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Chapter 3

The mixed Ax-Lindemann theorem

Convention: In this chapter we always consider a connected mixed Shimura variety S and its uniformization $\mathcal{X}^+ \xrightarrow{\text{unif}} S$. Unless stated otherwise, all closures taken in S are assumed to be Zariski closures and all closures taken in $\mathcal{X}^+$ are assumed to be closures in the archimedean topology. It happens that they often coincide by Chevalley's theorem in the situations we will consider. But for simplicity I will not discuss this.

3.1 Statement of the theorem

3.1.1 Four equivalent statements for Ax-Lindemann

There are several equivalent forms for the Ax-Lindemann theorem. In this section we will give four different statements and explain their equivalence. The proof for this theorem, being the core of this chapter, will be executed in the following sections.

We start from the most usual form of the Ax-Lindemann theorem. It is also this statement that we will prove afterwards.

Theorem 3.1.1. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\text{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let Y be an irreducible algebraic subvariety of S and let $\widetilde{Z}$ be an irreducible algebraic subset of $\mathcal{X}^+$ contained in* $\text{unif}^{-1}(Y)$, *maximal for these properties. Then $\widetilde{Z}$ is weakly special.*

The next statement we give shall be called the *semi-algebraic form of Ax-Lindemann*. In fact this and its direct variant Theorem 3.1.4 are the forms which will be adopted in all the applications in this dissertation. Recall that a connected semi-algebraic subset of $\mathcal{X}^+$ is called **irreducible** if its $\mathbb{R}$-Zariski closure in $\mathcal{X}^\vee$ is an irreducible real algebraic variety. Note that any connected semi-algebraic subset of $\mathcal{X}^+$ has only finitely many irreducible components.

Theorem 3.1.2. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\text{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let Y be an irreducible algebraic subvariety of S and let $\widetilde{Z}$ be a connected irreducible semi-algebraic subset of $\mathcal{X}^+$ contained in* $\text{unif}^{-1}(Y)$, *maximal for these properties. Then $\widetilde{Z}$ is complex analytic and each complex analytic irreducible component of $\widetilde{Z}$ is weakly special.*

The equivalence of Theorem 3.1.1 and Theorem 3.1.2 follows easily from [49, Lemma 4.1], which claims that **maximal connected irreducible semi-algebraic subsets of $\mathcal{X}^+$ which are contained in** $\mathrm{unif}^{-1}(Y)$ **are all algebraic in the sense of Definition 1.3.5** (there is a typo in the proof of [49, Lemma 4.1]: $\mathbb{C}^{2n}$ should be $\mathbb{C}^n$).

The next two forms of Ax-Lindemann have more "equidistributional" taste. Their equivalence with the two statements above is not hard to check (Theorem 3.1.3 with Theorem 3.1.1, Theorem 3.1.4 with Theorem 3.1.2).

Theorem 3.1.3. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let $\widetilde{Z}$ be any irreducible algebraic subset of $\mathcal{X}^+$. Then* $\overline{\mathrm{unif}(\widetilde{Z})}$ *is weakly special.*

Theorem 3.1.4. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S$ *be the uniformization. Let $\widetilde{Z}$ be any semi-algebraic subset of $\mathcal{X}^+$. Then every irreducible component of* $\overline{\mathrm{unif}(\widetilde{Z})}$ *is weakly special.*

Let us explain now why Theorem 3.1.1 implies Theorem 3.1.3. Let S, $(P, \mathcal{X}^+)$ and $\widetilde{Z}$ be as in Theorem 3.1.3. Let $Y := \overline{\mathrm{unif}(\widetilde{Z})}$ and let $\widetilde{W}$ be an irreducible algebraic subset of $\mathcal{X}^+$ which contains $\widetilde{Z}$ and is contained in $\mathrm{unif}^{-1}(Y)$, maximal for these properties. Such a $\widetilde{W}$ exists by, for example, dimension reason. Then $Y = \overline{\mathrm{unif}(\widetilde{W})}$ and $\widetilde{W}$ is a maximal irreducible algebraic subset of $\mathcal{X}^+$ which is contained in $\mathrm{unif}^{-1}(Y)$. Theorem 3.1.1 then implies that $\widetilde{W}$ is weakly special. Hence $\mathrm{unif}(\widetilde{W})$ is an irreducible subvariety of S by Corollary 2.3.3. So $Y = \overline{\mathrm{unif}(\widetilde{W})} = \mathrm{unif}(\widetilde{W})$ is weakly special since $\widetilde{W}$ is weakly special in $\mathcal{X}^+$. Theorem 3.1.2 implies Theorem 3.1.4 by a similar argument because any semi-algebraic subset of $\mathcal{X}^+$ has only finitely many connected irreducible components.

Let us explain now why Theorem 3.1.3 implies Theorem 3.1.1. Let S, $(P, \mathcal{X}^+)$, Y and $\widetilde{Z}$ be as in Theorem 3.1.1. Then Theorem 3.1.3 tells us that $\overline{\mathrm{unif}(\widetilde{Z})}$ is a weakly special subvariety of S, which we shall call Y_0. By assumption of Y and $\widetilde{Z}$, Y_0 is a subvariety of Y. Let $\widetilde{Y_0}$ be the complex analytic irreducible component of $\mathrm{unif}^{-1}(Y_0)$ containing $\widetilde{Z}$. Then $\widetilde{Y_0}$ is irreducible algebraic by Corollary 2.3.3. But then the maximality assumption on $\widetilde{Z}$ tells us that $\widetilde{Z} = \widetilde{Y_0}$. Hence $\widetilde{Z}$ is weakly special. Theorem 3.1.4 implies Theorem 3.1.2 by a similar argument.

3.1.2 Ax-Lindemann for the unipotent part

In this subsection we state Ax-Lindemann for the unipotent part. There is nothing new in the statement, but it is better to state it here because we will prove it separately in §3.4.

Given a connected mixed Shimura variety S, let S_G be its pure part. We have a projection $S \xrightarrow{[\pi]} S_G$. For any point $b \in S_G$, denote by E the fiber S_b. Suppose that S is associated with the mixed Shimura datum $(P, \mathcal{X}^+)$, which can be further assumed to satisfy $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. Let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Now $E = S_b \simeq \Gamma_W\backslash W(\mathbb{R})U(\mathbb{C})$ with the complex structure determined by $b \in S_G$ ($E = S_b = \Gamma_W\backslash W(\mathbb{C})/F_b^0W(\mathbb{C})$), where $\Gamma_W := \Gamma \cap W(\mathbb{Q})$.

By abuse of notation we denote by $\mathrm{unif}\colon W(\mathbb{R})U(\mathbb{C}) = W(\mathbb{C})/F_b^0W(\mathbb{C}) \to E$ for the uniformization of E. It is then the restriction of $\mathrm{unif}\colon \mathcal{X}^+ \to S$.

Theorem 3.1.5. *Let Y be an irreducible subvariety of E and let $\widetilde{Z}$ be a maximal irreducible algebraic subvariety which is contained in* $\mathrm{unif}^{-1}(Y)$*. Then $\widetilde{Z}$ is weakly special.*

Proof. If E is an algebraic torus over $\mathbb{C}$, this is a consequence of the Ax-Schanuel theorem [42, Corollary 3.6]. If E is an abelian variety, this is Pila-Zannier [51, pp9, Remark 1]. A proof using volume calculation and points counting method for these two cases can be found in the Appendix of this chapter. The general case will be proved in §3.4. □

3.2 Ax-Lindemann Part 1: Outline of the proof

In these three sections, we are going to prove Theorem 3.1.1. The organization of the proof is as follows: the outline of the proof is given in this section. After some preparation, the key proposition (Proposition 3.2.6) leading to the theorem will be stated and exploited (together with Theorem 3.1.5) to finish the proof in Theorem 3.2.8. We prove this key proposition in the next section using Pila-Wilkie's counting theorem and Theorem 3.1.5 will be proved in §3.4.

Now let us fix some notation which will be used through the whole proof:

Notation 3.2.1. *Consider the following diagram:*

$$\begin{array}{ccc} \mathcal{X}^+ & \xrightarrow{\quad\pi\quad} & \mathcal{X}_G^+ \\ \Big\downarrow{\scriptstyle \mathrm{unif}} & & \Big\downarrow{\scriptstyle \mathrm{unif}_G} \\ S = \Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi]} & S_G := \Gamma_G\backslash\mathcal{X}_G^+ \end{array}$$

Now we begin the proof of Theorem 3.1.1. Let us first of all do some reduction:

- Since every point of $\mathcal{X}^+$ is weakly special, we may assume $\dim(\widetilde{Z}) > 0$.
- Let $(Q, \mathcal{Y}^+)$ be the smallest mixed Shimura subdatum of $(P, \mathcal{X}^+)$ s.t $\widetilde{Z} \subset \mathcal{Y}^+$ and let S_Q be the corresponding special subvariety of S. Then $Q = \mathrm{MT}(\mathcal{Y}^+)$ by Proposition 1.1.19(1). If we replace $(P, \mathcal{X}^+)$ by $(Q, \mathcal{Y}^+)$, S by S_Q, $\mathrm{unif}\colon \mathcal{X}^+ \to S$ by $\mathrm{unif}_Q\colon \mathcal{Y}^+ \to S_Q$ and Y by an irreducible

component Y_0 of $Y \cap S_Q$, then $\widetilde{Z}$ is again a maximal irreducible algebraic subset of $\mathrm{unif}_Q^{-1}(Y_0)$. By definition, $\widetilde{Z}$ is weakly special in $\mathcal{X}^+$ iff it is weakly special in $\mathcal{Y}^+$. So we may assume $P = \mathrm{MT}(\mathcal{X}^+)$ and that $\widetilde{Z}$ is Hodge generic.

- Furthermore, let Y_0 by the minimal irreducible subvariety of S such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y_0)$, then $\widetilde{Z}$ is still maximal irreducible algebraic in $\mathrm{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. In fact it is not hard to see that after this reduction, $Y = \overline{\mathrm{unif}(\widetilde{Z})}$ and $\widetilde{Z}$ is weakly special iff Y is weakly special.

- By the previous reduction, there is a unique complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$. Denote it by $\widetilde{Y}$. Denote by $\widetilde{Y}_G := \pi(\widetilde{Y})$, $Y_G := [\pi](Y)$ and $\widetilde{Z}_G := \pi(\widetilde{Z})$. Remark that by Lemma 1.3.9, $\overline{\widetilde{Z}_G}$ is an algebraic subset of $\mathcal{X}_G^+$.

- Replacing Γ by a subgroup of finite index does not matter for this problem, so we may assume that Γ is neat and $\Gamma \subset P^{\mathrm{der}}(\mathbb{Q})$ (Remark 1.1.13(2)).

Let $\widetilde{F}$ be the smallest weakly special subset containing $\widetilde{Y}$. By Theorem 2.3.1, $\widetilde{F} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{z}$ some $\widetilde{z} \in \widetilde{Z} \subset \widetilde{Y}$, where N is the connected algebraic monodromy group associated with Y^{sm} and $U_N := U \cap N$. The set $\widetilde{F}$ is Hodge generic in $(P, \mathcal{X}^+)$ since $\widetilde{Z}$ is, so $N \lhd P$ and $N \lhd P^{\mathrm{der}}$ by Theorem 2.2.4.

Define

$$\Gamma_{\widetilde{Z}} := \{\gamma \in \Gamma | \gamma \cdot \widetilde{Z} = \widetilde{Z}\} \qquad (\text{resp. } \Gamma_{G,\overline{\widetilde{Z}_G}} := \{\gamma_G \in \Gamma_G | \gamma_G \cdot \overline{\widetilde{Z}_G} = \overline{\widetilde{Z}_G}\})$$

and

$$H_{\widetilde{Z}} := (\overline{\Gamma_{\widetilde{Z}}}^{\mathrm{Zar}})^\circ \qquad (\text{resp. } H_{\overline{\widetilde{Z}_G}} := (\overline{\Gamma_{G,\overline{\widetilde{Z}_G}}}^{\mathrm{Zar}})^\circ).$$

Define $U_{H_{\widetilde{Z}}} := U \cap H_{\widetilde{Z}}$ and $W_{H_{\widetilde{Z}}} := W \cap H_{\widetilde{Z}}$. Both of them are normal in $H_{\widetilde{Z}}$. Then $H_{\widetilde{Z}}$ (resp. $H_{\overline{\widetilde{Z}_G}}$) is the largest connected subgroup of P^{der} (resp. G^{der}) such that $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$ (resp. $H_{\overline{\widetilde{Z}_G}}(\mathbb{R})^+$) stabilizes $\widetilde{Z}$ (resp. $\overline{\widetilde{Z}_G}$).

Define $V_{H_{\widetilde{Z}}} := W_{H_{\widetilde{Z}}}/U_{H_{\widetilde{Z}}}$ and $G_{H_{\widetilde{Z}}} := H_{\widetilde{Z}}/W_{H_{\widetilde{Z}}} \hookrightarrow P/W = G$.

The following two lemmas were proved for the pure case in [50] and [29].

Lemma 3.2.2. *The set $\widetilde{Y}$ is stable under $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$.*

Proof. Every fiber of $\mathcal{X}^+ \to \mathcal{X}_{P/U}^+$ can be canonically identified with $U(\mathbb{C})$. So it is enough to prove that $\widetilde{Y}$ is stable under $H_{\widetilde{Z}}(\mathbb{R})^+$: If $U_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{y} \subset \widetilde{Y}$ for $\widetilde{y} \in \widetilde{Y}$, then $U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{y} \subset \widetilde{Y}$ because $\widetilde{Y}$ is complex analytic and $U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{y}$ is the smallest complex analytic subset of $\mathcal{X}^+$ containing $U_{H_{\widetilde{Z}}}(\mathbb{R})\widetilde{y}$.

If not, then since $H_{\widetilde{Z}}(\mathbb{Q})$ is dense (w.r.t. the archimedean topology) in $H_{\widetilde{Z}}(\mathbb{R})^+$, there exists $h \in H_{\widetilde{Z}}(\mathbb{Q})$ such that $h\widetilde{Y} \neq \widetilde{Y}$. The set $\widetilde{Z}$ is contained in $\widetilde{Y} \cap h\widetilde{Y}$ by definition of $H_{\widetilde{Z}}$, and hence contained in a complex analytic irreducible component $\widetilde{Y}'$ of it.

Consider the Hecke operator T_h. Then $T_h(Y) = \mathrm{unif}(h \cdot \mathrm{unif}^{-1}(Y))$. Hence

$$Y \cap T_h(Y) = \mathrm{unif}(\mathrm{unif}^{-1}(Y) \cap (h \cdot \mathrm{unif}^{-1}(Y))).$$

On the other hand, $T_h(Y)$ is equidimensional of the same dimension as Y by definition, hence by reason of dimension, $h\widetilde{Y}$ is an irreducible component of $\mathrm{unif}^{-1}(T_h(Y)) = \Gamma h \Gamma \widetilde{Y}$. So $\mathrm{unif}(h\widetilde{Y})$ is an irreducible component of $T_h(Y)$.

Since $\widetilde{Y}'$ is a complex analytic irreducible component of $\widetilde{Y} \cap h\widetilde{Y}$, it is also a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y) \cap (h\widetilde{Y}) = \Gamma\widetilde{Y} \cap h\widetilde{Y}$. So $Y' := \mathrm{unif}(\widetilde{Y}')$ is a complex analytic irreducible component of $Y \cap \mathrm{unif}(h\widetilde{Y})$. So Y' is a complex analytic irreducible component of $Y \cap T_h(Y)$, and hence is algebraic since $Y \cap T_h(Y)$ is.

Since $h\widetilde{Y} \neq \widetilde{Y}$ and Y is irreducible, $\dim(Y') < \dim(Y)$. But $\widetilde{Z} \subset \widetilde{Y} \cap h\widetilde{Y} \subset \mathrm{unif}^{-1}(Y')$. This contradicts the minimality of Y. □

Lemma 3.2.3. *$H_{\widetilde{Z}} \lhd N$.*

Proof. We have $\widetilde{Z} \subset \widetilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$, so the image of $\widetilde{Z}$ under the morphism

$$(P, \mathcal{X}^+) \to (P, \mathcal{X}^+)/N$$

is a point. But $H_{\widetilde{Z}}/(H_{\widetilde{Z}} \cap N)$ stabilizes this point which is Hodge generic (since $\widetilde{F}$ is Hodge generic in $\mathcal{X}^+$), and therefore is trivial by Remark 2.2.6. So $H_{\widetilde{Z}} < N$.

Let H' be the algebraic group generated by $\gamma^{-1} H_{\widetilde{Z}} \gamma$ for all $\gamma \in \Gamma_{Y^{\mathrm{sm}}}$, where $\Gamma_{Y^{\mathrm{sm}}}$ is the monodromy group of Y^{sm}. Since H' is invariant under conjugation by $\Gamma_{Y^{\mathrm{sm}}}$, it is invariant under $\overline{\Gamma_{Y^{\mathrm{sm}}}}^{\mathrm{Zar}}$, therefore invariant under conjugation by N.

By Lemma 3.2.2, $\widetilde{Y}$ is invariant under $H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})$. On the other hand, $\widetilde{Y}$ is also invariant under $\Gamma_{Y^{\mathrm{sm}}}$ by definition. So $\widetilde{Y}$ is invariant under the action of $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})$ where $U_{H'} := U \cap H'$. Since $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ is semi-algebraic, there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}$, which contains $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ and is contained in $\widetilde{Y}$ by [49, Lemma 4.1]. Now $\widetilde{Z} \subset \widetilde{E} \subset \widetilde{Y}$, so $\widetilde{Z} = \widetilde{E} = H'(\mathbb{R})^+ U_{H'}(\mathbb{C})\widetilde{Z}$ by maximality of $\widetilde{Z}$, and therefore $H' = H_{\widetilde{Z}}$ by definition of $H_{\widetilde{Z}}$. So $H_{\widetilde{Z}}$ is invariant under conjugation by N. Since $H_{\widetilde{Z}} < N$, $H_{\widetilde{Z}}$ is normal in N. □

Corollary 3.2.4.

$$G_{H_{\widetilde{Z}}},\ H_{\overline{\widetilde{Z}_G}} \lhd G^{\mathrm{der}} \text{ and } G_{H_{\widetilde{Z}}} \lhd H_{\overline{\widetilde{Z}_G}}.$$

Proof. We have $G_{H_{\widetilde{Z}}} \lhd G_N \lhd G^{\mathrm{der}}$, and so $G_{H_{\widetilde{Z}}} \lhd G^{\mathrm{der}}$ since all the three groups are reductive.

Working with $((G, \mathcal{X}_G^+),\ \overline{Y_G},\ \overline{\widetilde{Z}_G})$ instead of $((P, \mathcal{X}^+),\ Y,\ \widetilde{Z})$, we can prove (similar to Lemma 3.2.3) that $H_{\overline{\widetilde{Z}_G}} \lhd G_N$. Hence $H_{\overline{\widetilde{Z}_G}} \lhd G^{\mathrm{der}}$ by the same reason for $G_{H_{\widetilde{Z}}}$.

By definition $G_{H_{\widetilde{Z}}} < H_{\overline{\widetilde{Z}_G}}$. So $G_{H_{\widetilde{Z}}} \lhd H_{\overline{\widetilde{Z}_G}}$ since $G_{H_{\widetilde{Z}}} \lhd G^{\mathrm{der}}$. □

So far the proof looks similar to the pure case. From now on it will be quite different. For the readers' convenience, we list here some differences between the proof of Ax-Lindemann for mixed Shimura varieties and for the pure case:

- We shall prove that $\widetilde{Z}$ is an $H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})$-orbit. To prove this, it suffices to prove $\dim H_{\widetilde{Z}} > 0$ when S is a pure Shimura variety. However this is far from enough for the mixed case, since this does not exclude the naive counterexample when $\dim \widetilde{Z}_G > 0$ but $H_{\widetilde{Z}}$ is unipotent. To overcome it, we should at least prove $\dim G_{H_{\widetilde{Z}}} > 0$. In fact we shall directly prove $G_{H_{\widetilde{Z}}} = H_{\overline{\widetilde{Z}_G}}$ (Proposition 3.2.6). This equality is not obvious because, as appears in the proof of Lemma 3.2.5, there is no reason a priori why $\overline{\widetilde{Z}_G}$, which is obviously algebraic in $\mathrm{unif}^{-1}(Y_G)$, should be maximal for this property. If one could prove direcly this is the case, then Klingler-Ullmo-Yafaev [29, Theorem 1.3] would give directly the result.
- As mentioned in the Introduction, we shall make essential use of the "family" version of Pila-Wilkie's theorem (Remark 3.3.4);
- If $P = G$ is reductive, then $H_{\widetilde{Z}} \lhd N \lhd P$ implies directly $H_{\widetilde{Z}} \lhd P$. This is obviously false when P is not reductive.
- For a general mixed Shimura variety S, the fiber of $S \xrightarrow{[\pi]} S_G$ is not necessarily an algebraic group (Lemma 2.1.1), hence not a semi-abelian variety. We do not have Ax-Lindemann for the fiber for this case. Thus we should execute a proof of Ax-Lindemann for the fiber. As the readers will see in §3.4, the proof of this case calls for much more careful study of $\widetilde{Z}$. First of all, when doing the estimate and using the family version of Pila-Wilkie for the fiber (*Step I*), we should introduce a seemingly strange subgroup which serves as G_N in the section. The reason for this will be explained in Remark 3.4.1. Secondly, to prove that $W_{H_{\widetilde{Z}}}$ is normal in W is not trivial, and the key to the solution (*Step IV*) is a well-known fact: any holomorphic morphism from a complex abelian variety to an algebraic torus over $\mathbb{C}$ is trivial.

Before proceeding, we prove the following lemma:

Lemma 3.2.5. *1. $\overline{\widetilde{Y}_G}$ is weakly special. Hence $\overline{\widetilde{Y}_G} = G_N(\mathbb{R})^+ \widetilde{z}_G$ for any point $\widetilde{z}_G \in \widetilde{Z}_G$;*

2. $\overline{\mathrm{unif}_G(\widetilde{Z}_G)} = \overline{Y_G}$.

Proof. 1. Let $\widetilde{Z}'$ be an irreducible algebraic subset of $\mathcal{X}_G^+$ which contains $\overline{\widetilde{Z}_G}$ and is contained in $\mathrm{unif}^{-1}(\overline{Y_G})$, maximal for these properties. By [29, Theorem 1.3], $Z' := \mathrm{unif}_G(\widetilde{Z}')$ is weakly special, and therefore Zariski closed by definition. Now $\widetilde{Z} \subset \pi^{-1}(\widetilde{Z}') \cap \mathrm{unif}^{-1}(Y)$. However,

$$\mathrm{unif}(\pi^{-1}(\widetilde{Z}') \cap \mathrm{unif}^{-1}(Y)) = \mathrm{unif}(\pi^{-1}(\widetilde{Z}')) \cap Y = [\pi]^{-1}(Z') \cap Y.$$

Then we must have $Y \subset [\pi]^{-1}(Z')$ since Y is the minimal irreducible closed subvariety of S such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$. Therefore $\overline{Y_G} \subset Z'$. But $Z' \subset \overline{Y_G}$ by definition of Z', so $Z' = \overline{Y_G}$. This means that $\overline{Y_G}$ is weakly special.

2. Let $Y' := \overline{\mathrm{unif}_G(\widetilde{Z}_G)}$, then $\overline{\widetilde{Z}_G} \subset \mathrm{unif}_G^{-1}(Y')$. Then $\widetilde{Z} \subset \pi^{-1}(\mathrm{unif}_G^{-1}(Y')) = \mathrm{unif}^{-1}([\pi]^{-1}(Y'))$, and so

$$\widetilde{Z} \subset \mathrm{unif}^{-1}([\pi]^{-1}(Y')) \cap \mathrm{unif}^{-1}(Y) = \mathrm{unif}^{-1}([\pi]^{-1}(Y') \cap Y).$$

Hence there exists an irreducible component Y'' of $[\pi]^{-1}(Y') \cap Y$ such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y'')$. But

$$[\pi](Y'') \subset [\pi]([\pi]^{-1}(Y') \cap Y) = Y' \cap Y_G,$$

so $\dim([\pi](Y'')) \leqslant \dim(Y' \cap Y_G)$. If $Y' \neq \overline{Y_G}$, then $\dim(Y' \cap Y_G) < \dim(Y_G)$ and therefore $\dim(Y'') < \dim(Y)$, which contradicts the minimality of Y. So $Y' = \overline{Y_G}$.

□

Proposition 3.2.6 (key proposition)**.** *The set $\overline{\widetilde{Z}_G}$ is weakly special and $G_{H_{\widetilde{Z}}} = H_{\overline{\widetilde{Z}_G}}$. In other words,*

$$\overline{\widetilde{Z}_G} = G_{H_{\widetilde{Z}}}(\mathbb{R})^+ \widetilde{z}_G$$

for any point $\widetilde{z}_G \in \widetilde{Z}_G$.

Now let us show how this proposition together with Theorem 3.1.5 implies Theorem 3.1.1. Before proceeding to the final argument, we shall prove the following group theoretical lemma:

Lemma 3.2.7. *Fixing a Levi decomposition $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$, there exists a compatible Levi decomposition $P = W \rtimes G$.*

Proof. Suppose that the fixed Levi decomposition of $H_{\widetilde{Z}}$ is given by $s_1 \colon G_{H_{\widetilde{Z}}} \to H_{\widetilde{Z}}$. Define $P_* := \pi^{-1}(G_{H_{\widetilde{Z}}})$, then $H_{\widetilde{Z}} < P_*$. Now choose any Levi decomposition $P = W \rtimes G$ defined by $s_2 \colon G \to P$. Then $G_{H_{\widetilde{Z}}}$, being a subgroup of G, is realized as a subgroup of P via s_2. Hence s_2 induces a Levi-decomposition $P_* = W \rtimes^{s_2} G_{H_{\widetilde{Z}}}$. We have thus a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_{H_{\widetilde{Z}}} & \longrightarrow & H_{\widetilde{Z}} & \underset{\curvearrowleft s_1}{\longrightarrow} & G_{H_{\widetilde{Z}}} & \longrightarrow & 1 \\ & & \downarrow \cap & & \downarrow \cap & & \| & & \\ 1 & \longrightarrow & W & \longrightarrow & P_* & \underset{\curvearrowleft s_1}{\longrightarrow} & G_{H_{\widetilde{Z}}} & \longrightarrow & 1 \end{array},$$

where the morphism s_1 in the second line is induced by the one in the first line. Now s_1, s_2 define two Levi decompositions of P_*. They differ by the conjugation by an element w_0 of $W(\mathbb{Q})$ by [55, Theorem 2.3]. So replacing s_2 by its conjugation by w_0 we can find a Levi decomposition of P which is compatible with the fixed $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$. □

Theorem 3.2.8. *1. $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$ for any $\widetilde{z} \in \widetilde{Z}$;*

2. $H_{\widetilde{Z}} \lhd P$.

Hence $\widetilde{Z}$ is weakly special by definition.

Proof. 1. Consider a fibre of $\widetilde{Z}$ over a Hodge-generic point $\widetilde{z}_G \in \widetilde{Z}_G$ such that $\pi|_{\widetilde{Z}}$ is flat at $\widetilde{z}_G$ (such a point exists by [1, §4, Lemma 1.4] and generic flatness). Suppose that $\widetilde{W}$ is an irreducible algebraic component of $\widetilde{Z}_{\widetilde{z}_G}$ such that $\dim(\widetilde{Z}_{\widetilde{z}_G}) = \dim(\widetilde{W})$, then since $\pi|_{\widetilde{Z}}$ is flat at $\widetilde{z}_G$,

$$\dim(\widetilde{Z}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{Z}_{\widetilde{z}_G}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{W}).$$

Consider the set $\widetilde{E} := H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{W}$. It is semi-algebraic (since $\widetilde{W}$ is algebraic and the action of $P(\mathbb{R})^+U(\mathbb{C})$ on $\mathcal{X}^+$ is algebraic). The fact $\widetilde{W} \subset \widetilde{Z}$ implies that $\widetilde{E} \subset \widetilde{Z}$. By [49, Lemma 4.1], there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}_{\mathrm{alg}}$, which contains $\widetilde{E}$ and is contained in $\widetilde{Z}$. Now we have by Proposition 3.2.6

$$\pi(\widetilde{E}) = G_{H_{\widetilde{Z}}}(\mathbb{R})^+\widetilde{z}_G = H_{\overline{\widetilde{Z}_G}}(\mathbb{R})^+\widetilde{z}_G = \widetilde{Z}_G$$

and that the $\mathbb{R}$-dimension of every fiber of $\pi|_{\widetilde{E}}$ is at least $\dim_{\mathbb{R}}(\widetilde{W})$. So

$$\dim(\widetilde{E}_{\mathrm{alg}}) \geqslant \dim(\pi(\widetilde{E})) + \dim(\widetilde{W}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{W}) = \dim(\widetilde{Z}).$$

So $\widetilde{E} = \widetilde{Z}$ since $\widetilde{Z}$ is irreducible.

Next let $\widetilde{W}'$ be an irreducible algebraic subset which contains $\widetilde{Z}_{\widetilde{z}_G}$ and is contained in $\mathrm{unif}^{-1}(Y)_{\widetilde{z}_G}$, maximal for these properties. Then $\widetilde{W}'$ is weakly special by Theorem 3.1.5. We have $\widetilde{W}' \subset \widetilde{Y}$ since $\widetilde{Y}$ is an irreducible component of $\pi^{-1}(Y)$. Consider $\widetilde{E}' := H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{W}'$. Then $\widetilde{E}' \subset \widetilde{Y}$ by Lemma 3.2.2. But $\widetilde{E}'$ is semi-algebraic, so by [49, Lemma 4.1], there exists an irreducible algebraic subset of $\mathcal{X}^+$, say $\widetilde{E}'_{\mathrm{alg}}$ which contains $\widetilde{E}'$ and is contained in $\widetilde{Y}$. So $\widetilde{Z} = \widetilde{E} \subset \widetilde{E}'_{\mathrm{alg}} \subset \widetilde{Y}$, and hence $\widetilde{Z} = \widetilde{E}'_{\mathrm{alg}} = \widetilde{E}'$ by the maximality of $\widetilde{Z}$. So $\widetilde{Z}_{\widetilde{z}_G} = \widetilde{W}'$ is weakly special.

Write $\widetilde{Z}_{\widetilde{z}_G} = W'(\mathbb{R})U'(\mathbb{C})\widetilde{z}$ with $W' < W$, $U' = W' \cap U$ and $\widetilde{z} \in \widetilde{Z}_{\widetilde{z}_G}$. Then $W_{H_{\widetilde{Z}}} < W'$. The complex structure of $\pi^{-1}(\widetilde{z}_G)$ comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F^0_{\widetilde{z}_G}W(\mathbb{C})$, where $F^0_{\widetilde{z}_G}W(\mathbb{C}) = \exp(F^0_{\widetilde{z}_G}\,\mathrm{Lie}\,W_{\mathbb{C}})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that W'/U' is a $\mathrm{MT}(\widetilde{z}_G) = G$-module. Hence W' is a G-group.

Define $P' := W'H_{\widetilde{Z}}$, then P' is a subgroup of P since $W' > W_{H_{\widetilde{Z}}}$ and $G_{H_{\widetilde{Z}}}W' = W'$. Now we have

$$\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{Z}_{\widetilde{z}_G} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})W'(\mathbb{R})U'(\mathbb{C})\widetilde{z} = P'(\mathbb{R})^+U'(\mathbb{C})\widetilde{z}.$$

So $H_{\widetilde{Z}} = P'$ because $H_{\widetilde{Z}}$ is the largest subgroup of P^{der} such that $H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})$ stabilizes $\widetilde{Z}$. So we have $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$.

2. First of all, $U_{H_{\widetilde{Z}}} \lhd P$ by Proposition 1.1.19(2).

   Next consider the complex structure of $\pi^{-1}(\widetilde{z}_G)$. It comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F^0_{\widetilde{z}_G}W(\mathbb{C})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that $V_{H_{\widetilde{Z}}}$ is a $\mathrm{MT}(\widetilde{z}_G) = G$-module. Hence $W_{H_{\widetilde{Z}}}$ is a G-group. Besides, $G_{H_{\widetilde{Z}}} \lhd G$ by Proposition 3.2.6. In particular, $G_{H_{\widetilde{Z}}}$ is reductive.

   Then let us prove $W_{H_{\widetilde{Z}}} \lhd P$. It suffices to prove $W_{H_{\widetilde{Z}}} \lhd W$. For any $\widetilde{z} \in \widetilde{Z}$, we have proved in (1) that $\widetilde{Z}_{\widetilde{z}_G} = W_{H_{\widetilde{Z}}}(\mathbb{R})U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$ is weakly special. Hence by Proposition 1.2.4, there is a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $\widetilde{z} \in \mathcal{Y}^+$ and $W_{H_{\widetilde{Z}}} \lhd Q$. Define W^* to be the G-subgroup (of W) generated by $W_Q := \mathcal{R}_u(Q)$, then $W_{H_{\widetilde{Z}}} \lhd W^*$ since $W_{H_{\widetilde{Z}}}$ is a G-group.

   Fix a Levi decomposition $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$ and choose a compatible Levi decomposition $P = W \rtimes G$ (as is shown in Lemma 3.2.7). Let P^* be the group generated by GQ, then $\mathcal{R}_u(P^*) = W^*$ and $P^*/W^* = G$. The group P^* defines a connected mixed Shimura datum $(P^*, \mathcal{X}^{*+})$ with $\mathcal{X}^{*+} = P^*(\mathbb{R})^+U^*(\mathbb{C})\widetilde{z}$. Now $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z} \subset \mathcal{X}^{*+}$. But $\widetilde{Z}$ is Hodge generic in $\mathcal{X}^+$ by assumption, hence $P = P^*$ and $W = W^*$. So $W_{H_{\widetilde{Z}}} \lhd W$ and hence $W_{H_{\widetilde{Z}}} \lhd P$.

   Use the notation in §1.1.2.5. We are done if we can prove:

$$\forall u \in U,\ \forall v \in V,\ \textit{and}\ \forall g \in G_{H_{\widetilde{Z}}},\ (u, v, 1)(0, 0, g)(-u, -v, 1) \in H_{\widetilde{Z}}.$$

   By Corollary 1.1.37, there exist decompositions

$$U = U_N \oplus U_N^{\perp} \qquad V = V_N \oplus V_N^{\perp}$$

as G-modules such that G_N acts trivially on $U_N^{\perp}$ and $V_N^{\perp}$. Now

$$\begin{aligned}
&(u, v, 1)(0, 0, g)(-u, -v, 1) \\
&= (u, v, g)(-u, -v, 1) \\
&= (u - g \cdot u, v - g \cdot v, g) \\
&= ((u_N + u_N^{\perp}) - g \cdot (u_N + u_N^{\perp}), (v_N + v_N^{\perp}) - g \cdot (v_N + v_N^{\perp}), g) \\
&= (u_N - g \cdot u_N, v_N - g \cdot v_N, g) \\
&= (u_N, v_N, 1)(0, 0, g)(-u_N, -v_N, 1) \in H_{\widetilde{Z}},
\end{aligned}$$

where the last inclusion follows from Lemma 3.2.3.

□

3.3 Ax-Lindemann Part 2: Estimate

This section is devoted to prove Proposition 3.2.6. The proof uses essentially the "block family" version of Pila-Wilkie's counting theorem [48, Theorem 3.6].

Keep notation and assumptions as in the last section and denote by $\pi\colon (P, \mathcal{X}^+) \to (G, \mathcal{X}_G^+)$. The group $G = Z(G)^\circ H_1 ... H_r$ is an almost direct product, where H_i's are non-trivial simple groups and are normal in G. We have a decomposition

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^{r} (H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$$

by [39, 3.6]. Let $S_G^{\mathrm{ad}} := \Gamma_G^{\mathrm{ad}} \backslash \mathcal{X}_G^+$. Shrinking Γ_G^{ad} if necessary, we may assume $S_G^{\mathrm{ad}} \simeq \prod_{i=1}^r S_{H,i}$, where $S_{H,i}$ is a connected pure Shimura variety associated with $(H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$.

Without loss of generality we may assume $G_N = H_1 ... H_l$. It suffices to prove $H_i < G_{H_{\widetilde{Z}}}$ for each $i = 1, ..., l$. The case $l = 0$ is trivial, so we assume that $l \geqslant 1$. Define $Q_i := \pi^{-1}(H_i)$.

3.3.1 Fundamental set and definability

The goal of this subsection is to prove that there exists $\mathcal{F} \subset \mathcal{X}^+$ a fundamental set for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable.

First of all, by the Reduction Lemma (Lemma 1.1.35), it suffices to prove the existence of such a fundamental set for $(P, \mathcal{X}^+)$ pure and $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$ (see §3.5.1 for more details). The case where $(P, \mathcal{X}^+)$ is pure is guaranteed by Klingler-Ullmo-Yafaev [29, Theorem 4.1]. Now we prove the case $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$.

We draw the following diagram to make the notation more clear:

$$\begin{array}{ccc}
\mathcal{X}_{2g}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}_{2g,\mathrm{a}}^+ \\
{\scriptstyle \mathrm{unif}}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}_{P/U}} \\
S & \xrightarrow{[\pi_{P/U}]} & S_{P/U}
\end{array} .$$

In this case, $[\pi_{P/U}]\colon S \to S_{P/U}$ is an algebraic $\mathbb{G}_m$-torsor. By Peterzil-Starchenko [47, Theorem 1.3], there exists a fundamental set $\mathcal{F}_{P/U}$ for the action of Γ/Γ_U on $\mathcal{X}^+_{2g,\mathrm{a}}$ such that $\mathrm{unif}_{P/U}|_{\mathcal{F}_{P/U}}$ is definable (recall that if $g=0$, then $\mathcal{X}^+_{2g}=\mathbb{C}$, $S=\mathbb{C}^*$, $\mathrm{unif}=\exp$ and $S_{P/U}$ is a point). Let us now construct a fundamental set for the action of Γ on $\mathcal{X}^+_{2g}$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable and $\pi_{P/U}(\mathcal{F})=\mathcal{F}_{P/U}$.

Since any variety over a field is quasi-compact in the Zariski topology, there exists a finite Zariski open covering $\{V_\alpha\}_{\alpha\in\Lambda}$ of $S_{P/U}$ such that $S|_{V_\alpha}\simeq\mathbb{C}^*\times V_\alpha$ and these isomorphisms are algebraic. Define $U_\alpha := S|_{V_\alpha}=[\pi_{P/U}]^{-1}(V_\alpha)$ for every $\alpha\in\Lambda$. Then we have

$$\mathrm{unif}|_{\mathrm{unif}^{-1}(U_\alpha)}\colon\ \mathrm{unif}^{-1}(U_\alpha)\xrightarrow[\varphi]{\sim}U_{2g}(\mathbb{C})\times\mathrm{unif}^{-1}_{P/U}(V_\alpha)\to(\mathbb{C}^*)\times V_\alpha\simeq U_\alpha,$$

where φ is semi-algebraic (Proposition 1.3.3), the last isomorphism is algebraic and the middle morphism is $(\exp,\mathrm{unif}_{P/U}|_{\mathrm{unif}^{-1}_{P/U}}(V_\alpha))$. Let $\mathcal{F}_U:=\{s\in\mathbb{C}|\ -1<\Re e(s)<1\}$ and let $\mathcal{F}_\alpha:=\varphi^{-1}(\mathcal{F}_U\times\mathcal{F}_{P/U,\alpha})$. Then $\mathrm{unif}|_{\mathcal{F}_\alpha}$ is definable. Now $\mathcal{F}:=\cup\mathcal{F}_\alpha$ (remember that this is a finite union) satisfies the conditions we want.

Now we return to arbitrary $(P,\mathcal{X}^+)$. We have proved the existence of an $\mathcal{F}$ as stated at the beginning of this subsection. Let us choose such an $\mathcal{F}$ more carefully. First of all replace $\mathcal{F}$ by $\gamma\mathcal{F}$ if necessary to make sure $\mathcal{F}\cap\widetilde{Z}\neq\emptyset$. Next define $\mathcal{F}_G:=\pi(\mathcal{F})\subset\mathcal{X}^+_G\simeq\prod_{i=1}^r\mathcal{X}^+_{H,i}$. Denote by q_i the i-th projection and $\mathcal{F}_{H,i}:=q_i(\mathcal{F}_G)$. There exist some $\gamma_1=1,...,\gamma_s\in\Gamma_G<\Gamma$ such that $\prod_{i=1}^r\mathcal{F}_{H,i}\subset\cup_{j=1}^s\gamma_j\mathcal{F}_G$. Consider

$$\mathcal{F}':=(\bigcup_{j=1}^s\gamma_j\mathcal{F})\cap\pi^{-1}(\prod_{i=1}^r\mathcal{F}_{H,i}),$$

then $\mathcal{F}'$ is a fundamental set for the action of Γ on $\mathcal{X}^+$ and $\mathrm{unif}|_{\mathcal{F}'}$ is definable. Furthermore, $\pi(\mathcal{F}')=\prod_{i=1}^r\mathcal{F}_{H,i}$ and $\mathcal{F}_{H,i}=q_i\pi(\mathcal{F}')$. We still have $\mathcal{F}'\cap\widetilde{Z}\neq\emptyset$ since $\mathcal{F}\subset\mathcal{F}'$. Now replace $\mathcal{F}$ by $\mathcal{F}'$.

3.3.2 Counting points and conclusion

We shall work from now on with an $\mathcal{F}$ satisfying the conditions in the last paragraph of the previous subsection. By Lemma 3.2.5, $\overline{\widetilde{Y}_G}=\prod_{i=1}^l H_i(\mathbb{R})^+\widetilde{z}_G$. Fix a point $\widetilde{z}\in\mathcal{F}\cap\widetilde{Z}$. Define the following Shimura morphisms for each $i=1,...,l$

$$\begin{array}{ccc}(G,\mathcal{X}^+_G) & \xrightarrow{p_i} & (G_i,\mathcal{X}^+_{G,i}):=(G^{\mathrm{ad}},\mathcal{X}^+_G)/\prod_{j\neq i}H_j^{\mathrm{ad}}\\ \downarrow{\scriptstyle \mathrm{unif}_G} & & \downarrow{\scriptstyle \mathrm{unif}_{G,i}}\\ S_G & \xrightarrow{[p_i]} & S_{G,i}\end{array}.$$

Fix $i \in \{1, ..., l\}$. Define $\widetilde{Y}_{G,i} := p_i(\widetilde{Y}_G) = H_i^{\mathrm{ad}}(\mathbb{R})^+\pi_i(\widetilde{z}_G)$, $\widetilde{Z}_{G,i} := p_i(\widetilde{Z}_G)$ and $Y_{G,i} := [p_i](Y_G)$, then $\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})$ is Zariski dense in $\overline{Y_{G,i}}$ by Lemma 3.2.5. If $\dim(\widetilde{Z}_{G,i}) = 0$, then $\widetilde{Z}_{G,i}$ is a finite set of points since it is algebraic. But then $\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})$, and hence $\overline{Y_{G,i}} = \overline{\mathrm{unif}_{G,i}(\widetilde{Z}_{G,i})}$ is also a finite set of points. So $\dim(Y_{G,i}) = 0$, which contradicts $\widetilde{Y}_{G,i} = H_i^{\mathrm{ad}}(\mathbb{R})^+\pi_i(\widetilde{z}_G)$. To sum it up, $\dim(\widetilde{Z}_{G,i}) > 0$. For further convenience, we will denote by $\pi_i := p_i \circ \pi$.

Take an algebraic curve $C_{G,i} \subset \widetilde{Z}_{G,i}$ passing through $\pi_i(\widetilde{z})$. Now $\pi_i(\widetilde{Z} \cap \pi_i^{-1}(C_{G,i})) = \widetilde{Z}_{G,i} \cap C_{G,i} = C_{G,i}$, and hence there exists an algebraic curve $C \subset \widetilde{Z} \cap \pi_i^{-1}(C_{G,i})$ passing through $\widetilde{z}$ such that $\dim(\pi_i(C)) = 1$.

Let $\mathcal{F}_{G,i} := p_i(\mathcal{F}_G)$, then it is a fundamental set of $\mathrm{unif}_{G,i}$ and $\mathrm{unif}_{G,i}|_{\mathcal{F}_{G,i}}$ is definable. We define for any irreducible semi-algebraic subvariety A (resp. $A_{G,i}$) of $\mathrm{unif}^{-1}(Y)$ (resp. $\mathrm{unif}_{G,i}^{-1}(\overline{Y_{G,i}})$) the following sets: define

$$\Sigma^{(i)}(A) := \{g \in Q_i(\mathbb{R}) | \dim(gA \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(A)\}$$
$$(\text{resp. } \Sigma_G^{(i)}(A_{G,i}) := \{g \in H_i^{\mathrm{ad}}(\mathbb{R}) | \dim(gA_{G,i} \cap \mathrm{unif}_{G,i}^{-1}(\overline{Y_{G,i}}) \cap \mathcal{F}_{G,i}) = \dim(A_{G,i})\})$$

and

$$\Sigma'^{(i)}(A) := \{g \in Q_i(\mathbb{R}) | g^{-1}\mathcal{F} \cap A \neq \emptyset\}$$
$$(\text{resp. } \Sigma_G'^{(i)}(A_{G,i}) := \{g \in H_i^{\mathrm{ad}}(\mathbb{R}) | g^{-1}\mathcal{F}_{G,i} \cap A_{G,i} \neq \emptyset\}).$$

Then $\Sigma^{(i)}(A)$ and $\Sigma_G^{(i)}(A_{G,i})$ are by definition definable. Let $\Gamma_{G,i}^{\mathrm{ad}} := p_i(\Gamma_G^{\mathrm{ad}})$.

Lemma 3.3.1. *$\Sigma'^{(i)}(A) \cap \Gamma = \Sigma^{(i)}(A) \cap \Gamma$ (resp. $\Sigma_G'^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma_G^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}}$).*

Proof. The proof, which we include for completeness, is the same as [67, Lemma 5.2]. First of all $\Sigma^{(i)}(A) \cap \Gamma \subset \Sigma'^{(i)}(A) \cap \Gamma$ by definition. Conversely for any $\gamma \in \Sigma'^{(i)}(A) \cap \Gamma$, $\gamma^{-1}\mathcal{F} \cap A$ contains an open subspace of A since $\mathcal{F}$ is by choice open in $\mathcal{X}^+$. Hence $\gamma A \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F} = \gamma A \cap \mathcal{F}$ contains an open subspace of γA which must be of dimension $\dim(A)$. Hence $\gamma \in \Sigma^{(i)}(A) \cap \Gamma$. The proof for $A_{G,i}$ is the same. ☐

This lemma implies

$$\Sigma^{(i)}(C) \cap \Gamma = \Sigma'^{(i)}(C) \cap \Gamma \subset \Sigma'^{(i)}(\widetilde{Z}) \cap \Gamma = \Sigma^{(i)}(\widetilde{Z}) \cap \Gamma$$
$$(\text{resp. } \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma_G'^{(i)}(C_{G_i}) \cap \Gamma_{G,i}^{\mathrm{ad}} \subset \Sigma_G'^{(i)}(\overline{\widetilde{Z}_{G,i}}) \cap \Gamma_{G,i}^{\mathrm{ad}} = \Sigma^{(i)}(\overline{\widetilde{Z}_{G,i}}) \cap \Gamma_{G,i}^{\mathrm{ad}}) \quad (3.3.1)$$

Lemma 3.3.2. $\pi_i(\Gamma \cap \Sigma^{(i)}(C)) = \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G^{(i)}(C_{G,i})$.

Proof. By Lemma 3.3.1, it suffices to prove $\pi_i(\Gamma \cap \Sigma'^{(i)}(C)) = \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G'^{(i)}(C_{G,i})$. The inclusion $\subset$ is clear by definition. For the other inclusion, $\forall \gamma_{G,i} \in \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G'^{(i)}(C_{G,i})$, $\exists c_{G,i} \in C_{G,i}$ such that $\gamma_{G,i} \cdot c_{G,i} \in \mathcal{F}_{G,i}$.

Take a point $c \in C$ such that $\pi_i(c) = c_{G,i}$ and define $c_G := \pi(c) \in \mathcal{X}_G^+$. Suppose that under the decomposition

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^{r} (H_i^{\mathrm{ad}}, \mathcal{X}_{H,i}^+)$$

of [39, 3.6], $c_G = (c_{G,1}, ..., c_{G,r})$. Then by choice of $\mathcal{F}_G$, there exists $\gamma'_G \in \Gamma_G^{\mathrm{ad}}$ whose i-th coordinate is precisely the $\gamma_{G,i}$ in the last paragraph such that $\gamma'_G \cdot c_G \in \mathcal{F}_G$.

Let $\gamma_G \in \Gamma_G$ be such that its image under $\Gamma_G \to \Gamma_G^{\mathrm{ad}}$ is γ'_G, then $\gamma_G \cdot c \in \pi^{-1}(\mathcal{F}_G)$. Therefore there exist $\gamma_V \in \Gamma_V$, $\gamma_U \in \Gamma_U$ such that $(\gamma_U, \gamma_V, \gamma_G)c \in \mathcal{F}$. Denote by $\gamma = (\gamma_U, \gamma_V, \gamma_G)$, then $\gamma \in \Gamma \cap \Sigma'^{(i)}(C)$ and $\pi_i(\gamma) = \gamma_{G,i}$. □

For $T > 0$, define

$$\Theta_G^{(i)}(C_{G,i}, T) := \{\gamma_G \in \Gamma_{G,i}^{\mathrm{ad}} \cap \Sigma_G^{(i)}(C_{G,i}) | H(\gamma_G) \leqslant T\}.$$

Proposition 3.3.3. *There exists a constant $\delta > 0$ s.t. for all $T \gg 0$, $|\Theta_G^{(i)}(C_{G,i}, T)| \geqslant T^\delta$.*

Proof. This follows directly from [29, Theorem 1.3] applied to $((G_i, \mathcal{X}_{G,i}^+), S_{G,i}, \overline{\widetilde{Z}_{G,i}})$. □

Let us prove how these facts imply $H_i < G_{H_{\widetilde{Z}}}$.

Take a faithful representation $G^{\mathrm{ad}} \hookrightarrow \mathrm{GL}_n$ which sends Γ_G^{ad} to $\mathrm{GL}_n(\mathbb{Z})$. Consider the definable set $\Sigma_G^{(i)}(C_{G,i})$. By the theorem of Pila-Wilkie ([48, Theorem 3.6]), there exist $J = J(\delta)$ definable block families

$$B^j \subset \Sigma_G^{(i)}(C_{G,i}) \times \mathbb{R}^l, \qquad j = 1, ..., J$$

and $c = c(\delta) > 0$ such that for all $T \gg 0$, $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$ is contained in the union of at most $cT^{\delta/4n}$ definable blocks of the form B_y^j ($y \in \mathbb{R}^l$). By Proposition 3.3.3, there exist a $j \in \{1, ..., J\}$ and a block $B_{G,i} := B_{y_0}^j$ of $\Sigma_G^{(i)}(C_{G,i})$ containing at least $T^{\delta/4n}$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

Let $\Sigma^{(i)} := \Sigma^{(i)}(C) \cap \Sigma^{(i)}(\widetilde{Z})$, which is by definition a definable set. Consider $X^j := (\pi_i \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma^{(i)} \times \mathbb{R}^l)$, which is a definable family since π_i is algebraic.

By [69, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X_y^j has at most n_0 connected components. So the definable set $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ has at most n_0 connected components. Now

$$\pi_i(\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)} \cap \Gamma) = B_{G,i} \cap \pi_i(\Sigma^{(i)}(C) \cap \Gamma) = B_{G,i} \cap \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\mathrm{ad}} = B_{G,i} \cap \Gamma_{G,i}^{\mathrm{ad}}$$

by (3.3.1) and Lemma 3.3.2. So there exists a connected component B of $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ such that $\pi_i(B \cap \Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma^{(i)}(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by analytic continuation, and $\widetilde{Z} \subset \sigma^{-1}B\widetilde{Z}$ for any $\sigma \in B \cap \Gamma$. But B is connected, and therefore $\sigma^{-1}B\widetilde{Z} = \widetilde{Z}$ by maximality of $\widetilde{Z}$ and [49, Lemma 4.1]. So $\forall \sigma \in B \cap \Gamma$,

$$B \subset \sigma \, \mathrm{Stab}_{Q_i(\mathbb{R})}(\widetilde{Z}).$$

Fix a $\gamma_0 \in B \cap \Gamma$ such that $\pi_i(\gamma_0) \in \Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. We have already shown that $\pi_i(B \cap \Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. For any $\gamma'_{G,i} \in \pi_i(B \cap \Gamma) \cap \Theta_G^{(i)}(C_{G,i}, T^{1/2n})$, let γ' be one of its pre-images in $B \cap \Gamma$. Then $\gamma := \gamma'^{-1}\gamma_0$ is an element of $\Gamma \cap \mathrm{Stab}_{Q_i(\mathbb{R})}(\widetilde{Z}) = \Gamma_{\widetilde{Z}} \cap Q_i(\mathbb{R})$ such that $H(\pi_i(\gamma)) \ll T^{1/2}$. Therefore for $T \gg 0$, $\pi_i(\Gamma_{\widetilde{Z}}) \cap H_i^{\mathrm{ad}}(\mathbb{R})$ contains at least $T^{\delta/4n}/n_0$ elements $\gamma_{G,i}$ such that $H(\gamma_{G,i}) \leqslant T$. Hence $\dim(\pi_i(H_{\widetilde{Z}}) \cap H_i^{\mathrm{ad}}) > 0$ since $\pi_i(H_{\widetilde{Z}}) \cap H_i^{\mathrm{ad}}$ contains infinitely many rational points. But $\pi_i(H_{\widetilde{Z}}) = p_i\pi(H_{\widetilde{Z}}) = p_i(G_{H_{\widetilde{Z}}})$ by definition. So $H_i^{\mathrm{ad}} < p_i(G_{H_{\widetilde{Z}}})$ since H_i^{ad} is simple and $p_i(G_{H_{\widetilde{Z}}}) \cap H_i^{\mathrm{ad}} \lhd H_i^{\mathrm{ad}}$ by Corollary 3.2.4.

As a normal subgroup of G_N, $G_{H_{\widetilde{Z}}}$ is the almost direct product of some H_j's ($j = 1, ..., l$). So $H_i^{\mathrm{ad}} < p_i(G_{H_{\widetilde{Z}}})$ implies $H_i < G_{H_{\widetilde{Z}}}$. Now we are done.

Remark 3.3.4. *In the proof of the pure case by Klingler-Ullmo-Yafaev [29], it suffices to use a non-family version of Pila-Wilkie ([29, Theorem 6.1]). However this is not enough for our proof, since otherwise the n_0 would depend on T. Hence it is important to use a family version of Pila-Wilkie ([48, Theorem 3.6]).*

3.4 Ax-Lindemann Part 3: The unipotent part

We prove in this section Theorem 3.1.5.

We use the same notation as the first paragraph of §2.1 as well as the first paragraph of §3.1.2. Assume $\dim_{\mathbb{C}} T = m$ and $\dim_{\mathbb{C}} A = n$.

Proof of Theorem 3.1.5. First of all we may assume that $\widetilde{Z}$ is of positive dimension since every point is a weakly special subvariety of dimension 0. For any fundamental set $\mathcal{F}$ of the action of Γ_W on $W(\mathbb{R})U(\mathbb{C})$, define

$$\Sigma(\widetilde{Z}) := \{g \in W(\mathbb{R}) | \dim(g\widetilde{Z} \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(\widetilde{Z})\}$$

and

$$\Sigma'(\widetilde{Z}) := \{g \in W(\mathbb{R}) | g^{-1}\mathcal{F} \cap \widetilde{Z} \neq \emptyset\},$$

then by Lemma 3.3.1,

$$\Sigma(\widetilde{Z}) \cap \Gamma_W = \Sigma'(\widetilde{Z}) \cap \Gamma_W \tag{3.4.1}$$

Let $\Gamma_U := \Gamma \cap U(\mathbb{Q})$ and let $\Gamma_V := \Gamma_W/\Gamma_U$.

$\boxed{Case\ i:\ E{=}A.}$ This is [51, Theorem 2.1 and pp9 Remark 1]. A proof can be found in Appendix. In this case, $W = V$ and $\Gamma_V = \oplus_{i=1}^{2n}\mathbb{Z}e_i \subset \mathrm{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by unif: $\mathrm{Lie}(A) \to A$. Let $\mathcal{F}_V := \Sigma_{i=1}^{2n}(-1,1)e_i$, then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $\mathrm{Lie}(A)$ such that $\mathrm{unif}|_{\mathcal{F}_V}$ is definable.

$\boxed{Case\ ii:\ E{=}T.}$ This is a consequence of Ax's theorem [5] [42, Corollary 3.6]. A proof of this can be found in Appendix. In this case, $W = U$. Let

$\mathcal{F}_U := \{s \in \mathbb{C} | -1 < \Re e(s) < 1\}^m$, then $\mathcal{F}_U$ is a fundamental set for the action of Γ_U on $U(\mathbb{C})$ such that $\mathrm{unif}\,|_{\mathcal{F}_U}$ is definable.

Case iii : general E. Unlike the rest of the paper, the symbol π in this section denotes the map

$$\begin{array}{ccc} W(\mathbb{R})U(\mathbb{C}) & \xrightarrow{\pi} & V(\mathbb{R}) \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_V} \\ E & \xrightarrow{[\pi]} & A \end{array} . \tag{3.4.2}$$

Take $\mathcal{F}_V \subset V(\mathbb{R})$ any fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\mathrm{unif}_V\,|_{\mathcal{F}_V}$ is definable.We claim that:

$$\begin{gathered} \text{There exists a fundamental set } \mathcal{F} \text{ for the action of } \Gamma_W \text{ on } W(\mathbb{R})U(\mathbb{C}) \\ \text{such that } \mathrm{unif}\,|_{\mathcal{F}} \text{ is definable and } \pi(\mathcal{F}) = \mathcal{F}_V. \end{gathered} \tag{3.4.3}$$

By Reduction Lemma (Lemma 1.1.35), it suffices to prove this for $E = E_1 \times_A \dots \times_A E_m$ where E_i's are $\mathbb{G}_m$-torsors over A. But then it suffices to prove for the case $m = 1$. For this case, the proof is similar to §3.3.1.

Let Y_0 be the minimal closed irreducible subvariety of E such that $\widetilde{Z} \subset \mathrm{unif}^{-1}(Y_0)$, then $\widetilde{Z}$ is maximal irreducible algebraic in $\mathrm{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. Let N be the connected algebraic monodromy group of Y^{sm} and let $V_N := (N \cap W)/(N \cap U)$. Let $\widetilde{Y}$ be the complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$. For further convenience, we will denote by $\widetilde{Z}_V := \pi(\widetilde{Z})$, $\widetilde{Y}_V := \pi(\widetilde{Y})$ and $Y_V := [\pi](Y)$.

Repeating the proof of Lemma 3.2.5 (but using the conclusion of *Case i* instead of [29, Theorem 1.1]), we get that $\overline{\widetilde{Y}_V} = V_N(\mathbb{R}) + \widetilde{z}_V$ for some $\widetilde{z}_V \in \widetilde{Z}_V$ is weakly special, and $\overline{\mathrm{unif}_V(\widetilde{Z}_V)} = \overline{Y_V}$. Remark that by GAGA, these closures could be taken in the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) or the Zariski topology. If V_N is trivial, then we are actually in the situation of *Case ii*, and therefore $\widetilde{Z}$ is weakly special. From now on, suppose that $\dim(V_N) > 0$. Replace S by its smallest special subvariety containing Y_0, then $N \lhd P$ by Theorem 2.2.5. Hence V_N is a $G = \mathrm{MT}(b)$-submodule of V.

Define $W_0 := (\overline{\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})U(\mathbb{C})}(\widetilde{Z})}^{\mathrm{Zar}})^{\circ}$, $U_0 := W_0 \cap U$ and $V_0 := \pi(W_0) = W_0/U_0$. The proof is somehow technical, so we will divide it into several steps.

Step I. Let $V^{\dagger}$ be the smallest subgroup of V_N such that $\widetilde{Z}_V \subset V^{\dagger}(\mathbb{R}) + \widetilde{z}_V$. In Step I, we will prove $V^{\dagger} < V_0$.

Step I(i). We know that $A = \Gamma_V \backslash V(\mathbb{R})$ and $V(\mathbb{Q}) \simeq \Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider any $\mathbb{Q}$-quotient group V' of V of dimension 1

$$p' \colon V \to V'$$

such that $\dim(p'(V^\dagger)) = 1$. By abuse of notation, we shall denote its induced map $V(\mathbb{R}) \to V'(\mathbb{R})$ also by p'. Now let $\Gamma_{V'} := p'(\Gamma_V)$, then $\Gamma_{V'} \simeq \mathbb{Z}$ since p' is defined over $\mathbb{Q}$. Write $\Gamma_{V'} = \mathbb{Z}e'$, and let $\mathcal{F}_{V'} := (-1,1)e'$. Then $\mathcal{F}_{V'}$ is a fundamental set for the action of $\Gamma_{V'}$ on $V'(\mathbb{R})$. Define $A' = \Gamma_{V'}\backslash V'(\mathbb{R}) \simeq \mathbb{Z}\backslash\mathbb{R}$, $\mathrm{unif}_{V'}\colon V'(\mathbb{R}) \to A'$ the uniformization and $[p']\colon A \to A'$ the map induced by p'. Then $\mathrm{unif}_{V'}|_{\mathcal{F}_{V'}}$ is definable (even in $\mathbb{R}_{an}$). Define $Y_{V'} := [p'](Y_V)$ and $\widetilde{Y}_{V'} := p'(\widetilde{Y}_V)$.

Let $V'' := \mathrm{Ker}(p')$. The exact sequence of free $\mathbb{Z}$-modules

$$1 \to \Gamma_{V''} := \Gamma_V \cap V''(\mathbb{Q}) \simeq \mathbb{Z}^{2n-1} \to \Gamma_V \simeq \mathbb{Z}^{2n} \to \Gamma_{V'} \simeq \mathbb{Z} \to 1$$

splits, and hence $\Gamma_V \simeq \Gamma_{V''} \oplus \Gamma_{V'}$. This induces $V \simeq V'' \oplus V'$. Write $\Gamma_{V''} = \sum_{i=2}^{2n} \mathbb{Z}e''_i$ and take $\mathcal{F}_{V''} := \sum_{i=2}^{n}(-1,1)e''_i$. Define $\mathcal{F}_V := \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$. Then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\mathrm{unif}_V|_{\mathcal{F}_V}$ is definable (even in $\mathbb{R}_{an}$). Define $\mathcal{F}$ as in (3.4.3).

Since $p(V^\dagger) = V'$ by choice of V', $\dim_{\mathbb{R}} p'(\widetilde{Z}_V) > 0$ by minimality of $V^\dagger$. Hence $p'(\widetilde{Z}_V) = V'(\mathbb{R})$ since $p'(\widetilde{Z}_V)$ is connected.

Remark 3.4.1. *If we only request (V',p') to satisfy $p'(V_N) = 1$, then we do not know whether $\dim_{\mathbb{R}}(p'(\widetilde{Z}_V)) > 0$. This is because we are considering the real analytic topology (i.e. the topology whose closed sets are real analytic sets) on A' and the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) on A, and hence $\overline{\mathrm{unif}_V(\widetilde{Z}_V)} = \overline{Y_V}$ does NOT imply $\overline{\mathrm{unif}_{V'}(\widetilde{Z}_{V'})} = \overline{Y_{V'}}$. To overcome this problem, we introduce the seemingly strange subgroup $V^\dagger$ of V_N. We will prove (Step II) that V_0 is a* $\mathrm{MT}(b)$*-module with the help of $V^\dagger$. Then we prove the comparable result of Theorem 3.2.8(1) for the unipotent part in Step III.*

Let C be an $\mathbb{R}$-algebraic subvariety of $\widetilde{Z}$ of $\mathbb{R}$-dimension 1 such that $p'\pi(C) = V'(\mathbb{R})$. Define furthermore

$$\Sigma(C) := \{g \in W(\mathbb{R}) | \dim_{\mathbb{R}}(gC \cap \mathrm{unif}^{-1}(Y) \cap \mathcal{F}) = 1\}$$

and

$$\Sigma'(C) := \{g \in W(\mathbb{R}) | g^{-1}\mathcal{F} \cap C \neq \emptyset\}.$$

The set $\Sigma(C)$ is by definition definable. By Lemma 3.3.1,

$$\Sigma'(C) \cap \Gamma_W = \Sigma(C) \cap \Gamma_W \tag{3.4.4}$$

For $M > 0$, define

$$\Theta_{V'}(V'(\mathbb{R}), M) = \{\gamma_{V'} \in \Gamma_{V'} | H(\gamma_{V'}) \leqslant M\}.$$

Then

$$|\Theta_{V'}(V'(\mathbb{R}), M)| \gg M. \tag{3.4.5}$$

Step I(ii) is quite similar to the end of §3.3. Consider the definable set $V'(\mathbb{R})$. By the theorem of Pila-Wilkie ([48, Theorem 3.6]), there exist J definable block families

$$B^j \subset V'(\mathbb{R}) \times \mathbb{R}^l, \qquad j = 1, ..., J$$

and $c > 0$ such that for all $M \gg 0$, $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$ is contained in the union of at most $cM^{\delta/8}$ definable blocks of the form B^j_y ($y \in \mathbb{R}^l$). By (3.4.5), there exist a $j \in \{1, ..., J\}$ and a block $B_{V'} := B^j_{y_0}$ of $V'(\mathbb{R})$ containing at least $M^{\delta/8}$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

Let $\Sigma := \Sigma(C) \cap \Sigma(\widetilde{Z})$, which is by definition a definable set. Consider $X^j := ((p'\pi) \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma \times \mathbb{R}^l)$, which is a definable family since $p'\pi$ is $\mathbb{R}$-algebraic.

By [69, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X^j_y has at most n_0 connected components. So the definable set $\pi^{-1}(B_{V'}) \cap \Sigma$ has at most n_0 connected components. Now

$$p'\pi((p'\pi)^{-1}(B_{V'}) \cap \Sigma \cap \Gamma_W) = B_{V'} \cap p'\pi(\Sigma(C) \cap \Gamma_W) = B_{V'} \cap (V'(\mathbb{R}) \cap \Gamma_{V'}) = B_{V'} \cap \Gamma_{V'}$$

by (3.4.1), (3.4.4) and the choice of $\mathcal{F}$ (remember that $\Gamma_V = \Gamma_{V''} \oplus \Gamma_{V'}$ and $\mathcal{F}_V = \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$). So there exists a connected component B of $(p'\pi)^{-1}(B_{V'}) \cap \Sigma$ such that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $B \subset \Sigma(\widetilde{Z})$ by (complex) analytic continuation, and $\widetilde{Z} \subset \sigma_W^{-1} B\widetilde{Z}$ for any $\sigma_W \in B \cap \Gamma_W$. But B is connected, and therefore $\sigma_W^{-1} B\widetilde{Z} = \widetilde{Z}$ by maximality of $\widetilde{Z}$ and [49, Lemma 4.1]. So

$$B \subset \sigma_W \, \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z}).$$

Fix a $\sigma_W \in B \cap \Gamma_W$ such that $p'\pi(\sigma_W) \in \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. We have shown that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. For any $\sigma_{V'} \in p'\pi(B \cap \Gamma) \cap \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$, let σ'_W be one of its pre-images in $B \cap \Gamma_W$. Then $\gamma_W := \sigma_W^{-1}\sigma'_W$ is an element of $\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z})$ and $H(p'\pi(\gamma_W)) \ll M^{1/2}$. Therefore for $M \gg 0$, $p'\pi(\Gamma_W \cap \mathrm{Stab}_{W(\mathbb{R})}(\widetilde{Z}))$ contains at least $M^{\delta/8}/n_0$ elements $\gamma_{V'}$ such that $H(\gamma_{V'}) \leqslant M$. Therefore $\dim(p'\pi(W_0)) > 0$ since it is an infinite set. So $p'\pi(W_0) = V'$ since $\dim(V') = 1$. But V' is an arbitrary 1-dimensional quotient of V such that $p'(V^\dagger) = V'$. Therefore $V^\dagger < \pi(W_0) = V_0$.

Step II. We prove in this step that V_0 is a $\mathrm{MT}(b)$-module. This implies that W_0 is a $\mathrm{MT}(b)$-subgroup of W by Proposition 1.1.19(2).

By definition of $V^\dagger$, $\widetilde{Z}_V \subset V^\dagger(\mathbb{R}) + \widetilde{z}_V$. By definition of V_0, $V_0(\mathbb{R}) + \widetilde{z}_V \subset \widetilde{Z}_V$. Now the conclusion of *Step I* implies $V_0 = V^\dagger$ and $\widetilde{Z}_V = V_0(\mathbb{R}) + \widetilde{z}_V$. However $\widetilde{Z}_V$ is complex, so $V_0(\mathbb{R})$ is a complex subspace of $V(\mathbb{R})$. Therefore by considering the complex structure of $V(\mathbb{R})$, we get that $V_0(\mathbb{R})$ is a $\mathrm{MT}(b)(\mathbb{R})$-module. So V_0 is a $\mathrm{MT}(b)$-module.

Step III. can be seen as an analogue to the proof of Theorem 3.2.8(1). Consider a fibre of $\widetilde{Z}$ over a point $v \in \pi(\widetilde{Z})$ such that $\pi \colon W(\mathbb{C})/F_b^0W(\mathbb{C}) \to \mathrm{Lie}(A)$ is flat at v (such a point exists by generic flatness). Let $\widetilde{W}$ be an irreducible algebraic component of $\widetilde{Z}_v$ such that $\dim(\widetilde{Z}_v) = \dim(\widetilde{W})$, then since π is flat at v,

$$\dim(\widetilde{Z}) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{Z}_v) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{W}).$$

Consider the set $\widetilde{F} := W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{W}$. It is semi-algebraic. The fact $\widetilde{W} \subset \widetilde{Z}$ implies that $\widetilde{F} \subset \widetilde{Z}$. So by [49, Lemma 4.1], there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0W(\mathbb{C})$, say $\widetilde{F}_{\mathrm{alg}}$, which contains $\widetilde{F}$ and is contained in $\widetilde{Z}$. Since

$$\pi(\widetilde{F}) = \pi(W_0)(\mathbb{R}) + v = \overline{\pi(\widetilde{Z})}$$

and every fiber of $\pi|_{\widetilde{F}_{\mathrm{alg}}}$ has $\mathbb{R}$-dimension at least $\dim_{\mathbb{R}}(\widetilde{W})$, we have

$$\dim(\widetilde{F}_{\mathrm{alg}}) \geqslant \dim(\pi(\widetilde{F})) + \dim(\widetilde{W}) = \dim(\pi(\widetilde{Z})) + \dim(\widetilde{W}) = \dim(\widetilde{Z}).$$

So $\widetilde{F} = \widetilde{Z}$ since $\widetilde{Z}$ is irreducible. In other words, $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v$ and $\widetilde{Z}_v$ is irreducible for any $v \in \pi(\widetilde{Z})$.

Next for any $v \in \pi(\widetilde{Z})$, let $\widetilde{W}'$ be an irreducible algebraic subvariety which contains $\widetilde{Z}_v$ and is contained in $\mathrm{unif}^{-1}(Y)_v$, maximal for these properties. Then $\widetilde{W}'$ is weakly special by *Case ii.* Consider $\widetilde{F}' := W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{W}'$. Let $\widetilde{Y}$ be the irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{Z}$, then $\widetilde{W}' \subset \widetilde{Y}$ and so $\widetilde{F}' \subset \widetilde{Y}$ by Lemma 3.2.2. But $\widetilde{F}'$ is semi-algebraic, and hence by [49, Lemma 4.1] there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0W(\mathbb{C})$, say $\widetilde{F}'_{\mathrm{alg}}$, which contains $\widetilde{F}'$ and is contained in $\widetilde{Y}$. So $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v \subset \widetilde{F}'_{\mathrm{alg}} \subset \mathrm{unif}^{-1}(Y)$, and hence $\widetilde{Z} = \widetilde{F}'_{\mathrm{alg}} = \widetilde{F}'$ by the maximality of $\widetilde{Z}$. So $\widetilde{Z}_v = \widetilde{W}'$, i.e.

$$\text{For any } v \in \pi(\widetilde{Z}),\ \widetilde{Z}_v \text{ is a maximal irreducible algebraic subvariety of } W(\mathbb{C})/F^0W(\mathbb{C}) \text{ contained in } \mathrm{unif}^{-1}(Y)_v. \tag{3.4.6}$$

Now that $\widetilde{Z}_v = \widetilde{W}'$ is weakly special, we can write $\widetilde{Z}_v = U'(\mathbb{C}) + \widetilde{z}$ with $U' < U$ and $\widetilde{z} \in \widetilde{Z}_v$. Then $U_0 < U'$. The product $W' := W_0U'$ is a subgroup of W, and hence

$$\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{Z}_v = W_0(\mathbb{R})U'(\mathbb{C})\widetilde{z} = W'(\mathbb{R})U'(\mathbb{C})\widetilde{z}.$$

So $W_0 = W'$ and $U_0 = U'$. In other words,

$$\widetilde{Z} = \widetilde{E} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z} \tag{3.4.7}$$

for some point $\widetilde{z} \in \widetilde{Z}_v$.

Step IV. Let us now conclude that $\widetilde{Z}$ is weakly special.

First of all, $U_0 \lhd P$ by Proposition 1.1.19(2). Consider $(P, \mathcal{X}^+) \xrightarrow{\rho} (P, \mathcal{X}^+)/U_0$, then by definition $\widetilde{Z}$ is weakly special iff $\rho(\widetilde{Z})$ is. Replace $(P, \mathcal{X}^+)$ (resp. W, $\widetilde{Z}$, W_0, $\widetilde{z}$) by $(P, \mathcal{X}^+)/U_0$ (resp. W/U_0, $\rho(\widetilde{Z})$, $W_0/U_0 = V_0$, $\rho(\widetilde{z})$), then V_0 is a subgroup of W and $\widetilde{Z} = V_0(\mathbb{R})\widetilde{z}$. Use the notation of §1.1.2.5 and §1.3 and suppose $\widetilde{z} = (\widetilde{z}_U, \widetilde{z}_V)$. By Proposition 2.1.2, $\widetilde{Z}$ is weakly special iff $\widetilde{z}_V \in (N_W(V_0)/U)(\mathbb{R})$ iff $\Psi(V_0(\mathbb{R}), \widetilde{z}_V) = 0$. We shall prove the last claim.

Define $Z := \mathrm{unif}(\widetilde{Z})$, $z = \mathrm{unif}(\widetilde{z})$ and $z_V = [\pi](z) \in A$, then $\pi(\widetilde{Z}) = V_0(\mathbb{R}) + \widetilde{z}_V$ and $[\pi](Z) = A_0 + z_V$ where $A_0 = \Gamma_{V_0} \backslash V_0(\mathbb{R})$ is an abelian subvariety of A. We can compute the fiber

$$Z_{z_V} = \left(\mathrm{unif}(\Gamma_W \widetilde{Z})\right)_{z_V} = \widetilde{z}_U + \frac{1}{2}\Psi(\Gamma_V, \widetilde{z}_V) + \Gamma_U \quad \mathrm{mod}\ \Gamma_U. \tag{3.4.8}$$

We have $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset U(\mathbb{R})$ since Ψ is defined over $\mathbb{Q}$. Let us prove $\Psi(\Gamma_V, \widetilde{z}_V) \subset U(\mathbb{Q})$. Fix an isomorphism $\Gamma_U \simeq \mathbb{Z}^m$, which induces an isomorphism $U(\mathbb{Q}) \simeq \mathbb{Q}^m$. Suppose that there exists a $u \in \Psi(\Gamma_V, \widetilde{z}_V) \setminus U(\mathbb{Q})$, then at least one of the coordinates of u is irrational. Without loss of generality we may assume that its first coordinate $u_1 \in \mathbb{R} \setminus \mathbb{Q}$. Denote by U_1 the $\mathbb{Q}$-subgroup of U corresponding to the first factor of $U(\mathbb{Q}) \simeq \mathbb{Q}^m$, then

$$\mathrm{unif}\left(\widetilde{z}_U + U_1(\mathbb{R})\right) \subset \overline{Z_{z_V}}$$

since $\{lu_1 \bmod \mathbb{Z} | l \in \mathbb{Z}\}$ is dense in $[0, 1)$. So $\overline{Z_{z_V}}$ contains

$$\mathrm{unif}\left(\widetilde{z}_U + U_1(\mathbb{C})\right),$$

and so does Y_{z_V} since $\overline{Z} \subset Y$. Let $v := v_0 + \widetilde{z}_V \in V(\mathbb{R})$, then $\widetilde{z}_U + U_1(\mathbb{C}) \subset \mathrm{unif}^{-1}(Y)_v$. However $\widetilde{Z}_{\widetilde{z}_v} = \widetilde{z}_U$ by (3.4.7) (recall that we have reduced to $W_0 = V_0$ and $U_0 = 0$), which contradicts (3.4.6). Hence $\Psi(\Gamma_V, \widetilde{z}_V) \subset U(\mathbb{Q})$, and therefore $(1/2)\Psi(N\Gamma_V, \widetilde{z}_V) \subset \Gamma_U$ for some $N \gg 0$ (since $\mathrm{rank}\,\Gamma_V < \infty$). Now we can construct a new lattice Γ'_W with $N\Gamma_V$ and Γ_U. Γ'_W is of finite index in Γ_W. Replacing Γ_W by Γ'_W does not change the assumption or the conclusion of Ax-Lindemann, so we may assume $(1/2)\Psi(\Gamma_V, \widetilde{z}_V) \subset \Gamma_U$. Now we can define C^∞-morphisms

$$\begin{aligned} f\colon A_0 + z_V &\longrightarrow T \\ a_0 + z_V &\mapsto \widetilde{z}_U + (1/2)\Psi(v_0, \widetilde{z}_V) \bmod \Gamma_U \end{aligned}$$

and

$$\begin{aligned} s\colon A_0 + z_V &\longrightarrow E|_{A_0 + z_V} \\ a_0 + z_V &\mapsto (\widetilde{z}_U + (1/2)\Psi(v_0, \widetilde{z}_V), a_0 + z_V) \bmod \Gamma_W \end{aligned}$$

where v_0 is any point of $V_0(\mathbb{R})$ such that $\mathrm{unif}_V(v_0) = a_0$. But Z_a is a single point for all $a \in A_0 + z_V$ by (3.4.8), so s is the inverse of $[\pi]|_Z$, and therefore s is a holomorphic section of $E|_{A_0+z_V} \to A_0 + z_V$. Locally on $U_i \subset A_0 + z_V$,

s is represented by a holomorphic morphism $U_i \to T$, which must equal to $f|_{U_i}$ by definition. Hence f is holomorphic since being holomorphic is a local condition. So f is constant.

But $\Psi(0, \widetilde{z}_V) = 0$, and therefore $(1/2)\Psi(V_0(\mathbb{R}), \widetilde{z}_V) \subset \Gamma_U$. But $\Psi(V_0(\mathbb{R}), \widetilde{z}_V)$ is continuous and $\Psi(0, \widetilde{z}_V) = 0$, so $\Psi(V_0(\mathbb{R}), \widetilde{z}_V) = 0$. Hence we are done. $\square$

3.5 Appendix

3.5.1 About the definability

This subsection is devoted to explain more details for the definability in §3.3.1. For any connected mixed Shimura variety $S = \Gamma\backslash\mathcal{X}^+$ associated with $(P, \mathcal{X}^+)$ whose uniformization is denoted by $\mathrm{unif}\colon \mathcal{X}^+ \to S$, we have the following diagram by the reduction lemma (Lemma 1.1.35):

$$\begin{array}{ccc} (P', \mathcal{X}'^+) & \overset{i}{\hookrightarrow} & (G_0, \mathcal{D}^+) \times \prod_{j=1}^{r} (\mathrm{GSp}_{2g}, \mathcal{X}_{2g}^+) \\ {\scriptstyle p}\downarrow & & \\ (P, \mathcal{X}^+) & & \end{array}$$

where $\mathrm{Ker}(p\colon P' \to P) \subset U'$ is a $\mathbb{Q}$-vector group of dimension 1 or 0. Hence there exists a congruence group $\Gamma' \subset P'(\mathbb{Q})_+$ such that $p(\Gamma') = \Gamma$. Now in order to find a fundamental subset $\mathcal{F}$ for the action of Γ on $\mathcal{X}^+$ such that $\mathrm{unif}|_{\mathcal{F}}$ is definable, it suffices to find a fundamental subset $\mathcal{F}'$ for the action of Γ' on $\mathcal{X}'^+$ such that $\mathrm{unif}'|_{\mathcal{F}'}$ is definable (here $\mathrm{unif}\colon \mathcal{X}'^+ \to S' := \Gamma'\backslash\mathcal{X}'^+$).

By [53, 3.8], there exists a congruence subgroup $\Gamma^\dagger \subset (G_0 \times \prod_{j=1}^r \mathrm{GSp}_{2g})(\mathbb{Q})_+$ such that $\Gamma' = \Gamma^\dagger \cap P'(\mathbb{Q})_+$ and $S' \xrightarrow{[i]} S^\dagger := \Gamma^\dagger\backslash(\mathcal{D}^+ \times \prod \mathcal{X}_{2g}^+)$ is a closed immersion. Applying Lemma 3.5.1 to

$$((P', \mathcal{X}'^+), \Gamma') \hookrightarrow \left((G_0, \mathcal{D}^+) \times \prod_{j=1}^{r} (\mathrm{GSp}_{2g}, \mathcal{X}_{2g}^+), \Gamma^\dagger \right),$$

it suffices to find a fundamental subset $\mathcal{F}^\dagger$ for the action of $\Gamma^\dagger$ on $\mathcal{D}^+ \times \prod \mathcal{X}_{2g}^+$ such that $\mathrm{unif}^\dagger|_{\mathcal{F}^\dagger}$ is definable. Replacing $\Gamma^\dagger$ by a smaller congruence subgroup does not change the conclusion, hence we may furthermore assume $\Gamma^\dagger = \Gamma_0 \times \prod_{j=1}^r \Gamma_i$ such that Γ_0 is a congruence subgroup of $G_0(\mathbb{Q})_+$ and Γ_j is a congruence subgroup of the j-th $\mathrm{GSp}_{2g}(\mathbb{Q})_+$-factor. Hence we are reduced to the situation as stated in §3.3.1.

Lemma 3.5.1. *Consider the diagram*

$$\begin{array}{ccc} (P_1, \mathcal{X}_1^+) & \overset{i}{\hookrightarrow} & (P_2, \mathcal{X}_2^+) \\ {\scriptstyle \mathrm{unif}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{unif}_2} \\ S_1 := \Gamma_1\backslash\mathcal{X}_1^+ & \overset{[i]}{\hookrightarrow} & S_2 := \Gamma_2\backslash\mathcal{X}_2^+ \end{array}$$

where $\Gamma_1 = \Gamma_2 \cap P_1(\mathbb{Q})_+$. *If there exists a fundamental set* $\mathcal{F}_2$ *for the action of* Γ_2 *on* $\mathcal{X}_2^+$ *such that* $\mathrm{unif}_2|_{\mathcal{F}_2}$ *is definable, then there exists a fundamental set* $\mathcal{F}_1$ *for the action of* Γ_1 *on* $\mathcal{X}_1^+$ *such that* $\mathrm{unif}_1|_{\mathcal{F}_1}$ *is definable.*

Proof. One possible way to prove this lemma is to repeat the proof of Ullmo [64, Proposition 2.4] (remark that Théorème 2.6 of *loc.cit.* holds for arbitrary linear algebraic groups over $\mathbb{Q}$). The proof we present here, which uses the o-minimal theory, is due to Pila-Tsimerman [50, Section 4.2].

First of all, note that $\mathrm{unif}_2^{-1}(S_1)$ is the (not disjoint) union over $\gamma \in \Gamma_2$ of $\gamma\mathcal{X}_1^+$. Secondly consider $\mathrm{unif}_2^{-1}(S_1) \cap \mathcal{F}_2$. Since $\mathrm{unif}_2|_{\mathcal{F}_2}$ is definable, this intersection has only finitely many connected components. Therefore there are finitely many elements $\gamma_j \in \Gamma_2$ $(1 \leqslant j \leqslant m)$ such that

$$\mathrm{unif}_2\left(\bigcup_{j=1}^{m} \gamma_j^{-1}\mathcal{X}_1^+ \cap \mathcal{F}_2\right) = S_1$$

and thus

$$\mathrm{unif}_2\left(\bigcup_{j=1}^{m} \mathcal{X}_1^+ \cap \gamma_j\mathcal{F}_2\right) = S_1.$$

Define Γ_2^1 to be the subgroup of Γ_2 which stabilizes $\mathcal{X}_1^+$. Then $\Gamma_1 \subset \Gamma_2^1$.

Now for any $x \in \mathcal{X}_1^+$, there exists a $\gamma \in \Gamma_2$ such that $\gamma x \in \mathcal{F}_2$ because $\mathcal{F}_2$ is a fundamental set for the action of Γ_2 on $\mathcal{X}_2^+$. As above this means that there exists a j with $1 \leqslant j \leqslant m$ such that $\gamma x \in \gamma_j^{-1}\mathcal{X}_1^+$ and $\gamma\mathcal{X}_1^+ = \gamma_j^{-1}\mathcal{X}_1^+$. Therefore there exists a $\gamma' \in \Gamma_2^1$ with $\gamma = \gamma_j^{-1}\gamma'$. Therefore $\gamma_j^{-1}\gamma' x \in \mathcal{F}_2 \cap \gamma_j^{-1}\mathcal{X}_1^+$ and so $\gamma' x \in \gamma_j\mathcal{F}_2 \cap \mathcal{X}_1^+$. To sum it up, $\mathcal{X}_1^e := \bigcup_{j=1}^m(\mathcal{X}_1^+ \cap \gamma_j\mathcal{F}_2)$ contains a fundamental set for the action of Γ_2^1 on $\mathcal{X}_1^+$. Now by picking coset representatives for Γ_1 in Γ_2^1, we can find a finite union of elements $\alpha_l \in \Gamma_2$ such that $\bigcup_l(\alpha_l\mathcal{X}_1^e \cap \mathcal{X}_1^+)$ contains a fundamental set, which we call $\mathcal{F}_1$, for the action of Γ_1 on $\mathcal{X}_1^+$. Then $\mathcal{F}_1$ is what we desire. □

3.5.2 A simplified proof of flat Ax-Lindemann

We prove here Theorem 3.1.5 when $E = T$ is an algebraic torus over $\mathbb{C}$ (which corresponds to the case $W = U$) and when $E = A$ is a complex abelian variety (which corresponds to the case $W = V$). The proof is a rearrangement of existing proofs (combining the point counting of Pila-Zannier [51] and volume calculation of Ullmo-Yafaev [67]). We use the notation of §3.4.

Case i : E=A. In this case, $W = V$ and $\Gamma_V = \oplus_{i=1}^{2n}\mathbb{Z}e_i \subset \mathrm{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by $\mathrm{unif}\colon \mathrm{Lie}(A) \to A$. Let $\mathcal{F}_V := \Sigma_{i=1}^{2n}(-1,1)e_i$, then $\mathcal{F}_V$ is a fundamental set for the action of Γ_V on $\mathrm{Lie}(A)$ such that $\mathrm{unif}|_{\mathcal{F}_V}$ is definable. Define the norm of $z = (x_1, y_1, ..., x_n, y_n) \in \mathrm{Lie}(A) = \mathbb{R}^{2n}$ to be

$$\| z \| := \mathrm{Max}(|x_1|, |y_1|, ..., |x_n|, |y_n|).$$

It is clear that $\forall z \in \mathrm{Lie}(A)$ and $\forall \gamma_V \in \Gamma_V$ such that $\gamma_V z \in \mathcal{F}_V$,

$$H(\gamma_V) \ll \| x_V \| . \tag{3.5.1}$$

Let $\omega_V := dz_1 \wedge d\overline{z}_1 + ... + dz_n \wedge d\overline{z}_n$ be the canonical $(1,1)$-form of $\mathrm{Lie}(A) = \mathbb{C}^n$. Let p_i $(i = 1, ..., n)$ be the n natural projections of $\mathrm{Lie}(A) = \mathbb{C}^n$ to $\mathbb{C}$. Let C be an algebraic curve of $\widetilde{Z}$ and define $C_M := \{z \in C | \| z \| \leqslant M\}$. We have

$$\begin{aligned}\int_{C\cap\mathcal{F}_V} \omega_V &\leqslant d \sum_{i=1}^{n} \int_{p_i(C\cap\mathcal{F}_V)} dz_i \wedge d\overline{z}_i \\ &\leqslant d \sum_{i=1}^{n} \int_{p_i(\mathcal{F}_V)} dz_i \wedge d\overline{z}_i = d \cdot O(1)\end{aligned} \tag{3.5.2}$$

and

$$\int_{C_M} \omega_V \geqslant O(M^2) \tag{3.5.3}$$

with $d = \deg(C)$ by [27, Theorem 0.1].

By (3.5.1)

$$C_M \subset \bigcup_{\gamma_V \in \Theta(\widetilde{Z}, M)} (C \cap \gamma^{-1}\mathcal{F}).$$

Integrating both sides w.r.t. ω_V we have

$$M^2 \ll \#\Theta(\widetilde{Z}, M)$$

by (3.5.2) and (3.5.3).

Let $\mathrm{Stab}_V(\widetilde{Z}) := \overline{\Gamma_V \cap \mathrm{Stab}_{V(\mathbb{R})}(\widetilde{Z})}^{\mathrm{Zar}}$. Now by Pila-Wilkie [67, Theorem 3.4], there exists an semi-algebraic block $B \subset \Sigma(\widetilde{Z})$ of positive dimension containing arbtrarily many points $\gamma_V \in \Gamma_V$. We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by definition. Hence for any $\gamma_V \in \Gamma_V \cap B$, $\widetilde{Z} \subset \gamma_V^{-1} B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$, and therefore $\widetilde{Z} = \gamma_V^{-1} B\widetilde{Z}$ by maximality of $\widetilde{Z}$. So $\gamma_V^{-1}(B \cap \Gamma_V) \subset \mathrm{Stab}_V(\widetilde{Z})(\mathbb{Q})$, and hence $\dim(\mathrm{Stab}_V(\widetilde{Z})) > 0$. For any point $\widetilde{z} \in \widetilde{Z}$, $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R}) + \widetilde{z} \subset \widetilde{Z}$. By [51, Lemma 2.3], $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R})$ is full and complex. Define $V' := V/\mathrm{Stab}_V(\widetilde{Z})$ and $\Gamma_{V'} := \Gamma_V/(\Gamma_V \cap \mathrm{Stab}_V(\widetilde{Z})(\mathbb{Q}))$, and then $A' := V'(\mathbb{R})/\Gamma_{V'}$ is a quotient abelian variety of A. Let Y' (resp. $\widetilde{Z}'$) be the Zariski closure of the projection of Y (resp. $\widetilde{Z}$) in A' (resp. $V'(\mathbb{R})$). We prove that the image of $\widetilde{Z}'$ is a point. If not, then proceeding as before for the triple $(A', Y', \widetilde{Z}')$ can we prove $\dim(\mathrm{Stab}_{V'}(\widetilde{Z}')) > 0$. This contradicts the definition (maximality) of $\mathrm{Stab}_V(\widetilde{Z})$. Hence $\widetilde{Z}$ is a translate of $\mathrm{Stab}_V(\widetilde{Z})(\mathbb{R})$. So $\widetilde{Z}$ is weakly special.

Case ii : E=T. Define the norm of $x_U = (x_{U,1}, x_{U,2}, ..., x_{U,m}) \in U(\mathbb{C})$ to be

$$\| x_U \| := \mathrm{Max}(\| x_{U,1} \|, \| x_{U,2} \|, ..., \| x_{U,m} \|).$$

It is clear that for all $x_U \in U(\mathbb{C})$ and for all $\gamma_U \in \Gamma_U$ such that $\gamma_U x_U \in \mathcal{F}_U$,

$$H(\gamma_U) \ll \| x_U \| . \tag{3.5.4}$$

Let $\omega|_T = dz_1 \wedge d\overline{z}_1 + ... + dz_m \wedge d\overline{z}_m$ be the canonical $(1,1)$-form of $U(\mathbb{C}) \simeq \mathbb{C}^m$. Let p_i $(i = 1, ..., m)$ be the m natural projections of $U(\mathbb{C}) \simeq \mathbb{C}^m$ to $\mathbb{C}$. Let C be an algebraic curve of $\widetilde{Z}$ and define $C_M := \{x \in C | \| x \| \leqslant M\}$. We have

$$\begin{aligned}\int_{C_M \cap \mathcal{F}_U} \omega|_T &\leqslant d \sum_{i=1}^{m} \int_{p_i(C_M \cap \mathcal{F}_U)} dz_i \wedge d\overline{z}_i \\ &\leqslant d \sum_{i=1}^{m} \int_{\{s \in \mathbb{C} | -1 < \Re e(s) < 1, \|s\| \leqslant M\}} dz_i \wedge d\overline{z}_i = d \cdot O(M)\end{aligned} \tag{3.5.5}$$

where $d := \deg(C)$. On the other hand by [27, Theorem 0.1],

$$\int_{C_M} \omega|_T \geqslant O(M^2). \tag{3.5.6}$$

By (3.5.4)

$$C_M \subset \bigcup_{\gamma \in \Theta(\widetilde{Z}, M)} (C_M \cap \gamma^{-1} \mathcal{F}).$$

Integrating both side w.r.t. $\omega|_T$ and taking into account that

$$\gamma \cdot C_M \subset (\gamma C)_{2M} \qquad \text{if } H(\gamma) \leqslant M,$$

we have

$$M^2 \ll \# \Theta(\widetilde{Z}, M) \cdot M$$

by (3.5.5) and (3.5.6). Hence $\# \Theta(\widetilde{Z}, M) \gg M$.

Let $\mathrm{Stab}_U(\widetilde{Z}) := \overline{\Gamma_U \cap \mathrm{Stab}_{U(\mathbb{C})}(\widetilde{Z})}^{\mathrm{Zar}}$. Now by Pila-Wilkie [48, Theorem 3.6], there exists an semi-algebraic subset $B \subset \Sigma(\widetilde{Z})$ of positive dimension containing arbtrarily many points $\gamma_U \in \Gamma_U$. We have $B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ since $\Sigma(\widetilde{Z})\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$ by definition. Hence for any $\gamma_U \in \Gamma_U \cap B$, $\widetilde{Z} \subset \gamma_U^{-1} B\widetilde{Z} \subset \mathrm{unif}^{-1}(Y)$, and therefore $\widetilde{Z} = \gamma_U^{-1} B\widetilde{Z}$ by maximality of $\widetilde{Z}$. So $\gamma_U^{-1}(B \cap \Gamma_U) \subset \mathrm{Stab}_U(\widetilde{Z})(\mathbb{Q})$, and hence $\dim(\mathrm{Stab}_U(\widetilde{Z})) > 0$. Let $U' := U / \mathrm{Stab}_U(\widetilde{Z})$, $\Gamma_{U'} := \Gamma_U / (\Gamma_U \cap \mathrm{Stab}_U(\widetilde{Z})(\mathbb{Q}))$ and $T' := U'(\mathbb{C})/\Gamma_{U'}$. T' is an algebraic torus over $\mathbb{C}$. Let Y' (resp. $\widetilde{Z}'$) be the Zariski closure of the projection of Y (resp. $\widetilde{Z}$) in T' (resp. $U'(\mathbb{C})$). We prove that $\widetilde{Z}'$ is a point. If not, then proceeding as before for the triple $(T', Y', \widetilde{Z}')$ we can prove $\dim(\mathrm{Stab}_{U'}(\widetilde{Z}')) > 0$. This contradicts the definition (maximality) of $\mathrm{Stab}_U(\widetilde{Z})$. Hence $\widetilde{Z}$ is a translate of $\mathrm{Stab}_U(\widetilde{Z})(\mathbb{C})$. So $\widetilde{Z}$ is weakly special.