Q}})$.

Now (5.4.11), Lemma 5.4.6, (5.4.13), (5.4.14) and (5.4.15) together imply

$$\frac{\deg(\varphi_t)^{1/g}}{n_t^2}\deg(f_t)^{1/g}\widehat{h}_{\mathfrak{L}_g}(s) \leqslant (c_{15}+c_{17})c_{19}\log\Big(\max\Big(1, h_F(A)+\frac{1}{2}\log\deg(f_t)\Big)+2\Big) + \frac{c_{15}+c_{17}}{4}\log\deg(f_t) + \frac{c_{15}+c_{17}}{2}h_F(A) + c_{16} + c_{18}.$$

Since $\deg(\varphi_t) \geqslant 1$, we get that $\deg(f_t)$ is polynomially bounded by n_t from above.

On the other hand, letting $\deg(f_t) \to \infty$, we see that there exist two positive constants M_0 and c_{20} depending on nothing such that $\deg(\varphi_t)^{1/g} \leqslant c_{20}n_t^2$ for any $t \in Y(\overline{\mathbb{Q}}) \cap \Sigma$ with $\deg(f_t) > M_0$. But if $\deg(f_t) \leqslant M_0$, then $\deg(f_t)$ takes value in a finite set $\{1, ..., M_0\}$. So $\deg(\varphi_t)$ is bounded by n_t from above. ☐

5.4.4 Application of Pila-Wilkie

Keep the notation of the beginning of this section and §5.4.1.

Proposition 5.4.7. *Let Y and $\widetilde{s}$ be as in the beginning of this section. Let $\varepsilon > 0$. There exists a constant $C = C(Y, s, \varepsilon) > 0$ with the following property:*

For every $n \geqslant 1$, there exist at most Cn^ε definable blocks $B_i \subset \widetilde{Y}$ such that $\cup B_i$ contains all point of complexity n of $\widetilde{Y} \cap \widetilde{\Sigma}$.

Proof. The proof starts with the following lemma:

Lemma 5.4.8. *There exist constants C' and κ' depending only on g and $\widetilde{s}$ such that*

For any $\widetilde{t} \in \widetilde{Y} \cap \widetilde{\Sigma}$ of complexity n, there exists a $(v, h) \in P_{2g}(\mathbb{Q})^+$ such that $(v, h) \cdot \widetilde{s} = \widetilde{t}$ and $H((v, h)) \leqslant C'n^{\kappa'}$.

Proof. Let $t := \mathrm{unif}(\widetilde{t})$. Then $t \in \Sigma$ and therefore we have a relation as (5.4.1). Let $f'_t := f_t \circ \varphi_t$, then $f'_t \colon (A, \lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t})$ is a polarized isogeny. Moreover, there exists a $\delta'_t \in A(\overline{\mathbb{Q}})_{\mathrm{tor}}$ such that $N(\delta'_t) \leqslant N(\delta_t)\deg(\varphi_t)$ and

$$t = f'_t(s_t + \delta'_t). \tag{5.4.16}$$

Claim 5.4.9. *There exists a symplectic basis $\mathcal{B}'$ for $H_1(\mathfrak{A}_{[\pi]t}, \mathbb{Z})$ w.r.t. the polarization $\lambda_{[\pi]t}$ such that the height of $\gamma_{f'} \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ (the matrix expression of f'_t in coordinate $\mathcal{B}$ w.r.t. $\mathcal{B}'$) is polynomially bounded by $\deg(f'_t) = \deg(\varphi_t)\deg(f_t)$ from above (see the beginning of this section for $\mathcal{B}$).*

This claim follows from [43, Proposition 4.1]: remark that f'_t is a polarized isogeny instead of an arbitrary isogeny, hence the endomorphism $q \in \mathrm{End}(A)$ in [43, 4.3] equals $[\deg \varphi_t]^{1/g}$, and therefore the $u \in (\mathrm{End}\, A)^*$ in [43, 4.6] can be taken to be 1_A.

Then $\mathrm{unif}_G(\gamma_{f'} \cdot \widetilde{s}_G) = [\pi]s$. Besides let $\widetilde{\delta}'_t = (\widetilde{\delta}'_{t,V}, \widetilde{s}_G) \in \mathcal{F}$ be such that $\mathrm{unif}(\widetilde{\delta}'_t) = \delta'_t$. Then $\widetilde{\delta}'_{t,V} \in V_{2g}(\mathbb{Q})$ and, by (5.4.16) and (5.2.3),

$$\mathrm{unif}\left(\gamma_{f'}\left(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}, \widetilde{s}_G\right)\right) = t.$$

So there exists an element $\gamma = (\gamma_V, \gamma_G) \in \Gamma$ such that

$$\gamma\gamma_{f'}\left(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}, \widetilde{s}_G\right) = \widetilde{t},$$

i.e.

$$\widetilde{t} = \left(\gamma_V + \gamma_G\gamma_{f'}\left(\frac{\widetilde{s}_V}{n_t} + \widetilde{\delta}'_{t,V}\right), \gamma_G\gamma_{f'}\widetilde{s}_G\right) = \left(\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V}, \frac{\gamma_G\gamma_{f'}}{n_t}\right) \cdot \widetilde{s}.$$

Denote by

$$(v, h) := \left(\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V}, \frac{\gamma_G\gamma_{f'}}{n_t}\right),$$

then (v, h) is an element of $P_{2g}(\mathbb{Q})^+$ such that $(v, h)\widetilde{s} = \widetilde{t}$. Now we prove that $H((v, h))$ is polynomially bounded by the complexity n of $\widetilde{t}$. To prove this, it suffices to prove that n_t, $H(\widetilde{\delta}'_{t,V})$, $H(\gamma_{f'})$, $H(\gamma_G)$ and $H(\gamma_V)$ are all polynomially bounded by n.

The fact that n_t is bounded by n follows directly from the definition of complexity.

For $H(\widetilde{\delta}'_{t,V})$: because $\widetilde{\delta}'_t \in \mathcal{F} \simeq [0, N)^{2g} \times \mathcal{F}_G$ (where N is the level structure, and hence depend on nothing), we have $\widetilde{\delta}'_{t,V} \in [0, N)^{2g}$. Therefore $H(\widetilde{\delta}'_{t,V})$ is bounded up to a constant by the denominator of $\widetilde{\delta}'_{t,V}$, which equals $N(\delta'_t)$. But $N(\delta'_t) \leqslant \deg(\varphi_t)N(\delta_t)$, hence it suffices to bound both $\deg(\varphi_t)$ and $N(\delta_t)$ by n. Now $\deg(\varphi_t)$ is polynomially bounded by n_t, and hence by n, by Proposition 5.4.4. By definition of complexity, $N(\delta_t) \leqslant n$.

For $H(\gamma_{f'})$: by choice, $H(\gamma_{f'})$ is polynomially bounded by $\deg(f_t)\deg(\varphi_t)$, which is polynomially bounded by n_t by Proposition 5.4.4. Hence $H(\gamma_{f'})$ is polynomially bounded by n by definition of complexity.

For $H(\gamma_G)$: remark $\gamma_G\gamma_{f'}\widetilde{s}_G = \pi(\widetilde{t}) \in \mathcal{F}_G$. By [49, Lemma 3.2], $H(\gamma_G)$ is polynomially bounded by $||\gamma_{f'}\widetilde{s}_G||$. Therefore $H(\gamma_G)$ is polynomially bounded, with constants depending on $||\widetilde{s}_G||$, by n.

For $H(\gamma_V)$: remark $\gamma_V + \gamma_G\gamma_{f'}\widetilde{\delta}'_{t,V} + \gamma_G\gamma_{f'}\widetilde{s}_V/n_t = \widetilde{t}_V \in [0, N)^{2g}$ (where N is the level structure, and hence depend on nothing). Therefore $H(\gamma_V)$ is polynomially bounded by $||\gamma_G\gamma_{f'}\widetilde{\delta}_{t,V} + \gamma_G\gamma_{f'}\widetilde{s}_V/n_t||$. Therefore $H(\gamma_V)$ is polynomially bounded, with constants depending on $||\widetilde{s}_V||$, by n. □

Let $\sigma\colon P_{2g}(\mathbb{R})^+ \to \mathcal{X}^+_{2g,\mathrm{a}}$ be the map $(v,h) \mapsto (v,h)\cdot\widetilde{s}$.

The set $R = \sigma^{-1}(\widetilde{Y}) = \sigma^{-1}(\mathrm{unif}^{-1}(Y)\cap\mathcal{F})$ is definable because σ is semi-algebraic and $\mathrm{unif}\,|_{\mathcal{F}}$ is definable. Hence we can apply the family version of the Pila-Wilkie theorem ([48, 3.6]) to the definable set R: for every $\varepsilon > 0$, there are only finitely many definable block families $B^{(j)}(\varepsilon) \subset R\times\mathbb{R}^m$ and a constant $C_1'(R,\varepsilon)$ such that for every $T \geqslant 1$, the rational points of R of height at most T are contained in the union of at most $C_1'T^{\varepsilon}$ definable blocks $B_i(T,\varepsilon)$, taken (as fibers) from the families $B^{(j)}(\varepsilon)$. Since σ is semi-algebraic, the image under σ of a definable block in R is a finite union of definable blocks in $\widetilde{Y}$. Furthermore the number of blocks in the image is uniformly bounded in each definable block family $B^{(j)}(\varepsilon)$. Hence $\sigma(B_i(T,\varepsilon))$ is the union of at most $C_2'T^{\varepsilon}$ blocks in $\widetilde{Y}$, for some new constant $C_2'(Y,\widetilde{a},\varepsilon) > 0$.

By Lemma 5.4.8, for any point $\widetilde{t} \in \widetilde{Y}\cap\widetilde{\Sigma}$ of complexity n, there exists a rational element $\gamma \in R$ such that $\sigma(\gamma) = \widetilde{t}$ and $H(\gamma) \leqslant C'n^{\kappa'}$. By the discussion in the last paragraph, all such γ's are contained in the union of at most $C_1'(C'n^{\kappa'})^{\varepsilon}$ definable blocks. Therefore all points of $\widetilde{Y}\cap\widetilde{\Sigma}$ of complexity n are contained in the union of at most $C_1'C_2'C'^{\varepsilon}n^{\kappa'\varepsilon}$ blocks in $\widetilde{Y}$. □

5.4.5 End of proof of Theorem 5.1.5

Now we are ready to finish the proof of Theorem 5.1.5.

Let Σ_1 be the set of points $t \in Y\cap\Sigma$ such that there is a positive-dimensional block $B \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B)$. Let Y_1 be the Zariski closure of Σ_1. Let k be a number field such that both Y and Y_1 are defined over k.

Let t be a point in $Y\cap\Sigma$ of complexity n. By Proposition 5.4.2, there exist positive constants c_5 and c_6 depending only on (A,λ) and s such that

$$[k(t):k] \geqslant \frac{c_5}{[k:\mathbb{Q}]}n^{c_6}.$$

But all $\mathrm{Gal}(\overline{k}/k)$-conjugates of t are contained in $Y\cap\Sigma$ and have complexity n. By Proposition 5.4.7, the preimages in $\mathcal{F}$ of these points are contained in the union of $C(Y,s,c_6/2)n^{c_6/2}$ definable blocks, each of these blocks being contained in $\widetilde{Y}$.

For n large enough, $(c_5/[k:\mathbb{Q}])n^{c_6} > Cn^{c_6/2}$. Hence for $n \gg 0$, there exists a definable block $B \subset \widetilde{Y}$ such that $\mathrm{unif}(B)$ contains at least two Galois conjugates of t, and therefore $\dim B > 0$ since blocks are connected. So being in $\mathrm{unif}(B)$, those conjugates of t are in Σ_1. But Y_1 is defined over k, so $t \in Y_1$.

In summary, all points of $Y\cap\Sigma$ of large enough complexity are in Σ_1. This excludes only finitely many points of $Y\cap\Sigma$. So $Y_1 = Y$.

Let Σ_2 be the set of points $t \in Y\cap\Sigma$ such that there is a connected positive-dimensional semi-algebraic set $B' \subset \widetilde{Y}$ with $t \in \mathrm{unif}(B')$. Let Y_2 be the Zariski closure of Σ_2. By definition of blocks, $\Sigma_2 = \Sigma_1$, and hence $Y_2 = Y_1 = Y$.

Now the mixed Ax-Lindemann theorem (Theorem 3.1.4) yields the conclusion since $\dim(Y) = 1$. Alternatively, let $\widetilde{Y}'$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$. Then since $Y = Y_2$, there exists a positive-dimensional irreducible algebraic subset (in the sense of Definition 1.3.5) $\widetilde{Z}$ of $\mathcal{X}_{2g,\mathrm{a}}$ contained in $\widetilde{Y}'$ by [49, Lemma 4.1]. But $\dim \widetilde{Y}' = \dim \widetilde{Z} = 1$, therefore $\widetilde{Y}' = \widetilde{Z}$ is algebraic in the sense of Definition 1.3.5. In other words, Y is algebraic and a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ is also algebraic. Hence by Theorem 2.3.3, Y is weakly special.

5.5 Variants of the André-Pink-Zannier conjecture

In the previous sections we have discussed the intersection of a subvariety of $\mathfrak{A}_g$ with the set of division points of the polarized isogeny orbit of a given point (5.1.1). The goal of this section is twofold: one is to replace the given point by a finitely generated subgroup of one fiber of $\mathfrak{A}_g \to \mathcal{A}_g$ (remark that the fiber is an abelian variety), the other is to replace the polarized isogeny orbit by the isogeny orbit. In particular we will prove that although these changes to Conjecture 5.1.1 a priori seem to generalize the conjecture, both can actually be implied by Conjecture 5.1.1 itself.

In the rest of the section, fix a point $b \in \mathcal{A}_g$, which corresponds to a polarized abelian variety $(A, \lambda) := (\mathfrak{A}_{g,b}, \lambda_b)$. Let Λ be any finitely generated subgroup of A.

Theorem 5.5.1. *Let Y be an irreducible subvariety of $\mathfrak{A}_g$. Let Σ_0 be the set of division points of the polarized isogeny orbit of Λ, i.e.*

$$\Sigma_0 = \{t \in \mathfrak{A}_g | \ \exists n \in \mathbb{N} \text{ and a polarized isogeny } f\colon (A, \lambda) \to (\mathfrak{A}_{g,[\pi]t}, \lambda_{[\pi]t}) \text{ with } nt \in f(\Lambda)\}.$$

Assume that Conjecture 5.1.1 holds for all g. If $\overline{Y \cap \Sigma_0} = Y$, then Y is weakly special.

Proof. The proof is basically the same as Pink [54, Theorem 5.4] (how Conjecture 5.1.1 implies the Mordell-Lang conjecture).

Suppose $\operatorname{rank} \Lambda = r - 1$. Let V_{2g}^r be the direct sum of r copies of V_{2g} as a representation of GSp_{2g}. Then the connected mixed Shimura variety associated with $V_{2g}^r \rtimes \mathrm{GSp}_{2g}$ is the r-fold fiber product of $\mathfrak{A}_g$ over $\mathcal{A}_g$, and so its fiber over b is A^r. Denote by

$$\sigma\colon \mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g \to \mathfrak{A}_g$$

the summation map (remark that both varieties are abelian schemes over $\mathcal{A}_g$).

Now the homomorphisms

$$\begin{array}{rlll} P_{2g,\mathrm{a}} & = V_{2g} \rtimes \mathrm{GSp}_{2g} & \hookrightarrow V_{2g}^r \rtimes \mathrm{GSp}_{2g} \hookrightarrow & V_{2gr} \rtimes \mathrm{GSp}_{2gr} \\ & (v,h) & \mapsto ((v,...,v),h)) \mapsto & ((v,...,v),(h,...,h)) \end{array}$$

induce Shimura immersions

$$\begin{array}{ccccc} \mathfrak{A}_g & \longrightarrow & \mathfrak{A}_g \times_{\mathcal{A}_g} \dots \times_{\mathcal{A}_g} \mathfrak{A}_g & \longrightarrow & \mathfrak{A}_{gr} \\ {\scriptstyle [\pi]}\downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_g & \xrightarrow{=} & \mathcal{A}_g & \hookrightarrow & \mathcal{A}_{gr} \end{array}$$

For simplicity we shall not distinguish a point in $\mathfrak{A}_g$ (resp. $\mathcal{A}_g$) and its image in $\mathfrak{A}_{gr}$ (resp. $\mathcal{A}_{gr}$). Then $\mathfrak{A}_{gr,b} = A^r$.

Fix generators $a_1,...,a_{r-1}$ of Λ and set $a_r := -a_1 - ... - a_{r-1}$. Let Λ' be the division group of Λ, i.e. $\Lambda' = \{s|\ \exists n \in \mathbb{N}$ such that $ns \in \Lambda\} \subset A$. Then [54, Lemma 5.3] asserts that

$$\Lambda' = \Lambda^*_{a_1} + ... + \Lambda^*_{a_r} = \sigma(\Lambda^*_{a_1} \times ... \times \Lambda^*_{a_r}) \tag{5.5.1}$$

where (as Pink defined) $\Lambda^*_{a_i} := \{s \in A|\ \exists m, n \in \mathbb{Z} \setminus \{0\}$ such that $ns = ma_i\}$.

Now consider

$$\Lambda^\dagger := \sigma^{-1}(Y) \cap \{f^r(\Lambda^*_{a_1} \times ... \times \Lambda^*_{a_r})|\ f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'}) \text{ a polarized isogeny}\}.$$

We have

$$\begin{aligned} \sigma(\Lambda^\dagger) &= Y \cap \sigma(\{f^r(\Lambda^*_{a_1} \times ... \times \Lambda^*_{a_r})|\ f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'}) \text{ a polarized isogeny}\}) \\ &= Y \cap \{f^r\left(\sigma(\Lambda^*_{a_1} \times ... \times \Lambda^*_{a_r})\right) \mid f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'}) \text{ a polarized isogeny}\} \\ &= Y \cap \{f^r(\Lambda')|\ f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'}) \text{ a polarized isogeny}\} \qquad (5.5.1). \end{aligned}$$

Because $\overline{Y \cap \Sigma_0} = Y$, $Y \cap \{f(\Lambda')|\ f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'})$ a polarized isogeny$\}$ is Zariski dense in Y (as subsets of $\mathfrak{A}_g$). Therefore $\sigma(\Lambda^\dagger)$ is Zariski dense in Y (as subsets of $\mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g$, and hence as subsets of $\mathfrak{A}_{gr}$). Let $Y^\dagger$ be the Zariski closure of $\Lambda^\dagger$ in $\mathfrak{A}_g \times_{\mathcal{A}_g} ... \times_{\mathcal{A}_g} \mathfrak{A}_g$. Then $Y^\dagger$ is also a subvariety of $\mathfrak{A}_{gr}$. Since taking Zariski closures commutes with taking images under proper morphisms, we deduce that $\sigma(Y^\dagger) = Y$. So there exists an irreducible component Y' of $Y^\dagger$ such that $\sigma(Y') = Y$.

For any polarized isogeny $f\colon (A, \lambda) \to (\mathfrak{A}_{g,b'}, \lambda_{b'})$, the generalized Hecke orbit of $(a_1, ..., a_r) \in A^r$ as a point on $\mathfrak{A}_{gr}$ contains $f^r(\Lambda^*_{a_1} \times ... \times \Lambda^*_{a_r})$ by Corollary 5.2.5. Therefore the intersection of Y' with generalized Hecke orbit of $(a_1, ..., a_r)$ in $\mathfrak{A}_{gr}$ is Zariski dense in Y'. Hence Conjecture 5.1.1 for $\mathfrak{A}_{gr}$ implies that Y' is weakly special. Therefore $Y = \sigma(Y')$ is also weakly special by the geometric interpretation of weakly special subvarieties of $\mathfrak{A}_g$ and of $\mathfrak{A}_{gr}$ (Proposition 1.2.15). $\square$

Corollary 5.5.2. *Let Y be an irreducible subvariety of $\mathfrak{A}_g$. Let Σ'_0 be the set of division points of the isogeny orbit of Λ, i.e.*

$$\Sigma'_0 = \{t \in \mathfrak{A}_g|\ \exists n \in \mathbb{N} \text{ and an isogeny } f\colon A \to \mathfrak{A}_{g,[\pi]t} \text{ such that } nt \in f(\Lambda)\}.$$

Assume that Conjecture 5.1.1 holds for all g. If $\overline{Y \cap \Sigma'_0} = Y$, then Y is weakly special.

Proof. Recall Zarhin's trick (see Orr [42, Proposition 4.4]): for any isogeny $f\colon A \to A'$ between polarized abelian varieties, there exists $u \in \mathrm{End}(A^4)$ such that $f^4 \circ u\colon A^4 \to (A')^4$ is a polarized isogeny.

Now let $i\colon \mathfrak{A}_g \hookrightarrow \mathfrak{A}_{4g}$ be the natural embedding. Then $\Lambda_4 := \mathrm{End}(A^4)i(\Lambda)$ is a finitely generated subgroup of $A^4 = \mathfrak{A}_{4g,i(b)}$ and hence

$$\begin{aligned}\Sigma_0' \quad \subset \{t \in \mathfrak{A}_{4g} |\ &\exists n \in \mathbb{N} \text{ and a polarized isogeny} \\ &f\colon (A^4, \lambda^{\boxtimes 4}) \to (\mathfrak{A}_{4g,[\pi]t}, \lambda_{[\pi]t}) \text{ such that } nt \in f(\Lambda_4)\}.\end{aligned}$$

Now the conclusion follows from Theorem 5.5.1. □