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Chapter 4

From Ax-Lindemann to André-Oort

4.1 Distribution of positive-dimensional weakly special subvarieties

4.1.1 Weakly special subvarieties defined by a fixed $\mathbb{Q}$-subgroup

Let $S = \Gamma\backslash\mathcal{X}^+$ be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S$ be the uniformization. Suppose that N is a connected subgroup of P such that $N/(W \cap N) \hookrightarrow G$ is semi-simple. A subvariety of S is said to be **weakly special defined by** N if it is of the form $\mathrm{unif}(i(\varphi^{-1}(y')))$ under the notation of Definition 1.2.2 such that $N = \mathrm{Ker}(\varphi)$. Let $\mathfrak{F}(N)$ be the set of all weakly special subvarieties of S defined by N. The goal of this subsection is to prove:

Proposition 4.1.1. *If $\mathfrak{F}(N) \neq \emptyset$ and $N \ntriangleleft P$, then $\cup_{Z\in\mathfrak{F}(N)} Z$ is a finite union of proper special subvarieties of S.*

Proof. Take any $F \in \mathfrak{F}(N)$. Let $\mathcal{F}$ be a fundamental domain for the action Γ on $\mathcal{X}^+$. Suppose that $x' \in \mathcal{F}$ is such that $F = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})x')$. Consider $Q' := N_P(N)$, the normalizer of N in P. By definition of weakly special subvarieties, there exists $(R', \mathcal{Z}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $h_{x'} : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $R'_{\mathbb{C}}$ and $N \lhd R'$. Hence $R' < Q'$. Define $G_{Q'} := Q'/(W \cap Q')$. Then $G_{Q'}/(Z(G) \cap G_{Q'})$ is reductive by [15, Lemma 4.3] or [63, Proposition 3.28], and hence $G_{Q'}$ is reductive. Write

$$1 \to W \cap Q' \to Q' \xrightarrow{\pi_{Q'}} G_{Q'} \to 1.$$

The group $G_{Q'} = Z(G_{Q'})^\circ G^{\mathrm{nc}}_{Q'} G^{\mathrm{c}}_{Q'}$ is an almost-direct product, where $G^{\mathrm{nc}}_{Q'}$ (resp. $G^{\mathrm{c}}_{Q'}$) is the product of the $\mathbb{Q}$-simple factors whose set of $\mathbb{R}$-points is non-compact (resp. compact). Let $G_Q := Z(G_{Q'})^\circ G^{\mathrm{nc}}_{Q'}$ and then define $Q := \pi_{Q'}^{-1}(G_Q)$, then $h_{x'}$ factors through $Q_{\mathbb{C}}$ and $R' < Q$ by Definition 1.1.12(4). So $N \lhd Q$ and $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+U_Q(\mathbb{C})x'$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$. But then $F \subset \mathrm{unif}(\mathcal{Y}^+) \subset \cup_{Z\in\mathfrak{F}(N)} Z$.

Define $\mathfrak{Y}_Q := \{x \in \mathcal{X}^+ | h_x \text{ factors through } Q_{\mathbb{C}}\}$, then $Q(\mathbb{R})^+U_Q(\mathbb{C})\mathfrak{Y}_Q = \mathfrak{Y}_Q$. The discussion of last paragraph tells us that $F \subset \mathrm{unif}(\mathfrak{Y}_Q)$ for any $F \in \mathfrak{F}(N)$. On the other hand, for any $x \in \mathfrak{Y}_Q$, $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+U_Q(\mathbb{C})x$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$ and hence $\mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})x) \in \mathfrak{F}(N)$. Therefore $\mathrm{unif}(\mathfrak{Y}_Q) \subset \cup_{Z\in\mathfrak{F}(N)} Z$. To sum it up, $\cup_{Z\in\mathfrak{F}(N)} Z = \mathrm{unif}(\mathfrak{Y}_Q)$.

Now we are done if we can prove

Claim 4.1.2. *The set* $\mathfrak{Y}_Q$ *is a finite union of* $Q(\mathbb{R})^+U_Q(\mathbb{C})$*-conjugacy classes. In other words,* $\mathfrak{Y}_Q$ *is a finite union of connected mixed Shimura subdata of* $(P,\mathcal{X}^+)$.

Fix a special point x of $\mathcal{X}^+$ contained in $\mathfrak{Y}_Q$. There exists by definition a torus $T_x \subset Q$ such that $h_x : \mathbb{S}_\mathbb{C} \to Q_\mathbb{C}$ factors through $T_{x,\mathbb{C}}$. Furthermore, we may and do assume that $T_{x,\mathbb{C}}$ is a maximal torus of $Q_\mathbb{C}$. Let T be a maximal torus of $P_\mathbb{C}$ defined over $\mathbb{Q}$ such that $T > T_x$. Take a Levi decomposition $P = W \rtimes G$ such that $T < G < P$. Then the composite $\mathbb{S}_\mathbb{C} \xrightarrow{h_x} T_{x,\mathbb{C}} < P_\mathbb{C} \xrightarrow{\pi} G_\mathbb{C} < P_\mathbb{C}$ equals h_x and is defined over $\mathbb{R}$ by Definition 1.1.12(1).

For any other special point y of $\mathcal{X}^+$ contained in $\mathfrak{Y}_Q$, there exists $g \in Q(\mathbb{C})$ such that $gT_{x,\mathbb{C}}g^{-1} = T_{y,\mathbb{C}}$. The number of the $Q(\mathbb{R})$-conjugacy classes of maximal tori of $Q_\mathbb{R}$ defined over $\mathbb{R}$ is at most

$$\#(\mathrm{Ker}(H^1\left(\mathbb{R}, N_{Q(\mathbb{R})}(T_{x,\mathbb{R}})\right) \to H^1(\mathbb{R}, Q))) < \infty,$$

where $N_{Q(\mathbb{R})}(T_{x,\mathbb{R}})$ is the normalizer of $T_{x,\mathbb{R}}$ in $Q(\mathbb{R})$. So it is equivalent to prove the finiteness of the $Q(\mathbb{R})^+U_Q(\mathbb{C})$-conjugay classes in $\mathfrak{Y}_Q$ and to prove the finiteness of the $Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$. But $T_x < T < G$, so the $Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$ equals the $G_Q(\mathbb{R})^+$-conjugacy classes of the morphisms $\mathbb{S} \to T_{x,\mathbb{R}}$. In otherwords, it suffices to prove the claim for $(G, \mathcal{X}_G^+)$. Now the result follows from [15, Lemma 4.4(ii)] (or [39, 2.4] or [66, Lemma 3.7]).

□

4.1.2 The distribution theorem

Now we use the result of the previous subsection to prove the following theorem about the distribution of positive-dimensional weakly special subvarieties. This is a direct generalization of the comparative result of Ullmo for pure Shimura varieties [64, Théorème 4.1].

Theorem 4.1.3. *Let* $S = \Gamma\backslash\mathcal{X}^+$ *be a connected mixed Shimura variety associated with the connected mixed Shimura datum* $(P,\mathcal{X}^+)$*. Let* Y *be a Hodge generic irreducible subvariety of* S*. Then there exists an* $N \lhd P$ *such that for the diagram*

$$\begin{array}{ccc} (P,\mathcal{X}^+) & \xrightarrow{\rho} & (P',\mathcal{X}'^+) := (P,\mathcal{X}^+)/N \\ \downarrow \text{unif} & & \downarrow \text{unif}' \\ S & \xrightarrow{[\rho]} & S' \end{array} , \qquad (4.1.1)$$

- *the union of positive-dimensional weakly special subvarieties which are contained in* $Y' := \overline{[\rho](Y)}$ *is NOT Zariski dense in* Y'*;*
- $Y = [\rho]^{-1}(Y')$.

Proof. Without any loss of generality, we assume that the union of positive-dimensional weakly special subvarieties which are contained in Y is Zariski dense in Y.

Take a fundamental domain $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable. Such an $\mathcal{F}$ exists by §3.3.1.

By Reduction Lemma (Lemma 1.1.35), we may assume

$$(P, \mathcal{X}^+) \overset{\lambda}{\hookrightarrow} (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_{2g}, \mathcal{X}^+_{2g}),$$

i.e. replace $(P, \mathcal{X}^+)$ by $(P', \mathcal{X}'^+)$ in the reduction lemma if necessary. Identify $(P, \mathcal{X}^+)$ with its image under λ.

Let $\mathcal{T}$ be the set of the triples (U', V', G') consisting of an $\mathbb{R}$-subgroup of $U_{\mathbb{R}}$, an $\mathbb{R}$-sub-Hodge structure of $V_{\mathbb{R}}$ and a connected $\mathbb{R}$-subgroup of $G_{\mathbb{R}}$ which is semi-simple and has no compact factors. Let

$$\mathcal{G} := \mathbb{G}_m(\mathbb{R})^r \times \mathrm{GSp}_{2g}(\mathbb{R}) \times G(\mathbb{R}),$$

then $\mathcal{G}$ acts on $\mathcal{T}$ by $(g_U, g_V, g) \cdot (U', V', G') := (g_U U', g_V V', gG'g^{-1})$. Also we define the action of a triple $(U'(\mathbb{R}), V'(\mathbb{R}), G'(\mathbb{R}))$ on $\mathcal{X}^+ \simeq U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}^+_G$ as (1.3.2). This action is algebraic.

Lemma 4.1.4. *Up to the action of $\mathcal{G}$ on $\mathcal{T}$, there exist only finitely many such triples.*

Proof. First of all by root system theory and Galois cohomology, there exist only finitely many semi-simple subgroups of $G_{\mathbb{R}}$ up to conjugation by $G(\mathbb{R})$.

Secondly, V' is by definition a symplectic subspace of $V_{\mathbb{R}}$. Hence a symplectic base of V' extends to a symplectic base of $V_{\mathbb{R}} = V_{2g,\mathbb{R}}$. But $\mathrm{GSp}_{2g}(\mathbb{R})$ acts transitively on the set of symplectic bases of $V_{2g,\mathbb{R}}$, so there are only finitely many choices for V' up to the action of $\mathrm{GSp}_{2g}(\mathbb{R})$.

Finally, observe that for all $(\lambda_1, ..., \lambda_r) \in \mathbb{G}_m(\mathbb{R})^r$ and $(u_1, ..., u_r) \in U \simeq \oplus_{i=1}^r U^{(i)}_{2g}$,

$$(\lambda_1, ..., \lambda_r) \cdot (u_1, ..., u_r) = (\lambda_1 u_1, ..., \lambda_r u_r)$$

Now $(u_1, ..., u_r)$ and $(u'_1, ..., u'_r)$ are under the same orbit of the action of $\mathbb{G}_m(\mathbb{R})^r$ if and only if $u_i u'_i \geqslant 0$ with $u_i u'_i = 0 \Rightarrow u_i = u'_i = 0$ for all $i = 1, ..., r$. Hence up to the action of $\mathbb{G}_m(\mathbb{R})^r$, there are only finitely many U''s. ☐

Let $\mathfrak{W}(Y)$ (resp. $\mathfrak{W}_l(Y)$) be the union of weakly special subvarieties of positive dimension (resp. of real dimension l) contained in Y.

For any l with $\mathfrak{W}_l(Y) \neq \emptyset$, there exist by definition (and Proposition 1.2.4) a subgroup N_l of P^{der} and a point $x_0 \in \mathcal{F}$ such that $\mathrm{unif}(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C}) x_0)$ is a weakly special subvariety of dimension l contained in Y. Note that the triple $(U_{N_l,\mathbb{R}}, V_{N_l,\mathbb{R}}, G^{+\mathrm{nc}}_{N_l,\mathbb{R}}) \in \mathcal{T}$, where $G^{+\mathrm{nc}}_{N_l,\mathbb{R}}$ is the product of the $\mathbb{R}$-simple factors of $G^+_{N_l,\mathbb{R}}$ which are non-compact. We say that two such subgroups N_l, N'_l of P are equivalent if $(U_{N_l,\mathbb{R}}, V_{N_l,\mathbb{R}}, G^{+\mathrm{nc}}_{N_l,\mathbb{R}}) = (U_{N'_l,\mathbb{R}}, V_{N'_l,\mathbb{R}}, G^{+\mathrm{nc}}_{N'_l,\mathbb{R}})$. By condition

(4) of Definition 1.1.12, $\mathrm{unif}(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0) = \mathrm{unif}(N_l'(\mathbb{R})^+U_{N_l'}(\mathbb{C})x_0)$ iff N_l and N_l' are equivalent.

Define

$$B(N_{l,\mathbb{R}}, Y) := \{(g_U, g_V, g, x) \in \mathcal{G} \times \mathcal{F} |\ \mathrm{unif}((g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x) \text{ is contained in } Y \text{ and is not contained in } \cup_{l'>l} \mathfrak{W}_{l'}(Y)\}.$$

Then by analytic continuation,

$$B(N_{l,\mathbb{R}}, Y) = \{(g_U, g_V, g, x) \in \mathcal{G} \times \mathcal{F} |\ \mathrm{unif}|_{\mathcal{F}}((g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x) \text{ is contained in } Y \text{ and is not contained in } \cup_{l'>l} \mathfrak{W}_{l'}(Y)\}. \tag{4.1.2}$$

Lemma 4.1.5. *For any $(g_U, g_V, g, x) \in B(N_{l,\mathbb{R}}, Y)$, define*

$$\widetilde{Z} := (g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})x.$$

Then $\mathrm{unif}(\widetilde{Z})$ *is a weakly special subvariety of* Y.

Proof. The set $\widetilde{Z}$ is a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$ (see the paragraph before Theorem 3.1.2 for the definition of "connected irreducible semi-algebraic subsets of $\mathcal{X}^+$"). Let $\widetilde{Z}^\dagger$ be a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$ and which contains $\widetilde{Z}$, maximal for these properties. By Ax-Lindemann (here we use Theorem 3.1.2), $\widetilde{Z}^\dagger$ is complex analytic and each of its complex analytic irreducible component is weakly special. But $\widetilde{Z}$ is smooth, so $\widetilde{Z}$ is contained in one complex analytic irreducible component of $\widetilde{Z}^\dagger$ which we denote by $\widetilde{Z}'$. Now we have

$$\begin{aligned}\dim(\widetilde{Z}) - \dim(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0) &= \dim(gG_{N_l}(\mathbb{R})^+g^{-1}\cdot x_G) - \dim(G_{N_l}(\mathbb{R})^+x_{0,G}) \\ &= \dim(\mathrm{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G})) - \dim(\mathrm{Stab}_{gG_{N_l}(\mathbb{R})^+g^{-1}}(x_G)) \\ &\geqslant 0\end{aligned}$$

because $\mathrm{Stab}_{gG_{N_l}(\mathbb{R})^+g^{-1}}(x_G)$ is a compact subgroup of $gG_{N_l}(\mathbb{R})^+g^{-1}$ and $\mathrm{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G})$ is a maximal compact subgroup of $G_{N_l}(\mathbb{R})^+$. Hence

$$\dim(\widetilde{Z}') \leqslant l = \dim(N_l(\mathbb{R})^+U_{N_l}(\mathbb{C})x_0) \leqslant \dim(\widetilde{Z}) \leqslant \dim(\widetilde{Z}')$$

where the first inequality follows from the definition of $B(N_{l,\mathbb{R}}, Y)$. Therefore $\widetilde{Z} = \widetilde{Z}'$ is weakly special. So $\mathrm{unif}(\widetilde{Z})$ is weakly special. □

Define

$$C(N_{l,\mathbb{R}}, Y) := \{\underline{t} := (g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) | (g_U, g_V, g) \in \mathcal{G} \text{ such that } \exists x \in \mathcal{F} \text{ with } \mathrm{unif}(\underline{t}\cdot x) \subset Y \text{ and is not contained in } \cup_{l'>l} \mathfrak{W}_{l'}(Y)\}.$$

Let ψ_l be the morphism from $B(N_{l,\mathbb{R}}, Y)$ to

$$(\mathbb{G}_m(\mathbb{R})^r / \mathrm{Stab}_{\mathbb{G}_m(\mathbb{R})^r} U_{N_l}(\mathbb{R})) \times \mathrm{GSp}_{2g}(\mathbb{R}) / \mathrm{Stab}_{\mathrm{GSp}_{2g}(\mathbb{R})} V_{N_l}(\mathbb{R}) \times G(\mathbb{R}) / N_{G(\mathbb{R})}G_{N_l}(\mathbb{R})^{+\mathrm{nc}},$$

sending $(g_U, g_V, g, x) \mapsto (g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1})$. Then there is a bijection between $\psi_l(B(N_{l,\mathbb{R}}, Y))$ and $C(N_{l,\mathbb{R}}, Y)$.

Lemma 4.1.6. *The set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y))$) is countable.*

Proof. By Lemma 4.1.5, $\mathrm{unif}((g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) \cdot x)$ is weakly special. Hence by Proposition 1.2.4 there exists a $\mathbb{Q}$-subgroup N' of P^{der} such that

$$(g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) = (U_{N'}(\mathbb{C}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\mathrm{nc}}). \tag{4.1.3}$$

But $g_U U_{N_l}(\mathbb{R}) = g_U U_{N_l}(\mathbb{C}) \cap U(\mathbb{R})$ and $U_{N'}(\mathbb{R}) = U_{N'}(\mathbb{C}) \cap U(\mathbb{R})$, so

$$(g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), gG_{N_l}(\mathbb{R})^{+\mathrm{nc}}g^{-1}) = (U_{N'}(\mathbb{R}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\mathrm{nc}}).$$

So $C(N_{l,\mathbb{R}}, Y)$, and therefore $\psi_l(B(N_{l,\mathbb{R}}, Y))$ is countable. □

Proposition 4.1.7. *For any $l > 0$ and N_l,*

1. *the set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y))$) is finite;*
2. *the set $\cup_{l' \geqslant l} \mathfrak{W}_{l'}(Y)$ is definable;*

Proof. We prove the two statements together by induction on l.

Step I. Let d be the maximum of the dimensions of weakly special subvarieties of positive dimension contained in Y. For any N_d, $B(N_{d,\mathbb{R}}, Y)$ is definable by (4.1.2), and hence $\psi_d(B(N_{d,\mathbb{R}}, Y))$ is definable since ψ_d is algebraic. So $\psi_d(B(N_{d,\mathbb{R}}, Y))$, and therefore $C(N_{d,\mathbb{R}}, Y)$, is finite by Lemma 4.1.6.

Consider all the triples

$$\mathfrak{W}_d(Y, \mathcal{T}) := \{(U', V', G') \in \mathcal{T} | \ \exists x \in \mathcal{F} \text{ with } \mathrm{unif}((U'(\mathbb{C}), V'(\mathbb{R}), G'(\mathbb{R})^+)x) \text{ weakly special of dimension } d \text{ contained in } Y\}.$$

By Lemma 4.1.4, there exist finitely many triples $(U'_i, V'_i, G'_i) \in \mathcal{T}$ $(i = 1, ..., n)$ such that any $\underline{t} \in \mathfrak{W}_d(Y, \mathcal{T})$ is of the form $\underline{g} \cdot (U'_i, V'_i, G'_i)$ for some $\underline{g} \in \mathcal{G}$ and some i. Furthermore, by Proposition 1.2.4, we may assume

$$(U'_i, V'_i, G'_i) = (U_{N'_i,\mathbb{R}}, V_{N'_i,\mathbb{R}}, G^{+\mathrm{nc}}_{N'_i,\mathbb{R}})$$

for some $N'_i < Q$ $(i = 1, ..., n)$. But we just proved that $C(N'_{i,\mathbb{R}}, Y)$ is finite $(\forall i = 1, ..., n)$. Hence $\mathfrak{W}_d(Y, \mathcal{T})$ is a finite set. Again by Propostition 1.2.4, each triple of $\mathfrak{W}_d(Y, \mathcal{T})$ equals $(U_{N',\mathbb{R}}, V_{N',\mathbb{R}}, G^{+\mathrm{nc}}_{N',\mathbb{R}})$ for some $N' < P$. We shall denote this triple by N' for simplicity.

Hence

$$\mathfrak{W}_d(Y) = \bigcup_{N' \in \mathfrak{W}_d(Y, \mathcal{T})} \bigcup_{\substack{(1,1,1,x) \\ \in B(N'_{\mathbb{R}}, Y)}} \mathrm{unif}((N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable.

Step II. For any l and N_l, $B(N_{l,\mathbb{R}}, Y)$ is definable by (4.1.2) and induction hypothesis (2). Arguing as in the previous case we get that $C(N_{l,\mathbb{R}}, Y)$ is finite. Define

$$\mathfrak{W}_l(Y, \mathcal{T}) := \{(U', V', G') \in \mathcal{T} \mid \exists x \in \mathcal{F} \text{ with } \operatorname{unif}((U'(\mathbb{C}), V'(\mathbb{R}), G'(\mathbb{R})^+)x) \text{ weakly special of dimension } l \text{ contained in } Y \text{ but not contained in } \cup_{l'>l} \mathfrak{W}_{l'}(Y)\}.$$

Arguing as in the previous case we can get that $\mathfrak{W}_l(Y, \mathcal{T})$ is a finite set and each element of it equals $(U_{N',\mathbb{R}}, V_{N',\mathbb{R}}, G^{+\mathrm{nc}}_{N',\mathbb{R}})$ for some $N' < P$. Hence

$$\bigcup_{l' \geqslant l} \mathfrak{W}_{l'}(Y) = \bigcup_{l'>l} \mathfrak{W}_{l'}(Y) \cup \bigcup_{N' \in \mathfrak{W}_l(Y,\mathcal{T})} \bigcup_{\substack{(1,1,1,x) \\ \in B(N'_{\mathbb{R}},Y)}} \operatorname{unif}(N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable by induction hypothesis (2). □

From now on, for any connected subgroup $N^\dagger$ of P, we will denote by $\mathfrak{F}(N^\dagger)$ the set of all weakly special subvarieties of S defined by the group $N^\dagger$ (see the beginning of this section) and $\mathfrak{F}(N^\dagger, Y) := \{Z \in \mathfrak{F}(N^\dagger) \text{ s.t. } Z \subset Y\}$. Remark that when proving Proposition 4.1.7, we have also given the following description of $\mathfrak{W}(Y) = \cup_{l=1}^d \mathfrak{W}_l(Y)$:

$$\mathfrak{W}(Y) = \bigcup_{N'} \operatorname{unif}(N'(\mathbb{R})^+ U'_N(\mathbb{C})\text{-orbits contained in } \operatorname{unif}^{-1}(Y)) = \bigcup_{N'} \bigcup_{Z \in \mathfrak{F}(N',Y)} Z \tag{4.1.4}$$

which is a finite union on N''s and each N' is of positive dimension. We have assumed that $\mathfrak{W}(Y)$ is Zariski dense in Y (otherwise there is nothing to prove). Therefore by (4.1.4), there exists an N_1 of positive dimension such that

$$\bigcup_{Z \in \mathfrak{F}(N_1,Y)} Z \tag{4.1.5}$$

is Zariski dense in Y.

We now prove $N_1 \lhd P$. If not, then by Proposition 4.1.1, $\cup_{Z \in \mathfrak{F}(N_1)} Z$ equals a finite union of proper special subvarieties of S. The intersection of this union and Y is not Zariski dense in Y since Y is Hodge generic in S. This is a contradiction. Hence $N_1 \lhd P$.

Consider the diagram

$$\begin{array}{ccc} (P, \mathcal{X}^+) & \xrightarrow{\rho_1} & (P_1, \mathcal{X}_1^+) := (P, \mathcal{X}^+)/N_1 \\ \downarrow{\scriptstyle \operatorname{unif}} & & \downarrow{\scriptstyle \operatorname{unif}_1} \\ S & \xrightarrow{[\rho_1]} & S_1 \end{array} \tag{4.1.6}$$

and let $Y_1 := \overline{[\rho_1](Y)}$, which is Hodge generic in S_1. Since $\dim(N_1) > 0$, $\dim(S_1) < \dim(S)$. It is not hard to prove $[\rho]^{-1}(Y_1) = Y$ by the fact (4.1.5). If the union of positive-dimensional weakly special subvarieties contained in

Y_1 is not Zariski dense in Y_1, then take $N = N_1$. Otherwise by the same argument, there exists a normal subgroup $N_{1,2}$ of P_1 such that $\dim(N_{1,2}) > 0$ and $\cup_{Z \in \mathfrak{F}(N_{1,2},Y_1)} Z$ is Zariski dense in Y_1. Let $N_2 := \rho_1^{-1}(N_{1,2})$, then $N_2 \lhd P$. Draw the same diagram (4.1.6) with N_2 instead of N_1, then we get a mixed Shimura variety S_2 with $\dim(S_2) < \dim(S_1)$ and a Hodge generic subvariety Y_2 of S_2. Continue the process (if the union of positive-dimensional weakly special subvarieties contained in Y_2 is Zariski dense in Y_2).

Since $\dim(S) < \infty$, this process will end in a finite step. Hence there exists a number $k > 0$ such that the union of positive-dimensional weakly special subvarieties contained in Y_k is not Zariski dense in Y_k. Then $N := N_k$ is the desired subgroup of P. □

4.2 Lower bound for Galois orbits of special points

For pure Shimura varieties, Ullmo and Pila-Tsimerman have explained separately in [64, §5] [50, §7] how to deduce the André-Oort Conjecture from Ax-Lindemann with a suitable lower bound for Galois orbits of special points. In this section we prove that in order to get a suitable lower bound for Galois orbits of special points for an arbitrary mixed Shimura variety, it is enough to have one for its pure part.

In this section, we will consider mixed Shimura data (resp. varieties) instead of only connected ones. See Definition 1.1.12.

Let $(P, \mathcal{X})$ be a mixed Shimura datum. Let $\pi\colon (P, \mathcal{X}) \to (G, \mathcal{X}_G)$ be the projection to its pure part. We use the notation of §1.1.2.5. In particular, we fix a Levi decomposition $P = W \rtimes G$ and an embedding $(G, \mathcal{X}_G) \hookrightarrow (P, \mathcal{X})$ as in [71, pp 6].

Let K be an open compact subgroup of $P(\mathbb{A}_f)$ defined as follows: for $M > 3$ even, $K_U := MU(\widehat{\mathbb{Z}})$, $K_V := MV(\widehat{\mathbb{Z}})$, $K_W := K_U \times K_V$ with the group law as in §1.1.2.5, $K_G := \{g \in G(\widehat{\mathbb{Z}}) | g \equiv 1 \bmod M\}$ and $K := K_W \rtimes K_G$.

Let s be a special point of $M_K(P, \mathcal{X})$ which corresponds to a special point $x \in \mathcal{X}$. The group $\mathrm{MT}(x)$ is of the form wTw^{-1} for a torus $T \subset G$ and $w \in W(\mathbb{Q})$. Let $\mathrm{ord}(w) \in \mathbb{Z}_{>0}$ be the smallest integer such that $\mathrm{ord}(w)w \in W(\mathbb{Z})$. Define *the order of* s to be $N(s) := \mathrm{ord}(w)$.

Remark 4.2.1. *It is not hard to show that if the fiber of* $S \xrightarrow{[\pi]} S_G$ *is a semi-abelian variety, then* $N(s)$ *coincides with the order of* s *as a torsion point on the fiber (up to a constant).*

Attached to $(P, \mathcal{X})$ there is a number field $E = E(P, \mathcal{X})$ called the **reflex field** and $M_K(P, \mathcal{X})$ is defined over E (cf. [53, 11.5]). We want a comparison of $|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|$ and $|\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|$.

Define $(G^w, \mathcal{X}_{G^w}) := (wGw^{-1}, w^{-1} \cdot \mathcal{X}_G)$, $K_{G^w} := G^w(\mathbb{A}_f) \cap K$ and $K'_G :=$

$w^{-1}K_{G^w}w$, then we have the following commutative diagram:

$$\begin{array}{ccc} M_{K_{G^w}}(G^w, \mathcal{X}_{G^w}) & \hookrightarrow & M_K(P, \mathcal{X}) \\ \wr \downarrow [w^{-1}\cdot] & & \downarrow [\pi] \\ M_{K'_G}(G, \mathcal{X}_G) & \xrightarrow{\rho} & M_{K_G}(G, \mathcal{X}_G) \end{array} .$$

All the morphisms in this diagram are defined over E since the reflex field of $(P, \mathcal{X})$, $(G, \mathcal{X}_G)$ and $(G^w, \mathcal{X}_{G^w})$ are all E. Denote by $s' := [w^{-1}\cdot](s)$. Let $T^w := wTw^{-1}$. Let $K'_T := K \cap T^w(\mathbb{A}_f)$ and let $K_T := K \cap T(\mathbb{A}_f)$. The following inequality follows essentially from [66, §2.2] (note that we do not need GRH for this inequality since [66, Lemma 2.13, 2.14] are not used!). We refer to the Appendix of this chapter, or more concretely Theorem 4.4.1, for a more precise version.

$$\begin{aligned} |\mathrm{Gal}(\overline{\mathbb{Q}}/E)s| &= |\mathrm{Gal}(\overline{\mathbb{Q}}/E)s'| \\ &\geqslant B^{i(T)}|K_T/K'_T||\mathrm{Gal}(\overline{\mathbb{Q}}/E)\rho(s')| \\ &= B^{i(T)}|K_T/K'_T||\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)| \end{aligned} \tag{4.2.1}$$

for some $B \in (0,1)$ depending only on $(P, \mathcal{X})$.

Write $w = (u, v)$ under the identification $W \simeq U \times V$ in §1.1.2.5. All elements of $w^{-1}Kw$ are of the form

$$(-u, -v, 1)(u', v', g')(u, v, 1) = (u' - (u - g'u) - \Psi(v, v'), v' - (v - g'v), g')$$

with $(u', v', g') \in K$. Since $K'_T = w^{-1}K_{T^w}w = w^{-1}Kw \cap T(\mathbb{A}_f)$, this element is in K'_T iff

- $u' = u - g'u + \Psi(v, v') \in K_U$
- $v' = v - g'v \in K_V$
- $g' \in T(\mathbb{A}_f) \cap K_G = K_T$.

So

$$t \in w^{-1}K_{T^w}w \iff \begin{cases} t \in K_T; \\ v - tv \in K_V = MV(\widehat{\mathbb{Z}}); \\ u - tu + \Psi(v, v - tv) \in K_U = MU(\widehat{\mathbb{Z}}). \end{cases} \tag{4.2.2}$$

Lemma 4.2.2. $|K_T/K'_T| \geqslant \mathrm{ord}(w)\prod_{p|\,\mathrm{ord}(w)}(1 - \frac{1}{p})$.

Proof. Let T' be the image of $\mathbb{G}_{m,\mathbb{R}} \xrightarrow{\omega} \mathbb{S} \xrightarrow{w^{-1}\cdot x} G_{\mathbb{R}}$, then it is an algebraic torus defined over $\mathbb{Q}$ by Remark 1.1.13(1). We always have $T' < T$. If T' is trivial, then $P = G$ is adjoint by reason of weight, and $\mathrm{ord}(w) = 1$. If not, $T' \simeq \mathbb{G}_{m,\mathbb{Q}}$ and

$$T'(M) := \{t' \in T'(\widehat{\mathbb{Z}}) | t' \equiv 1 \bmod (M)\} \subset K_G \cap T(\mathbb{A}_f) = K_T.$$

So
$$T'(M)/(T'(M)\cap w^{-1}K_{T^w}w)\hookrightarrow K_T/w^{-1}K_{T^w}w.$$
Hence it is enough to prove that LHS is of cardinality$\geqslant$ ord(w).

Since T' acts on V and U via a scalar, $t'\in T'(M)\cap w^{-1}K_{T^w}w$ iff

1. $t'\in T'(M)$
2. $v-t'v\in MV(\widehat{\mathbb{Z}})$
3. $u-t'u\in MU(\widehat{\mathbb{Z}})$.

Let $t'\in T'(M)\subset T'(\widehat{\mathbb{Z}})=\widehat{\mathbb{Z}}^*$. Suppose $\mathrm{ord}(w)=\prod p^{n_p}$ and $M=\prod p^{m_p}$. If $n_p=0$, then condition (2) and (3) are automatically satisfied. If $n_p>0$, then condition (2) and (3) imply that $t'_p=1+a_{n_p+m_p}p^{n_p+m_p}+...\in\mathbb{Z}_p^*$, hence

$$|T'(\mathbb{Z}_p)\cap T'(M)/(T'(\mathbb{Z}_p)\cap T'(M)\cap w^{-1}K_{T^w,p}w)|=p^{n_p-1}(p-1).\quad (4.2.3)$$

To sum up,

$$|T'(M)/(T'(M)\cap w^{-1}K_{T^w}w)|=\mathrm{ord}(w)\prod_{p|\,\mathrm{ord}(w)}(1-\frac{1}{p}).\quad (4.2.4)$$

□

Theorem 4.2.3. *For any $\varepsilon\in(0,1)$, there exist a positive constant C_ε (depending only on $(P,\mathcal{X})$ and ε) such that*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|\geqslant C_\varepsilon N(s)^{1-\varepsilon}|\mathrm{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|.$$

Proof. We have proved in Lemma 4.2.1

$$p|\,\mathrm{ord}(w)\iff K_{T,p}\neq K'_{T,p}.\quad (4.2.5)$$

Hence denoting by $\varsigma(M):=|\{p,p|M\}|$ for any $M\in\mathbb{Z}_{>0}$, we have by Lemma 4.2.1

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/E)s|\geqslant B^{\varsigma(N(s))}N(s)\prod_{p|N(s)}(1-\frac{1}{p})|\mathrm{Gal}(\overline{\mathbb{Q}}/E)\rho(s')|$$

by Lemma 4.2.2. Now the theorem follows from the basic facts of elementary math:

$$\forall\varepsilon\in(0,1),\text{ there exists }C_\varepsilon>0\text{ such that }B^{\varsigma(N(s))}N(s)^\varepsilon\geqslant C_\varepsilon.\quad (4.2.6)$$

$$\forall\varepsilon\in(0,1),\text{ there exists }C'_\varepsilon>0\text{ such that }N(s)^\varepsilon\prod_{p|N(s)}(1-\frac{1}{p})\geqslant C'_\varepsilon.\quad (4.2.7)$$

□

Corollary 4.2.4. *For A an abelian variety over a number field $k \subset \mathbb{C}$ and t a torsion point of $A(\mathbb{C})$, denote by $N(t)$ its order and $k(t)$ the field of definition of t over k.*

Let $g, d \in \mathbb{N}_+$ and let $\varepsilon \in (0,1)$. There exists $c > 0$ such that for all number fields $k \subset \mathbb{C}$ of degree d over $\mathbb{Q}$, all g-dimensional CM abelian varieties A over k and all torsion points t in $A(\mathbb{C})$,

$$[k(t) : k] \geqslant cN(t)^{1-\varepsilon}.$$

Proof. (compare with [59]) By Zarhin's trick, it suffices to give a proof for A principally polarized. Such an A can be realized as a fiber of $\mathfrak{A}_g(4) \to \mathcal{A}_g(4)$, and any torsion point t of A is a special point of $\mathfrak{A}_g(4)$. Now this result is a direct consequence of Proposition 4.2.3. □

Remark 4.2.5. *The lower bound of the Galois orbit of a special point for pure Shimura varieties is given by [64, Conjecture 2.7]. It has been proved under the Generalized Riemann Hypothesis by Ullmo-Yafaev [66]. For the case of $\mathcal{A}_g$, it is equivalent to the following conjectural lower bound (suggested and proved for $g = 2$ by Edixhoven [19, 18]): suppose that $x \in \mathcal{A}_g$ is a special point. Let A_x denote the CM abelian variety parametrised by x and let R_x be the center of $\mathrm{End}(A_x)$, then there exists $\delta(g) > 0$ such that*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_g |\,\mathrm{disc}(R_x)|^{\delta(g)}. \tag{4.2.8}$$

For their equivalence see [62, Theorem 7.1]. The best unconditional result is given by Tsimerman [62, Theorem 1.1]: (4.2.8) is true when $g \leqslant 6$ (and for $g \leqslant 3$ by a similar method in [68]).

Hence for a mixed Shimura variety of Siegel type of genus g and any special point x, Theorem 4.2.3 tells us that if [64, Conjecture 2.7] is verified for the pure part, then for any $\varepsilon \in (0,1)$, there exists $\delta(g) > 0$ such that

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_{g,\varepsilon} N(x)^{1-\varepsilon}|\,\mathrm{disc}(R_{[\pi](x)})|^{\delta(g)}.$$

4.3 The André-Oort conjecture and its weak form

4.3.1 The André-Oort conjecture

Theorem 4.3.1. *Let S be a connected mixed Shimura variety of abelian type (i.e. its pure part is of abelian type). Let Y be an irreducible subvariety of S containing a Zariski-dense set of special points. If (4.2.8) holds for the pure part of S (this is true if we assume GRH), then Y is special.*

In particular, by [62, Theorem 1.1], the André-Oort Conjecture holds unconditionally for any mixed Shimura variety whose pure part is a subvariety of $\mathcal{A}_6^n$.

Proof. Suppose S is associated with $(P, \mathcal{X}^+)$. Replacing Γ by a neat subgroup does not change the assumption or the conclusion, so we may assume that $\Gamma = \{\gamma \in P(\mathbb{Z}) | \gamma \equiv 1 \bmod M\}$ for some $M > 3$ even. Replacing S by the smallest connected mixed Shimura subvariety does not change the assumption or the conclusion, so we may assume that Y is Hodge generic in S. Since Y contains a Zariski-dense set of special points, we may assume that Y is defined over a number field k. Suppose that Y is not special.

If the set of positive-dimensional weakly special subvarieties of Y is Zariski dense in Y, then let N be the normal subgroup P as in Theorem 4.1.3. Consider the diagram (4.1.1), then Y is special iff $Y' := \overline{[\rho](Y)}$ is. The connected mixed Shimura variety S' is again of abelian type. Replacing (S, Y) by (S', Y'), we may assume that the set of positive-dimensional special subvarieties of Y is not Zariski dense in Y.

Now we are left prove that the set of special points of Y which do not lie in any positive-dimensional special subvariety is finite.

By definition, there exists a Shimura morphism $(G, \mathcal{X}_G^+) \rightarrow \prod_{i=1}^r (\mathrm{GSp}_{2g}^{(i)}, \mathbb{H}_g^{+(i)})$ (the upper-index (i) is to distinguish different factors) such that $G \rightarrow \prod_{i=1}^r \mathrm{GSp}_{2g}^{(i)}$ has a finite kernel (contained in the center) and $\mathcal{X}_G^+ \hookrightarrow \prod_{i=1}^r \mathbb{H}_g^{+(i)}$. Therefore under Proposition 1.3.3, we can identify $\mathcal{X}^+$ as a subspace of $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathbb{H}_g^{+r}$. Then any special point is contained in $U(\mathbb{Q}) \times V(\mathbb{Q}) \times (\mathbb{H}_g^{+r} \cap M_{2g}(\overline{\mathbb{Q}})^r)$ and hence we can define its height (for $\overline{\mathbb{Q}}$-points, see [12, Definition 1.5.4 multiplicative height]).

Now take $\mathcal{F}$ as in §3.3.1. For any special point $x \in S$, take a representative $\widetilde{x} \in \mathrm{unif}^{-1}(x)$ in $\mathcal{F}$, then by [49, Theorem 3.1], $H(\widetilde{x}_{G,i}) \ll |\operatorname{disc}(R_{[\pi](x)_i})|^{B_g}$ for a constant B_g ($\forall i = 1, ..., r$). By choice of $\mathcal{F}$, $H(\widetilde{x}_V)$, $H(\widetilde{x}_U) \ll N(x)$ (see Remark 1.3.4). If (4.2.8) holds, then by Proposition 4.2.3

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/k)x| \gg_g H(\widetilde{x})^{\varepsilon(g)}$$

for some $\varepsilon(g) > 0$. Hence for $H(\widetilde{x}) \gg 0$, Pila-Wilkie [48, 3.2] implies that $\exists \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ such that $\widetilde{\sigma(x)}$ is contained in a connected semi-algebraic subset $\widetilde{Z}$ of $\mathrm{unif}^{-1}(Y) \cap \mathcal{F}$ of positive dimension. Let Z' be an irreducible component of $\overline{\mathrm{unif}(\widetilde{Z})}$ containing $\mathrm{unif}(\widetilde{\sigma(x)})$. Theorem 3.1.4 tells us that Z' is weakly special. Hence $\sigma^{-1}(Z')$ is weakly special containing a special point x, and therefore is special. But $\dim(Z') > 0$ since $\dim(\widetilde{Z}) > 0$. Hence $\sigma^{-1}(Z')$ is special of positive dimension. To sum up, the heights of the elements of

$$\{\widetilde{x} \in \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ special and } \mathrm{unif}(\widetilde{x}) \text{ is not contained in any positive-dimensional special subvariety}\}$$

is uniformly bounded, and hence this set is finite by Northcott's theorem [12, Theorem 1.6.8]. □

4.3.2 The weak form of the André-Oort conjecture

By the proof of Theorem 4.3.1, we can see that the only obstacle left to claim the whole André-Oort conjecture for mixed Shimura varieties of abelian type is the lower bound (4.2.8). However if we consider a weaker version of the André-Oort conjecture, this obstacle is removed by a series of work of Habegger-Pila [24] and Orr [43]. Thus by a similar proof to Theorem 4.3.1, we can prove the following theorem unconditionally. This theorem generalizes the previous work of Edixhoven-Yafaev [72, 20] (for curves in pure Shimura varieties) and Klingler-Ullmo-Yafaev [66, 30] (for pure Shimura varieties). Its p-adic version for $\mathfrak{A}_g$ has been proved by Scanlon [58] based on the result of Moonen for $\mathcal{A}_g$ [40].

Theorem 4.3.2. *Let S be a connected mixed Shimura variety whose pure part S_G is a subvariety of $\mathcal{A}_g$ for some g. Denote by $S \xrightarrow{[\pi]} S_G$. Let Y be an irreducible subvariety of S and let a be a special point of $\mathcal{A}_g$ whose corresponding abelian variety is denoted by A_a. Consider the set*

$$\Sigma'_a := \{s \in S \text{ special such that } A_{[\pi]s} \text{ is isogenous to } A_a, \text{ where } A_{[\pi]s} \text{ is the abelian variety represented by } [\pi]s\}.$$

If $\overline{Y \cap \Sigma'_a} = Y$, then Y is a special subvariety.

Proof. We may assume $a \in [\pi]Y$. Suppose S is associated with $(P, \mathcal{X}^+)$. Replacing Γ by a neat subgroup does not change the assumption or the conclusion, so we may assume that $\Gamma = \{\gamma \in P(\mathbb{Z}) | \gamma \equiv 1 \bmod M\}$ for some $M > 3$ even. Replacing S by the smallest connected mixed Shimura subvariety does not change the assumption or the conclusion, so we may assume that Y is Hodge generic in S.

Let $(G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/\mathcal{R}_u(P)$. By Theorem 4.1.3, such a group N (which may be trivial) exists: N is the maximal normal subgroup of P such that the followings hold:

- there exists a diagram of Shimura morphisms

$$\begin{array}{ccccc} (P, \mathcal{X}^+) & \xrightarrow{\rho} & (P', \mathcal{X}'^+) := (P, \mathcal{X}^+)/N & \xrightarrow{\pi'} & (G', \mathcal{X}'^+_G) := (P', \mathcal{X}'^+)/\mathcal{R}_u(P') \\ \Big\downarrow{\scriptstyle \mathrm{unif}} & & \Big\downarrow{\scriptstyle \mathrm{unif}'} & & \Big\downarrow{\scriptstyle \mathrm{unif}'_G} \\ S & \xrightarrow{[\rho]} & S' & \xrightarrow{[\pi']} & S'_G \end{array}$$

- the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is not Zariski dense in Y';

- $Y = [\rho]^{-1}(Y')$.

Suppose that Y is not special. Then Y' is not a special subvariety of S'. On the other hand, Y' is defined over a number field since it contains a Zariski dense subset of special points.

Define $W_N := \mathcal{R}_u(N) < W := \mathcal{R}_u(P)$ and $G_N := N/W_N \lhd G < \mathrm{GSp}_{2g}$. The reductive group G decomposes as an almost direct product $Z(G)^\circ H_1...H_r$ with all H_i's simple. Without any loss of generality, we may assume that H_1,...,H_l are the simple factors of G which appear in the decomposition of G_N. Define $G_N^\perp := H_{l+1}...H_r$. Define $T := \mathrm{MT}(a)$, then T is a torus since a is a special point of $\mathcal{A}_g$.

Let $G_1 := G_N^\perp T$. This is a subgroup of G (and therefore a subgroup of GSp_{2g}). Moreover, it defines a connected Shimura subdatum $(G_1, \mathcal{X}_{G_1}^+)$ of $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$ and hence its associated connected Shimura subvariety S_{G_1} of $\mathcal{A}_g$ such that $a \in S_{G_1}$. Recall that $(P', \mathcal{X}'^+) = (P, \mathcal{X}^+)/N$ and $(G', \mathcal{X}'^+_G) = (G, \mathcal{X}_G^+)/G_N$. Therefore the natural Shimura morphisms

$$(G_1, \mathcal{X}_{G_1}^+) \hookrightarrow (G, \mathcal{X}_G^+) \twoheadrightarrow (G', \mathcal{X}'^+_G)$$

identify $\mathcal{X}_{G_1}^+$ and $\mathcal{X}'^+_G$.

Consider the connected mixed Shimura datum $(P, \mathcal{X}^+)$. Then $W := \mathcal{R}_u(P)$ is a G_1-module such that the action of G_1 on W induces a Hodge-structure of type $\{(-1,0),(0,-1),(-1,-1)\}$ on $\mathrm{Lie}\,W$. Therefore by Proposition 1.1.23, there exists a connected mixed Shimura datum $(P_1, \mathcal{X}_1^+)$ such that $P_1 = W \rtimes G_1$ and $(G_1, \mathcal{X}_{G_1}) = (P_1, \mathcal{X}_1^+)/W$. Now $(P_1, \mathcal{X}_1^+)$ is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$. Since $N \lhd P$, we have $W_N \lhd P_1$. Now we have the following diagram of Shimura morphisms:

$$\begin{array}{ccccccc}
(P_2, \mathcal{X}_2^+) := (P_1, \mathcal{X}_1^+)/W_N & \xleftarrow{\rho'} & (P_1, \mathcal{X}_1^+) & \xhookrightarrow{j} & (P, \mathcal{X}^+) & \xrightarrow{\rho} & (P', \mathcal{X}'^+) = (P, \mathcal{X}^+)/N \\
\downarrow \mathrm{unif}_2 & & \downarrow & & \downarrow & & \downarrow \mathrm{unif}' \\
S_2 & \xleftarrow{[\rho']} & S_1 & \xrightarrow{[j]} & S & \xrightarrow{[\rho]} & S'
\end{array}.$$

Then the map $\rho \circ j \circ \rho'^{-1} \colon (P_2, \mathcal{X}_2^+) \to (P', \mathcal{X}'^+)$ is well-defined and is a Shimura morphism. Hence Y' is a special subvariety of S' iff $Y_2 := ([\rho]\circ[j]\circ[\rho']^{-1})^{-1}(Y')$ is a special subvariety of S_2. Hence it suffices to prove that Y_2 is special. But $\mathcal{X}_2^+$ and $\mathcal{X}'^+$ are identified under $\rho \circ j \circ \rho'^{-1}$ by the discussion in the last paragraph, so the union of positive-dimensional weakly special subvarieties of Y_2 is not Zariski dense in Y_2 by choice of Y'. Therefore we are left to prove that the set of special points of Y_2 which do not lie in any positive-dimensional special subvariety is finite. Remark that Y_2 is defined over a number field (which we call k) since Y' is.

Take the pure part of the diagram above, we get the following diagram of

Shimura morphisms between pure Shimura data and pure Shimura varieties:

$$\begin{array}{ccccccc} (G_2,\mathcal{X}_{G_2}^+) & \xleftarrow[\sim]{\rho'_G} & (G_1,\mathcal{X}_{G_1}^+) & \xhookrightarrow{j_G} & (G,\mathcal{X}_G^+) & \xrightarrow{\rho_G} & (G',\mathcal{X}_G'^+) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_{G_2} & \xleftarrow[\sim]{[\rho'_G]} & S_{G_1} & \xrightarrow{[j_G]} & S_G & \xrightarrow{[\rho_G]} & S'_G \end{array}.$$

Therefore $\mathcal{X}_{G_2}^+$ can be seen as a subset of $\mathcal{X}_G^+$, and hence of $\mathbb{H}_g^+$. Denote by $[\pi_2]\colon S_2 \to S_{G_2}$. Let

$$\Sigma''_a := \{t \in S_2 \text{ special such that } A_{[\pi_2]t} \text{ is isogenous to } A_a, \text{ where } A_{[\pi_2]t} \text{ is the abelian variety represented by } [\pi_2]t\}.$$

Since $\overline{Y \cap \Sigma'_a} = Y$, we have $\overline{Y' \cap [\rho](\Sigma'_a)} = Y'$. But then by the identification of $\mathcal{X}_2^+$ and $\mathcal{X}'^+$, we get that

$$\overline{Y_2 \cap \Sigma''_a} = Y_2.$$

For any $t \in \Sigma''_a$, take a representative $\widetilde{t} \in \mathrm{unif}_2^{-1}(t)$ in the fundamental set $\mathcal{F}$ as in §3.3.1. Then $\widetilde{t} = (\widetilde{t}_U, \widetilde{t}_V, \widetilde{t}_G) \in U_2(\mathbb{Q}) \times V_2(\mathbb{Q}) \times (\mathbb{H}_g^+ \cap M_{2g}(\overline{\mathbb{Q}}))$ and hence we can define its height. By choice of $\mathcal{F}$, both $H(\widetilde{t}_U)$ and $H(\widetilde{t}_V)$ are bounded by $N(t)$ which is defined as in the paragraph above Remark 4.2.1 (see Remark 1.3.4). But up to constants depending only on a (or more explicitely, only on $H(\widetilde{a})$), $H(\widetilde{t}_G)$ is polynomially bounded from above by the minimum degree of the isogenies $A_{[\pi_2]t} \to A_a$. This follows from [43, Proposition 4.1, Section 4.2]. But the minimum degree of the isogenies $A_{[\pi_2]t} \to A_a$ is polynomially bounded from above by $|\mathrm{Gal}(\overline{\mathbb{Q}}/k)[\pi_2]t|$. This follows from [43, Theorem 5.1]. Hence by Theorem 4.2.3,

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/k)t| \gg_{g,\widetilde{a}} H(\widetilde{t})^{\mu(g,\widetilde{a})}$$

for some $\mu(g,\widetilde{a}) > 0$. Hence for $H(\widetilde{t}) \gg 0$, Pila-Wilkie [48, 3.2] implies that there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ such that $\widetilde{\sigma(t)}$ is contained in a connected semi-algebraic subset $\widetilde{Z}$ of $\underline{\mathrm{unif}_2^{-1}(Y_2)} \cap \mathcal{F}$ of positive dimension. Let Z' be an irreducible component of $\mathrm{unif}(\widetilde{Z})$ containing $\mathrm{unif}(\widetilde{\sigma(t)})$. Theorem 3.1.4 tells us that Z' is weakly special. Hence $\sigma^{-1}(Z')$ is weakly special containing a special point t, and therefore is special. But $\dim(Z') > 0$ since $\dim(\widetilde{Z}) > 0$. Hence $\sigma^{-1}(Z')$ is special of positive dimension. To sum it up, the heights of the elements of

$$\{\widetilde{t} \in \mathrm{unif}_2^{-1}(Y_2) \cap \mathcal{F} \text{ special and } \mathrm{unif}_2(\widetilde{t}) \text{ is not contained in a positive-dimensional special subvariety of } S_2\}$$

is uniformly bounded from above. Therefore this set is finite by Northcott's theorem. □

4.4 Appendix: comparison of Galois orbits of special points of pure Shimura varieties

Let $(G, \mathcal{X}_G)$ be a pure Shimura datum satisfying

$$\begin{array}{c} Z(G)^\circ \text{ is an almost direct product of a } \mathbb{Q}\text{-split torus } Z_G^s \\ \text{with a torus of compact type } Z_G^c \text{ defined over } \mathbb{Q} \end{array} \tag{SV5}$$

In this case, G is an almost direct product of Z_G^s with $G^c := Z_G^c G^{\mathrm{der}}$. Let $E = E(G, \mathcal{X}_G)$ be its reflex field and let $K' = \prod_p K'_p \subset K = \prod_p K_p$ be two neat open compact subgroups of $G(\mathbb{A}_f)$. We have a natural morphism

$$\rho\colon M_{K'}(G, \mathcal{X}_G) \to M_K(G, \mathcal{X}_G). \tag{4.4.1}$$

By [37, Theorem 5.5, Proposition 5.2], $M_{K'}(G, \mathcal{X}_G)$, $M_K(G, \mathcal{X}_G)$ and ρ can all be defined over E.

Let s be a special point of $M_{K'}(G, \mathcal{X}_G)$, then $s \in M_{K'}(G, \mathcal{X}_G)(\overline{E})$. The goal of this section is to compare $|\mathrm{Gal}(\overline{E}/E)s|$ and $|\mathrm{Gal}(\overline{E}/E)\rho(s)|$. Let $T := \mathrm{MT}(s)$ be the Mumford-Tate group of s. Define $K'_T := K' \cap T(\mathbb{A}_f)$ and $K_T := K \cap T(\mathbb{A}_f)$. Then $K'_T = \prod_p K'_{T,p}$ and $K_T = \prod_p K_{T,p}$. Now we can state our theorem:

Theorem 4.4.1. *There exists a constant $B \in (0, 1)$ depending only on $(G, \mathcal{X})$ such that*

$$|\mathrm{Gal}(\overline{E}/E)s| \geqslant B^{i(T)} |K_T/K'_T| |\mathrm{Gal}(\overline{E}/E)\rho(s)|$$

where $i(T) = |\{p : K_{T,p} \neq K'_{T,p}\}|$.

Proof. This is a direct consequence of Lemma 4.4.4, (4.4.2), Lemma 4.4.6 and Lemma 4.4.7. □

Remark 4.4.2. *This theorem has essentially been proved by Ullmo-Yafaev [66, §2.2]: the authors proved this result for a less general $(G, \mathcal{X}_G)$ and a particular K_T, but their proof also works for our $(G, \mathcal{X}_G)$ and arbitrary K_T as long as it is neat. To make the demonstration more clear, we summarize their results and arguments and see how they apply to our $(G, \mathcal{X}_G)$ and a general K_T.*

Lemma 4.4.3. *For any point $y \in M_K(G, \mathcal{X}_G)$, K acts transitively on the right on $\rho^{-1}(y)$ and the stabilizer of any point of $\rho^{-1}(y)$ is K'. By consequence ρ is étale of degree $|K/K'|$.*

Proof. (cf. [66, Lemma 2.11]) Let $y = \overline{(x, g)}$ be a point of $M_K(G, \mathcal{X})$, then $\rho^{-1}(y) = \overline{(x, gK)}$. We first prove that $\forall a \in K$,

$$\overline{(x, ga)} = \overline{(x, gak)} \text{ in } M_{K'}(G, \mathcal{X}) \iff k \in K'.$$

The direction $\Leftarrow$ is trivial. Now let us prove $\Rightarrow$. Suppose

$$\overline{(x, ga)} = \overline{(x, gak)} \in M_{K'}(G, \mathcal{X})$$

with $k \in K$. There exist $q \in G(\mathbb{Q})$ and $k' \in K'$ such that $x = qx$ and $ga = qgakk'$. The second condition implies $q \in gKg^{-1}$.

Define $G' := G/Z_G^s$, then $(G, \mathcal{X}_G)/Z_G^s = (G', \mathcal{X}_G)$ is a Shimura datum such that $Z(G')(\mathbb{R})$ is compact. Now we have $x = \overline{q}x$ and $\overline{q} \in \overline{gKg^{-1}}$ where we add $^{-}$ to denote elements and subsets of G'. The set $\overline{gKg^{-1}}$ is a neat open compact subgroup of $G'(\mathbb{A}_f)$ and $\overline{q} \in G'(\mathbb{Q})$. Since $Z(G')(\mathbb{R})$ is compact, $\mathrm{Stab}_{G'(\mathbb{R})}(x)$ is compact (see e.g. [66, Remark 2.3]). But $G'(\mathbb{Q}) \cap \overline{gKg^{-1}}$ is a lattice of $G'(\mathbb{R})$, so $\mathrm{Stab}_{G'(\mathbb{R})}(x) \cap G'(\mathbb{Q}) \cap \overline{gKg^{-1}}$ is finite. Furthermore the latter intersection must be $\{1\}$ since $\overline{gKg^{-1}}$ is neat. Therefore as an element of the latter intersection, $\overline{q} = 1$. Hence $q \in Z_G^s(\mathbb{Q}) \simeq (\mathbb{Q}^*)^n$. This implies also $q \in Z_G^s(\mathbb{A}_f) \cap gKg^{-1}$, which is a neat open compact subgroup of $Z_G^s(\mathbb{A}_f) \simeq (\mathbb{A}_f^*)^n$. But the intersection of $(\mathbb{Q}^*)^n$ with any neat open compact subgroup of $(\mathbb{A}_f^*)^n$ is trivial, hence $q = 1$.

Now $ga = gakk'$ implies $k = (k')^{-1} \in K'$. So K acts transitively on the right on $\rho^{-1}(y)$ and the stabilizer of any point of $\rho^{-1}(y)$ is K'. □

Lemma 4.4.4. $|\mathrm{Gal}(\overline{E}/E)s| \geqslant |\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \cdot |\mathrm{Gal}(\overline{E}/E)\rho(s)|$.

Proof. (cf. [66, Lemma 2.12]) Because ρ is defined over E, $|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}(\sigma(\rho(s)))|$ is independent of $\sigma \in \mathrm{Gal}(\overline{E}/E)$. This allows us to conclude. □

To give a lower bound for $|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)|$, we shall work with the Shimura subdatum (T, x) of $(G, \mathcal{X}_G)$. The Shimura subdatum (T, x) is defined as follows: $T = \mathrm{MT}(s)$. By [38, Lemma 5.13], $M_{K'}(G, \mathcal{X}_G) = \coprod \Gamma(g)\backslash \mathcal{X}^+$, where $\Gamma(g) = G(\mathbb{Q})_+ \cap gK'g^{-1}$ is a congruence subgroup of $G(\mathbb{Q})$. Choose $x \in \mathcal{X}^+$ such that s is the image of x under the uniformization. The Shimura datum (T, x) still satisfies (SV5) (see e.g. [66, Remark 2.3]).

Let F be the reflex field of (T, x), then F is a finite extension of E. Define

$$\rho' \colon M_{K'_T}(T, x) \to M_{K_T}(T, x),$$

which is the restriction of ρ, then ρ' is defined over F. We have

$$|\mathrm{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \geqslant |\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \tag{4.4.2}$$

Let $\pi_0(M_{K'_T}(T, x))$ be the set of geometric components of $M_{K'_T}(T, x)$. Recall that

$$\pi_0(M_{K'}(T, x)) = T(\mathbb{Q})_+ \backslash T(\mathbb{A}_f)/K'_T.$$

This is a finite abelian group. The action of $\mathrm{Gal}(\overline{E}/F)$ on $\pi_0(M_{K'_T}(T, x))$ is given by the reciprocity morphism

$$r \colon \mathrm{Gal}(\overline{E}/F) \to \pi_0(M_{K'_T}(T, x)).$$

Let us describe this action more explicitly. Denote for any $\alpha \in T(\mathbb{A}_f)$ by $\overline{(x, \alpha)}$ the image of (x, α) in $M_{K'_T}(T, x)$. It is a connected component of $M_{K'_T}(T, x)$.

As sets we have the following identification:

$$\begin{array}{ccc} \{\overline{(x,\alpha)} \mid \alpha \in T(\mathbb{A}_f)\} & \xrightarrow{\sim} & \pi_0(M_{K'_T}(T,x)) \\ \overline{(x,\alpha)} & \mapsto & \overline{\alpha} \end{array}.$$

Let $\sigma \in \mathrm{Gal}(\overline{E}/F)$ and let $t \in T(\mathbb{A}_f)$ such that $\overline{t} = r(\sigma)$, then $\forall \alpha \in T(\mathbb{A}_f)$,

$$\sigma(\overline{(x,\alpha)}) = \overline{(x,t\alpha)} = \overline{(x,\alpha t)}. \tag{4.4.3}$$

Recall the following result from Ullmo-Yafaev [66, Proposition 2.9]:

Lemma 4.4.5. *There exists a positive integer A depending only on $(G,\mathcal{X})$ such that $\forall m \in T(\mathbb{A}_f)$, the image of m^A in $\pi_0(M_{K'_T}(T,x))$ is $r(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{E}/F)$.*

Proof. [66, Proposition 2.9], which follows from Lemma 2.4-Lemma 2.8 of *loc.cit.*, announces this result when $Z(G)(\mathbb{R})$ is compact. However the only role this hypothesis plays is to guarantee that $T(\mathbb{Q})$ is discrete (hence closed) in $T(\mathbb{A}_f)$ in Lemma 2.8 of *loc.cit.*. Our hypothesis for $Z(G)$ at the beginning of this section implies that T is an almost product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$ (see e.g. [66, Remark 2.3]), and hence $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ ([38, Theorem 5.26]). □

Lemma 4.4.6. *Let Θ_A be the image of the morphism $k \mapsto k^A$ on K_T/K'_T. We have*

1. $\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)$;
2. $|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A|$.

Proof. (cf. [66, Lemma 2.15 & 2.16])

1. We have $\rho'(\Theta_A \cdot s) = \rho'(s)$. So $\Theta_A \cdot s \subset \rho'^{-1}\rho'(s)$. Moreover similar to Lemma 4.4.3, K_H/K'_H acts simply transitively on $\rho'^{-1}\rho'(s)$. For any $\overline{(x,\alpha)} \in \rho'^{-1}\rho'(s)$ and $k \in K_T/K'_T$, this action is given by

$$\overline{(x,\alpha)}k = \overline{(x,\alpha k)}. \tag{4.4.4}$$

Let $m \in K_T$, then the image of m^A in $\pi_0(M_{K'_T}(T,x))$ is $r(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{E}/F)$ by Lemma 4.4.5. It follows that the image of Θ_A in $\pi_0(M_{K'_T}(T,x)) = T(\mathbb{Q})_+\backslash T(\mathbb{A}_f)/K'_T$ is contained in the image of $\mathrm{Gal}(\overline{E}/F)$. So for $s = \overline{(x,\beta)}$, we have $\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s$ by (4.4.4) and (4.4.3). To sum it up,

$$\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s).$$

2. By (1) we have $|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A \cdot s|$. Moreover we have

$$|\rho'^{-1}\rho'(s)| = |(K_T/K'_T) \cdot s| \leqslant \frac{|K_T/K'_T|}{|\Theta_A|}|\Theta_A \cdot s|$$

and

$$|K_T/K'_T| = |\rho'^{-1}\rho'(s)|$$

by the same argument for Lemma 4.4.3. These three (in)equalities yield the desired inequality. Remark that we have also proved $|\Theta_A \cdot s| = |\Theta_A|$.

□

Lemma 4.4.7. *There exists an integer $r > 0$ depending only on $(G, \mathcal{X})$ such that*

$$|\Theta_A| \geqslant \prod_{\{p : K_{T,p} \neq K'_{T,p}\}} \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

Proof. (cf. [66, Lemma 2.18]) Since $K_T/K'_T = \prod_p K_{T,p}/K'_{T,p}$, we have

$$\Theta_A = \prod_{\{p : K_{T,p} \neq K'_{T,p}\}} \Theta_{A,p}.$$

Let L_T be the splitting field of T and let $d := \dim(T)$. Then $[L_T : \mathbb{Q}]$ is the size of the image of the representation of $\mathrm{Gal}(\overline{E}/\mathbb{Q})$ on the character group $X^*(T)$ of T. This is a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$ and hence its size is bounded from above in terms of d only. But d is bounded from above in terms of $\dim(G)$ only, so $[L_T : \mathbb{Q}]$ is bounded from above in terms of $\dim(G)$ only.

Using a basis of the character group of T one can embed T into $\mathrm{Res}_{L_T/\mathbb{Q}}\mathbb{G}_{m,L_T}$. Via this embedding, K_T and K'_T are both subgroups of the product of $(\mathbb{Z}_p \otimes O_{L_T})^*$. The group $(\mathbb{Z}_p \otimes O_{L_T})^*$ is the direct product of the groups of units of E_v, completion of E at the place v with $v|p$. By the local unit theorem, the group of units of such an E_v is a dirct product of a cyclic group and $\mathbb{Z}_p^{[E_v:\mathbb{Q}_p]}$.

It follows that there exists a constant r depending only on $(G, \mathcal{X})$ such that $K_{T,p}/K'_{T,p}$ is a finite abelian group which is the product of at most r cyclic factors. Therefore the size of the kernel of the A-th power map on $K_{T,p}/K'_{T,p}$ is bounded by A^r, i.e.

$$\Theta_{A,p} \geqslant \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

□