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Chapter 2

Ax's theorem of log type

2.1 Results for the unipotent part

Given a connected mixed Shimura variety S, let S_G be its pure part. We have a projection $S \xrightarrow{[\pi]} S_G$. For any point $b \in S_G$, denote by E the fiber S_b. Suppose that S is associated with the mixed Shimura datum $(P, \mathcal{X}^+)$, which can be further assumed to satisfy $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. Let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Now $E = S_b \simeq \Gamma_W\backslash W(\mathbb{R})U(\mathbb{C})$ with the complex structure determined by $b \in S_G$ ($E = S_b = \Gamma_W\backslash W(\mathbb{C})/F_b^0 W(\mathbb{C})$), where $\Gamma_W := \Gamma \cap W(\mathbb{Q})$. Write $T := \Gamma_U\backslash U(\mathbb{C})$ and $A := \Gamma_A\backslash V(\mathbb{C})/F_b^0 V(\mathbb{C})$ where $\Gamma_U := \Gamma \cap U(\mathbb{Q})$ and $\Gamma_V := \Gamma_W/\Gamma_U$, then A is a complex abelian variety and E is an algebraic torus over A whose fibers are isomorphic to T.

Lemma 2.1.1. *If E admits a structure of algebraic group whose group law is compatible with the group law of W, then W (hence E) is commutative. In this case E is a semi-abelian variety.*

Proof. If E is an algebraic group, then T is a normal subgroup of E. Hence E acts on T by conjugation, and this action factors via A, and then it is trivial by [13, 8.10 Proposition]. Therefore T is in the center of E. Now consider the commutator pairing $E \times E \to E$. This factors through a morphism $A \times A \xrightarrow{f} T$. But as a morphism from an abelian variety to an algebraic torus over $\mathbb{C}$, f is then constant. So the commutator pairing $E \times E \to E$ is trivial, and hence E is commutative.

The commutator pairing $W \times W \to W$ induces an alternating form $\Psi\colon V \times V \to U$ (see §1.1.2.5) which induces the morphism f above. We have proved in the last paragraph that $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset \Gamma_U$ with $\Gamma_U := \Gamma \cap U(\mathbb{Q})$. But $\Psi(V(\mathbb{R}), v)$ is continuous for any $v \in V(\mathbb{R})$ and $\Psi(0, V(\mathbb{R})) = 0$, so $\Psi(V(\mathbb{R}), V(\mathbb{R})) = 0$. Hence the commutator pairing $W \times W \to W$ is trivial, and therefore W is commutative. □

2.1.1 Weakly special subvarieties of a complex semi-abelian variety

Proposition 2.1.2. *Use the notation as at the beginning of the section. Weakly special subvarieties of E are precisely the subsets of E of the form*

$$\mathrm{unif}(W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z})$$

where W_0 is a $\mathrm{MT}(b)$*-subgroup of W (i.e. a subgroup of W normalized by* $\mathrm{MT}(b)$*), $U_0 := W_0 \cap U$,* $\mathrm{unif}(\widetilde{z}_G) = b$ *and* $\widetilde{z}_V \in (N_W(W_0)/U)(\mathbb{R})$ *($\widetilde{z} = (\widetilde{z}_U, \widetilde{z}_V, \widetilde{z}_G)$ under (1.3.1)).*

In particular, if E can be given the structure of an algebraic group whose group law is compatible with that of W (i.e. W is commutative), then the weakly special subvarieties of E are precisely the translates of subgroups of E.

Proof. Let Z be a weakly special variety of E and let $\widetilde{Z}$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Z)$, then there exists a diagram as in Definition 1.2.2 such that $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $Q_{\mathbb{C}}$, $N \lhd Q$ and $\widetilde{Z} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{z}$ for some $\widetilde{z} \in \widetilde{Z}$. As is explained in [54, paragraph 2, pp 265], $G_N = 1$. We prove that $N = W_N$ satisfies the conditions which we require. Let $U_N := W_N \cap U$, then U_N is a $\mathrm{MT}(b)$-module by Proposition 1.1.19(2). Denote by $V_N := W_N/U_N$, $\pi_{P/U}\colon (P, \mathcal{X}^+) \to (P/U, \mathcal{X}^+_{P/U})$ and $[\pi_{P/U}]\colon S \to S_{P/U}$. Then $[\pi_{P/U}](Z)$ is a subvariety of A since Z is a subvariety of E. So $\pi_{P/U}(\widetilde{Z}) = V_N(\mathbb{R}) + \pi_{P/U}(\widetilde{z})$ is the translate of a complex subspace of $V(\mathbb{R}) = V(\mathbb{C})/F^0_b V(\mathbb{C})$, and therefore V_N is a $\mathrm{MT}(b)$-module. So W_N is stable under the action of $\mathrm{MT}(b)$. Now $\widetilde{z}_V \in (N_W(N)/U)(\mathbb{R})$ since $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $N_P(N)_{\mathbb{C}}$.

Conversely let $\widetilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\widetilde{z}$ with W_0, $\widetilde{z}$ as stated. Fix a Levi decomposition $P = W \rtimes G$. Let $G' := \mathrm{MT}(b)$, let $W' := N_W(W_0)$ and let $Q := W' \rtimes G'$. Then $W_0 \lhd Q$ and hence $\widetilde{z}\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $Q_{\mathbb{C}}$. Therefore $(Q, \mathcal{Y}^+)$, where $\mathcal{Y}^+ := Q(\mathbb{R})^+(U \cap Q)(\mathbb{C})\widetilde{z}$, is a connected mixed Shimura subdatum of $(P, \mathcal{X}^+)$ such that $b \in \mathrm{unif}(\mathcal{Y}^+)$. Now consider the morphisms of connected mixed Shimura data

$$(Q, \mathcal{Y}^+)/W_0 \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+).$$

In the fibres above the point $b \in S_G$ these maps are simply

$$S_{Q,b}/Z \twoheadleftarrow S_{Q,b} \hookrightarrow E = S_b.$$

Hence Z is a weakly special subvariety by definition. □

Corollary 2.1.3. *1. Weakly special subvarieties of a complex abelian variety are precisely the translates of its abelian subvarieties;*

2. Weakly special subvarieties of an algebraic torus over $\mathbb{C}$ are precisely the translates of its subtori.

Proof. This is a direct consequence of Proposition 2.1.2. □

2.1.2 Smallest weakly special subvariety containing a given subvariety of an abelian variety or an algebraic torus over $\mathbb{C}$

Proposition 2.1.4. *1. Let X be a complex abelian variety and let Z be an irreducible subvariety of X. Denote by*

$$\widetilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{R} = H_1(X, \mathbb{R}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\mathrm{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes \mathbb{R})$.

2. Let X be an algebraic torus over $\mathbb{C}$ and let Z be an irreducible subvariety of X. Denote by

$$\widetilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{C} = H_1(X, \mathbb{C}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\mathrm{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes \mathbb{C})$.

Proof. 1. If X is a complex abelian variety, then the result is due to Ullmo-Yafaev. Their proof of [65, Proposition 5.1] has in fact revealed this property. Here we restate the proof with more details.

Let $Z^{\mathrm{de}} \xrightarrow{s} Z$ be a desingularization of Z^{de} such that there exists a Zariski open subset Z_0^{de} of Z^{de} such that $Z_0^{\mathrm{de}} \xrightarrow[s]{\sim} Z^{\mathrm{sm}}$. By the commutative diagram

$$\begin{array}{ccccc} \pi_1(Z_0^{\mathrm{de}}, z) & \xrightarrow{\sim} & \pi_1(Z^{\mathrm{sm}}, z) & & \\ \downarrow & & \downarrow & \searrow & \\ \pi_1(Z^{\mathrm{de}}, z) & \longrightarrow & \pi_1(Z, z) & \longrightarrow & \pi_1(X, z) \end{array},$$

where $z \in Z^{\mathrm{sm}}$ (the surjectivity on the left is due to [31, 2.10.1]), we know that the image of $\pi_1(Z^{\mathrm{de}}, z)$ and the image of $\pi_1(Z^{\mathrm{sm}}, z)$ in $\pi_1(X, z)$ are the same.

Let $\mathrm{Alb}(Z^{\mathrm{de}})$ be the Albanese variety of Z^{de} normalized by z, then the map $\tau\colon Z^{\mathrm{de}} \to Z \to X$ factors uniquely through the Albanese morphism([70, Theorem 12.15]):

$$\begin{array}{ccccc} Z^{\mathrm{de}} & \twoheadrightarrow & Z & \hookrightarrow & X \\ & \searrow^{\mathrm{alb}} & & \nearrow_{\Gamma} & \\ & & \mathrm{Alb}(Z^{\mathrm{de}}) & & \end{array}$$

Let $A := \Gamma(\mathrm{Alb}(Z^{\mathrm{de}}))$, then it is the smallest weakly special subvariety (i.e. the translate of an abelian subvariety) of X containing Z since $\mathrm{alb}(Z^{\mathrm{de}})$ generates $\mathrm{Alb}(Z^{\mathrm{de}})$ ([70, Lemma 12.11]).

It suffices to prove that the image of $\pi_1(Z^{\mathrm{de}}, z)$ in $\pi_1(X, z) \simeq H_1(X, \mathbb{Z})$ is of finite index in $H_1(A, \mathbb{Z})$. This is true since the image of $\pi_1(Z^{\mathrm{de}}, z)$ in $H_1(X, \mathbb{Z})$ contains

$$(\Gamma \circ \mathrm{alb})_* H_1(Z^{\mathrm{de}}, \mathbb{Z}) \simeq \Gamma_* H_1(\mathrm{Alb}(Z^{\mathrm{de}}), \mathbb{Z}) \simeq \Gamma_* \pi_1(\mathrm{Alb}(Z^{\mathrm{de}}))$$

(the first isomorphism is given by the definition of Albanese varieties via Hodge theory, see e.g. the proof of [70, Lemma 12.11]), which is of finite index in $\pi_1(A, z) \simeq H_1(A, \mathbb{Z})$ by [31, 2.10.2].

2. If X is an algebraic torus over $\mathbb{C}$, then we can first of all translate Z by a point such that the translate contains the origin of X. Now we are done if we can prove that the smallest subtorus containing this translate of Z is $u(\pi_1(Z^{\mathrm{sm}}, z) \otimes_{\mathbb{Z}} \mathbb{C})$.

   Suppose $T \simeq (\mathbb{C}^*)^m$ is the smallest sub-torus of X containing Z with $j\colon Z^{\mathrm{sm}} \hookrightarrow T$ the inclusion. We are done if we can prove $[\pi_1(T, z) : j_* \pi_1(Z^{\mathrm{sm}}, z)] < \infty$. If not, then

$$j_* \pi_1(Z^{\mathrm{sm}}, z) \;\subset\; \mathrm{Ker}(\mathbb{Z}^m \overset{\rho}{\twoheadrightarrow} \mathbb{Z}) \tag{2.1.1}$$

   for some map ρ. Since the covariant functor $T \mapsto X_*(T)$ ($X_*(T)$ is the co-character group of T) is an equivalence between the category {algebraic tori over $\mathbb{C}$ and their morphisms as algebraic groups} and the category {free $\mathbb{Z}$-modules of finite rank}, the map ρ corresponds to a surjective map (with connected kernel) of tori $p\colon T \twoheadrightarrow T'$. The composition of the maps $Z^{\mathrm{sm}} \xrightarrow{j} T \xrightarrow{p} T' = \mathbb{G}_{m,\mathbb{C}}$ should be dominant by the choice of T. But then we have

$$[\pi_1(T', p(z)) : (p \circ j)_* \pi_1(Z^{\mathrm{sm}}, z)] < \infty$$

   ([31, 2.10.2]), which contradicts (2.1.1) by the following lemma.

Lemma 2.1.5. *For any $\mathbb{C}$-split torus $T \simeq (\mathbb{C}^*)^n$, we have a canonical isomorphism*

$$X_*(T) \xrightarrow[\sim]{\psi_T} \pi_1(T, 1).$$

Here "canonical" means that for any morphism (between algebraic groups) $f\colon T \to T'$ between two such $\mathbb{C}$-split tori, the following diagram commutes:

$$\begin{array}{ccc} X_*(T) & \xrightarrow[\sim]{\psi_T} & \pi_1(T, 1) \\ {\scriptstyle X_*(f)}\downarrow & & \downarrow{\scriptstyle f_*} \\ X_*(T') & \xrightarrow[\sim]{\psi_{T'}} & \pi_1(T', 1) \end{array}$$

Proof. Denote by $U_1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and $i\colon U_1 \hookrightarrow \mathbb{C}^*$ the inclusion. Then the map ψ_T is defined by

$$\begin{array}{ccc} X_*(T) & \xrightarrow{\psi_T} & \pi_1(T,1) \\ \nu & \mapsto & [\nu \circ i] \end{array}.$$

This is a group homomorphism. It is surjective since a representative of the generators of $\pi_1(T,1)$ is given by the n coordinate embeddings $U_1 \hookrightarrow \mathbb{C}^* \hookrightarrow T = (\mathbb{C}^*)^n$. ψ_T is injective since $X_*(T) \simeq \pi_1(T,1) \simeq \mathbb{Z}^n$ is torsion-free. The rest of the lemma is immediate by the construction of ψ_T. □

□

2.2 Monodromy groups of admissible variations of mixed Hodge structures

2.2.1 Arbitrary variation of mixed $\mathbb{Z}$-Hodge structures

Let $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ be a variation of mixed $\mathbb{Z}$-Hodge structures over a complex manifold S (see §1.1.1.4 for definition). Let $\pi\colon \widetilde{S} \to S$ be a universal covering and choose a trivialization $\pi^*\mathbb{V} \simeq \widetilde{S} \times V$. For $s \in S$, $\mathrm{MT}_s \subset \mathrm{GL}(\mathcal{V}_s)$ denote the Mumford-Tate group of its fibre. The choice of a point $\widetilde{s} \in \widetilde{S}$ with $\pi(\widetilde{s}) = s$ gives an identification $\mathcal{V}_s \simeq V$, whence an injective homomorphism $i_{\widetilde{s}}\colon \mathrm{MT}_s \hookrightarrow \mathrm{GL}(V)$. By [1, §4, Lemma 4], on $S^\circ := S \setminus \Sigma$ where Σ is a meager subset of S, $M := \mathrm{Im}(i_{\widetilde{s}}) \subset \mathrm{GL}(V)$ does not depend on s, nor on the choice of $\widetilde{s}$. We call S° the **Hodge-generic locus** and the group M the **generic Mumford-Tate group** of $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$.

On the other hand, if we choose a base-point $s \in S$ and a point $\widetilde{s} \in \widetilde{S}$ with $\pi(\widetilde{s}) = s$, then then local system $\mathbb{V}$ corresponds to a representation $\rho\colon \pi_1(S,s) \to \mathrm{GL}(V)$, called the monodromy representation. The algebraic monodromy group is defined as the smallest algebraic subgroup of $\mathrm{GL}(V)$ over $\mathbb{Q}$ which contains the image of ρ. We write H_s^{mon} for its connected component of the identity, called the **connected algebraic monodromy group**. Given the trivialization of $\pi^*\mathbb{V}$, the group $H_s^{\mathrm{mon}} \subset \mathrm{GL}(V)$ is independent of the choice of s and $\widetilde{s}$.

Suppose now that $(\mathbb{V}, W_\cdot, \mathcal{F}^\cdot)$ is graded-polarizable, then $H_s^{\mathrm{mon}} < M$ for any $s \in S^\circ$ by [1, §4, Lemma 4].

2.2.2 Admissible variations of $\mathbb{Z}$-mixed Hodge structures

We now recall the concept of "admissible" variations of mixed Hodge structures which was introduced by Steenbrick-Zucker and studied by Kashiwara and Hain-Zucker. We give the definition here, but instead of the exact definition, we shall only use the notion of "admissibility" and the fact that it can be defined using "curve test". We will use Δ (resp. Δ^*) to denote the unit disc (resp. punctured unit disc).

Definition 2.2.1. *(see [45, Definition 14.49])*

1. *A variation of mixed Hodge structures* $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ *over the punctured unit disc* Δ^* *is called* ***admissible*** *if*
   - *it is graded-polarizable;*
   - *the monodromy* T *is unipotent and the weight filtration* $M(N, W_.)$ *of* $N := \log T$ *relative to* $W_.$ *exists;*
   - *the filtration* $\mathcal{F}^\cdot$ *extends to a filtration* $\widetilde{\mathcal{F}}^\cdot$ *of* $\widetilde{\mathcal{V}}$ *which induced* ${}^k\widetilde{\mathcal{F}}$ *on* $\mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ *for each* k.
2. *Let* S *be a smooth connected complex algebraic variety and let* $\overline{S}$ *be a compactification of* S *such that* $\overline{S} \setminus S$ *is a normal crossing divisor. A graded-polarizable variation of mixed Hodge structures* $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ *on* S *is called* ***admissible*** *if for every holomorphic map* $i\colon \Delta \to \overline{S}$ *which maps* Δ^* *to* S *and such that* $i^*\mathbb{V}$ *has unipotent monodromy, the variation* $i^*(\mathbb{V}, W_., \mathcal{F}^\cdot)$ *is admissible. (This definition is sometimes called the "curve test" version).*

Remark 2.2.2. *This definition is equivalent to the one in [25, 1.5]. See [61, Properties 3.13 and Appendix], [28, §1 and Theorem 4.5.2] and [25, 1.5] for details.*

The following lemma is an easy property of admissibility and is surely known by many people, but I cannot find any reference, so I give a proof here.

Lemma 2.2.3. *Let* S *be a smooth connected complex algebraic variety and let* $(\mathbb{V}, W_., \mathcal{F}^\cdot)$ *be an admissible variation of mixed Hodge structures on* S. *Then for any smooth connected (not necessarily closed) subvariety* $j\colon Y \hookrightarrow S$, $j^*(\mathbb{V}, W_., \mathcal{F}^\cdot)$ *is also admissible on* Y.

Proof. Take smooth compactifications $\overline{Y}$ of Y and $\overline{S}$ of S such that $\overline{Y} \setminus Y$ and $\overline{S} \setminus S$ are normal crossing divisors and such that $j\colon Y \hookrightarrow S$ extends to a morphism $\overline{j}\colon \overline{Y} \to \overline{S}$. This can be done by first choosing any compactifications of Y^{cp} of Y and S^{cp} of S with normal crossing divisors and then taking a suitable resolution of singularities of the closure of the graph of j in $Y^{\mathrm{cp}} \times S^{\mathrm{cp}}$. Now the conclusion follows from our "curve test" version of the definition. ☐

2.2.3 Consequences of admissibility

Y.André proved:

Theorem 2.2.4. *Let* $(\mathbb{V}, W_{\cdot}, \mathcal{F}^{\cdot})$ *be an admissible variation of mixed Hodge structures over a smooth connected complex algebraic variety* S*, then for any* $s \in S$*, the connected monodromy group* H_s^{mon} *is a normal subgroup of the generic Mumford-Tate group* M *and also its derived group* M^{der}*.*

Proof. [1, §5, Theorem 1] states that $H_s^{\mathrm{mon}} \lhd M^{\mathrm{der}}$, and in the proof he first proved that $H_s^{\mathrm{mon}} \lhd M$. □

Now we state a theorem which roughly says that all the variations of mixed Hodge structure obtained from representations of the underlying group of a connected mixed Shimura datum are admissible. Explicitly, let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Suppose that Γ is neat. Consider any $\mathbb{Q}$-representation $\xi\colon P \to \mathrm{GL}(V)$. By [55, Proposition 4.2], there exists a Γ-invariant lattice $V_{\mathbb{Z}}$ of V. Now ξ and $V_{\mathbb{Z}}$ together give rise to a VMHS on S whose underlying local system is $\Gamma\backslash(\mathcal{X}^+ \times V_{\mathbb{Z}})$. This variation is (graded-)polarizable by [53, 1.18(d)]. Wildeshaus proved:

Theorem 2.2.5. *Let* S*,* $(P, \mathcal{X}^+)$*,* $\xi\colon P \to \mathrm{GL}(V)$ *and* $V_{\mathbb{Z}}$ *be as in the paragraph above, then the variation of mixed Hodge structures obtained as above is admissible.*

Proof. [71, Theorem 2.2] says that the corresponding $\mathbb{Q}$-variation is admissible, and Γ gives a $\mathbb{Z}$-structure as in the discussion above. □

Remark 2.2.6. *In this language, we can rephrase Definition 1.1.18 as:* P *is the generic Mumford-Tate group (of the variation in Theorem 2.2.5). For any Hodge generic point* $x \in \mathcal{X}^+$*, the only* $\mathbb{Q}$*-subgroup* N *of* P^{der} *such that* $N(\mathbb{R})^+U_N(\mathbb{C})$*, where* $U_N := U \cap N$*, stabilizes* x *is the trivial group.*

2.3 The smallest weakly special subvariety containing a given subvariety

In this section, our goal is to prove a theorem (Theorem 2.3.1) which (in some sense) generalizes [39, 3.6, 3.7]. In particular, we get a criterion of weak specialness as a corollary (Corollary 2.3.3) which generalizes [65, Theorem 4.1].

2.3.1 Connected algebraic monodromy group associated with a subvariety of a mixed Shimura variety

Before the proof, let us do some technical preparation at first.

Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. We may assume $P = \mathrm{MT}(\mathcal{X}^+)$ by Proposition 1.1.19. There exists a $\Gamma' \leqslant \Gamma$ of finite index such that Γ' is neat. Let $S' := \Gamma'\backslash\mathcal{X}^+$ and let

$\mathrm{unif}'\colon \mathcal{X}^+ \to S'$ be its uniformization. Choose any faithful $\mathbb{Q}$-representation $\xi\colon P \to \mathrm{GL}(M)$ of P, then Theorem 2.2.5 claims that ξ (together with a choice of a Γ'-invariant lattice of M) gives rise to an admissible variation of mixed Hodge structure on S'. The generic Mumford-Tate group of this variation is P.

Suppose that Y is an irreducible subvariety of S. Let Y' be an irreducible component of $p^{-1}(Y)$ under $p\colon S' = \Gamma'\backslash\mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$, then Y' is an irreducible subvariety of S' which maps surjectively to Y under p. The variation we constructed above can be restricted to Y'^{sm}, and this restriction is still admissible by Lemma 2.2.3. The **connected algebraic monodromy group associated with** Y^{sm} is defined to be the connected algebraic monodromy group of the restriction of the VMHS defined in the last paragraph to Y'^{sm}, i.e. the neutral component of the Zariski closure of the image of $\pi_1(Y'^{\mathrm{sm}}, y') \to \pi_1(S', y') \to P$.

Let us briefly prove that the connected algebraic monodromy group associated with Y^{sm} is well-defined. Suppose that we have another covering $S'' \xrightarrow{p'} S'$ with S'' smooth. Let Y'' be an irreducible component of $p'^{-1}(Y')$. Let $Y_0''^{\mathrm{sm}} := Y''^{\mathrm{sm}} \cap p'^{-1}(Y'^{\mathrm{sm}})$, then by the commutative diagram

$$\begin{array}{ccccc}
\pi_1(Y_0''^{\mathrm{sm}}, y'') = \pi_1(Y''^{\mathrm{sm}}, y'') & \longrightarrow & \pi_1(S'', y'') & \longrightarrow & P \\
\downarrow & & \downarrow & & \Big\downarrow{=} \\
\pi_1(Y'^{\mathrm{sm}}, y') & \longrightarrow & \pi_1(S', y') & \longrightarrow & P
\end{array},$$

where the equality in the top-left cornor is given by [31, 2.10.1] and the morphism on the left is of finite index by [31, 2.10.2], the neutral components of the Zariski closures of the images of $\pi_1(Y''^{\mathrm{sm}}, y'')$ and $\pi_1(Y'^{\mathrm{sm}}, y')$ in P coincide.

2.3.2 Ax's theorem of log type

Theorem 2.3.1 (Ax of log type)**.** *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ be the uniformization. Let Y be an irreducible subvariety of S and*

- *let $\widetilde{Y}$ be a complex analytic irreducible component of* $\mathrm{unif}^{-1}(Y)$*;*
- *take $\widetilde{y}_0 \in \widetilde{Y}$;*
- *let N be the connected algebraic monodromy group associated with Y^{sm}.*

Then

1. *The set $\widetilde{F} := N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y}_0$, where $U_N := U \cap N$, is a weakly special subset of $\mathcal{X}^+$ (or equivalently, $F :=$ $\mathrm{unif}(\widetilde{F})$ is a weakly special subvariety of S). Moreover N is the largest subgroup of Q such that $N(\mathbb{R})^+U_N(\mathbb{C})$ stabilizes $\widetilde{F}$, where $(Q, \mathcal{Y}^+)$ is the smallest connected mixed Shimura subdatum with $\widetilde{F} \subset \mathcal{Y}^+$;*

2. *The Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ (which means the complex analytic irreducible component of the intersection of the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$ and $\mathcal{X}^+$ which contains $\widetilde{Y}$) is $\widetilde{F}$;*

3. *The smallest weakly special subset containing $\widetilde{Y}$ is $\widetilde{F}$ and F is the smallest weakly special subvariety of S containing Y.*

Proof. 1. Let S_Y be the smallest special subvariety containing Y. Such an S_Y exists since the irreducible components of intersections of special subvarieties are special (which can easily be shown by means of generic Mumford-Tate group). By definition of special subvarieties, there exists a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+)$ such that S_Y is the image of $\Gamma_Q \backslash \mathcal{Y}^+$ in S where $\Gamma_Q := \Gamma \cap Q(\mathbb{Q})$. We may furthermore assume $(Q, \mathcal{Y}^+)$ to have generic Mumford-Tate group by Proposition 1.1.19.

Let N be the connected algebraic monodromy group associated with Y^{sm}, then $N \lhd Q$ (and also $N \lhd Q^{\mathrm{der}}$) by the discussion at the beginning of this section (which claims that the variation we use to define N is admissible), Remark 2.2.6 (which claims that the generic Mumford-Tate group of this variation is Q) and Theorem 2.2.4.

Then $\widetilde{F}$ is a weakly special subset of $\mathcal{Y}^+$ since it is the inverse image of the point $\varphi(\widetilde{y}_0)$ under the Shimura morphism $(Q, \mathcal{Y}^+) \xrightarrow{\varphi} (Q, \mathcal{Y}^+)/N$. Then $\widetilde{F}$ is also a weakly special subset of $\mathcal{X}^+$ by definition. By the choice of $(Q, \mathcal{Y}^+)$, $\widetilde{F}$ is Hodge generic in $\mathcal{Y}^+$, and hence $\varphi(\widetilde{F})$ is a Hodge generic point in $\mathcal{Y}'^+$. Now $\mathrm{Stab}_{Q^{\mathrm{der}}(\mathbb{Q})}(\widetilde{F})^\circ = N(\mathbb{Q})$ by Remark 2.2.6.

2. We prove that $\widetilde{F}$ is the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$. We first show that the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ defined as in the statement of the theorem exists. To see this, denote by $\widetilde{Y}^\vee$ the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$. Recall that $\mathcal{X}^+$ is realized as a semi-algebraic open subset (w.r.t. the archimedean topology) of $\mathcal{X}^\vee$ as in §1.3.1. Hence $\widetilde{Y}^\vee \cap \mathcal{X}^+$ has only finitely many complex analytic irreducible components[1], which we call $I_1, ..., I_r$. If $\widetilde{Y}$ is contained in both I_i and I_j where I_i and I_j are distinct, then

$$\widetilde{Y} \subset I_i \cap I_j \subset (\widetilde{Y}^\vee \cap \mathcal{X}^+)^{\mathrm{sing}} \subset (\widetilde{Y}^\vee)^{\mathrm{sing}} \cap \mathcal{X}^+ \subsetneq \widetilde{Y}^\vee \cap \mathcal{X}^+$$

[1]This is true for any irreducible subvariety Z of $\mathcal{X}^\vee$ by induction on $\dim Z$: since the collection of all semi-algebraic sets forms an o-minimal theory, $(Z \cap \mathcal{X}^+)^{\mathrm{sm}}$ decomposes into finitely many connected components, each of which semi-algebraic (To better understand this, recall the theorem of Klingler-Ullmo-Yafaev [29, Appendix] which says that for $(P, \mathcal{X}^+)$ pure, a subset of $\mathcal{X}^+$ is irreducible algebraic iff it is semi-algebraic and complex analytic irreducible. Their argument can be generalized to the mixed case without much difficulty.). Remark that these connected components are also precisely the complex analytic irreducible components since the ambient subset of $\mathcal{X}^+$ is smooth. Now $(Z \cap \mathcal{X}^+)^{\mathrm{sing}} = Z^{\mathrm{sing}} \cap \mathcal{X}^+$ also has only finitely many complex analytic irreducible components by induction hypothesis. So we can conclude.

But $(\widetilde{Y}^\vee)^{\mathrm{sing}}$ is an algebraic subvariety of $\mathcal{X}^\vee$. So this contradicts the fact that $\widetilde{Y}^\vee$ is the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^\vee$. Hence $\widetilde{Y}$ is contained in a unique complex analytic irreducible component of $\widetilde{Y}^\vee \cap \mathcal{X}^+$. So the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$ defined as in the statement of the theorem exists.

Next we prove that it suffices to prove $\widetilde{Y} \subset \widetilde{F}$. Assume this. Let $\overline{\widetilde{Y}}$ be the Zariski closure of $\widetilde{Y}$ in $\mathcal{X}^+$, then $\overline{\widetilde{Y}} \subset \widetilde{F}$ since $\widetilde{Y} \subset \widetilde{F}$ and $\widetilde{F}$ is algebraic (Lemma 1.3.8). On the other hand, $\Gamma_{Y^{\mathrm{sm}}} := \mathrm{Im}(\pi_1(Y^{\mathrm{sm}}) \to \pi_1(S) \to P)$ stabilizes $\widetilde{Y}$, so $\Gamma_{Y^{\mathrm{sm}}}\widetilde{y}_0 \subset \widetilde{Y}$. The group $\Gamma_{Y^{\mathrm{sm}}}$ is Zariski dense in N, and hence Zariski dense in $N_{\mathbb{C}}$. But $\widetilde{F}$ is a complex analytic irreducible component of $N(\mathbb{C})\widetilde{y}_0 \cap \mathcal{X}^+$, so $\Gamma_{Y^{\mathrm{sm}}}\widetilde{y}_0$ is Zariski dense in $\widetilde{F}$. Hence we have $\widetilde{F} \subset \overline{\widetilde{Y}}$. As a result, $\widetilde{F} = \overline{\widetilde{Y}}$.

Now we prove that $\widetilde{Y} \subset \widetilde{F}$ (or equivalently, $Y \subset F$).

The fact that $\widetilde{Y} \subset \widetilde{F}$ has nothing to do with the level structure. Hence we may assume $\Gamma = \Gamma_W \rtimes \Gamma_G$ with $\Gamma_W \subset W(\mathbb{Z})$, $\Gamma_U := \Gamma_W \cap U \subset U(\mathbb{Z})$, $\Gamma_V := \Gamma_W/\Gamma_U \subset V(\mathbb{Z})$ and $\Gamma_G \subset G(\mathbb{Z})$ small enough such that they are all neat and such that $\Gamma \subset P^{\mathrm{der}}(\mathbb{Q})$ (Remark 1.1.13(2)). We write $\Gamma_{P/U} := \Gamma/\Gamma_U$.

We may replace $(P, \mathcal{X}^+)$ by $(Q, \mathcal{Y}^+)$ and S by S_Y (same notation as in (1)) since $\widetilde{Y}$, $\widetilde{F} \subset \mathcal{Y}^+$ and Y, $F \subset S_Y$. In other words, we may assume that Y is Hodge generic in S and $(P, \mathcal{X}^+)$ is irreducible.

Consider the following diagram:

$$
\begin{array}{ccccc}
\mathcal{X}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}^+_{P/U} & \xrightarrow{\pi_G} & \mathcal{X}^+_G \\
\downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_{P/U}} & & \downarrow{\scriptstyle \mathrm{unif}_G} \\
S = \Gamma\backslash\mathcal{X}^+ & \xrightarrow{[\pi_{P/U}]} & S_{P/U} := \Gamma_{P/U}\backslash\mathcal{X}^+_{P/U} & \xrightarrow{[\pi_G]} & S_G := \Gamma_G\backslash\mathcal{X}^+_G
\end{array}
$$

Denote by π and $[\pi]$ the composites of the maps in the two lines respectively. Denote by $\widetilde{Y}_G := \pi(\widetilde{Y})$, $Y_G := [\pi](Y)$ and $\widetilde{Y}_{P/U} := \pi_{P/U}(\widetilde{Y})$, $Y_{P/U} := [\pi_{P/U}](Y)$; $\widetilde{F}_G := \pi(\widetilde{F})$, $F_G := [\pi](F)$ and $\widetilde{F}_{P/U} := \pi_{P/U}(\widetilde{F})$, $F_{P/U} := [\pi_{P/U}](F)$. Denote by $\widetilde{y}_{0,P/U} := \pi_{P/U}(\widetilde{y}_0)$ and $\widetilde{y}_{0,G} := \pi(\widetilde{y}_0)$.

Now to make the proof more clear, we divide it into several steps.

Step I. Prove that $\widetilde{Y}_G \subset \widetilde{F}_G$.

We begin the proof with the following lemma:

Lemma 2.3.2. *In the context above, the connected algebraic monodromy group associated with* $\overline{Y_G}^{\mathrm{sm}}$ *(resp.* $\overline{Y_{P/U}}^{\mathrm{sm}}$*) is* G_N *(resp.* N/U_N *where* $U_N := U \cap N$*).*

Proof. We only prove the statement for $\overline{Y_G}^{\mathrm{sm}}$. The proof for $\overline{Y_{P/U}}^{\mathrm{sm}}$ is similar. Take $Y_0^{\mathrm{sm}} := Y^{\mathrm{sm}} \cap \pi^{-1}(Y_G^{\mathrm{sm}})$, then we have the commutative diagram below:

$$\begin{array}{ccccc}
\pi_1(Y_0^{\mathrm{sm}}, y) & \longrightarrow & \pi_1(Y_G^{\mathrm{sm}}, y_G) & \twoheadrightarrow & \pi_1(\overline{Y_G}^{\mathrm{sm}}, \zeta_G) \\
\downarrow & & & \searrow & \downarrow \\
\pi_1(Y^{\mathrm{sm}}, y) & \longrightarrow & \pi_1(S, y) & \longrightarrow & \pi_1(S_G, y_G) \\
 & & \downarrow & & \downarrow \\
 & & P & \longrightarrow & G
\end{array}$$

Here, the morphism on the left and the right morphism on the top are surjective since $\mathrm{codim}_{Y^{\mathrm{sm}}}(Y^{\mathrm{sm}} - Y_0^{\mathrm{sm}}) \geqslant 1$ and $\mathrm{codim}_{\overline{Y_G}^{\mathrm{sm}}}(\overline{Y_G}^{\mathrm{sm}} - Y_G^{\mathrm{sm}}) \geqslant 1$ ([31, 2.10.1]). Now [31, 2.10.2] shows that the image of $\pi_1(Y_0^{\mathrm{sm}}, y)$ is of finite index in $\pi_1(Y_G^{\mathrm{sm}}, y_G)$, so the neutral components of the Zariski closures of $\pi_1(Y^{\mathrm{sm}}, y)$ and $\pi_1(\overline{Y_G}^{\mathrm{sm}}, y_G)$ in G coincide. Hence we are done. □

Let $\widetilde{Z}$ be the closure (w.r.t. archimedean topology) of $\widetilde{Y}_G$ in $\mathcal{X}_G^+$, then $\widetilde{Z}$ is a complex analytic irreducible component of $\mathrm{unif}_G^{-1}(\overline{Y_G})$. For the pure connected Shimura datum $(G^{\mathrm{ad}}, \mathcal{X}_G^+)$, we have a decomposition ([39, 3.6])

$$(G^{\mathrm{ad}}, \mathcal{X}_G^+) = (G_N^{\mathrm{ad}}, \mathcal{X}_{G,1}^+) \times (G_2, \mathcal{X}_{G,2}^+).$$

By [39, 3.6, 3.7] and Lemma 2.3.2, $\widetilde{Z} \subset \mathcal{X}_{G,1}^+ \times \{\widetilde{y_{G,2}}\}$, i.e. $\widetilde{Z} \subset G_N(\mathbb{R})^+\widetilde{x}_G$ for some $\widetilde{x}_G \in \mathcal{X}_G^+$. But $\widetilde{y}_{0,G} \in \widetilde{Y}_G \subset \widetilde{Z}$, so $\widetilde{F}_G = G_N(\mathbb{R})^+\widetilde{y}_{0,G} \subset G_N(\mathbb{R})^+\widetilde{x}_G$. This implies that $\widetilde{F}_G = G_N(\mathbb{R})^+\widetilde{x}_G$. As a result, $\widetilde{Y}_G \subset \widetilde{Z} \subset \widetilde{F}_G$.

<u>Step II.</u> Consider the Shimura morphism

$$(P, \mathcal{X}^+) \overset{\rho}{\twoheadrightarrow} (P', \mathcal{X}^{+\prime}) := (P, \mathcal{X}^+)/N.$$

Then $\widetilde{F} = \rho^{-1}(\rho(\widetilde{F}))$ by definition of ρ. So in order to prove $\widetilde{Y} \subset \widetilde{F}$, it is enough to show that $\rho(\widetilde{Y}) \subset \rho(\widetilde{F})$. Hence we may replace $(P, \mathcal{X}^+)$ by $(P', \mathcal{X}^{+\prime})$. In other words, we may assume $N = \mathbf{1}$.

In this case $\widetilde{F}$ is just a point $\widetilde{x} \in \mathcal{X}^+$. Call $\widetilde{x}_{P/U} := \pi_{P/U}(\widetilde{x})$, $\widetilde{x}_G := \pi(\widetilde{x})$ and $x := \mathrm{unif}(\widetilde{x})$, $x_{P/U} := \mathrm{unif}_{P/U}(\widetilde{x}_{P/U})$, $x_G := \mathrm{unif}_G(\widetilde{x}_G)$. Then since $Y_G \subset F_G$, we have $Y \subset E$ where E is the fibre of $S \xrightarrow{[\pi]} S_G$ over x_G. Denote by A the fibre of $S_{P/U} \xrightarrow{[\pi]_G} S_G$ over x_G and T the fibre of $S \xrightarrow{[\pi_{P/U}]} S_{P/U}$ over $x_{P/U}$, then by [53, 3.13, 3.14] A is an abelian variety and T is an algebraic torus.

Step III. Prove that $\widetilde{Y}_{P/U} \subset \widetilde{F}_{P/U}$, i.e. $\widetilde{Y}_{P/U} = \{\widetilde{x}_{P/U}\}$.

By Step I, $Y_{P/U} \subset A$. We have the following morphisms

$$\pi_1(Y^{\mathrm{sm}}_{P/U}) \to \pi_1(A) \to \pi_1(S_{P/U}) = \Gamma_{P/U} \to P/U = V \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y^{\mathrm{sm}}_{P/U})$ (resp. $\pi_1(A)$) in $P/U = V \rtimes G$ is $\mathbf{1}$ (resp. V), so the image of

$$\pi_1(Y^{\mathrm{sm}}_{P/U}) \to \pi_1(A)$$

is a finite group.

Now $Y_{P/U}$ is irreducible since Y is irreducible. So by Proposition 2.1.4, $Y_{P/U} \subset A$ is a point. Equivalently, $\widetilde{Y}_{P/U}$ is a point. So $\widetilde{Y}_{P/U} \subset \widetilde{F}_{P/U}$ since $\widetilde{Y}_{P/U} \cap \widetilde{F}_{P/U} \neq \emptyset$ (both of them contain $\widetilde{y}_{0,P/U}$).

Step IV. Prove that $\widetilde{Y} \subset \widetilde{F}$, i.e. $\widetilde{Y} = \{\widetilde{x}\}$.

By Step I, $Y \subset E$. By Step III, $Y_{P/U} = \{x_{P/U}\}$. So $Y \subset T$. We have the following morphisms

$$\pi_1(Y^{\mathrm{sm}}) \to \pi_1(T) \to \pi_1(S) = \Gamma \to P = W \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y^{\mathrm{sm}})$ (resp. $\pi_1(T)$) in $P = W \rtimes G$ is $\mathbf{1}$ (resp. U), so the image of

$$\pi_1(Y^{\mathrm{sm}}) \to \pi_1(T)$$

is a finite group.

Now since Y is irreducible, by Proposition 2.1.4, $Y \subset T$ is a point. Equivalently, $\widetilde{Y}$ is a point. So $\widetilde{Y} \subset \widetilde{F}$ since $\widetilde{Y} \cap \widetilde{F} \neq \emptyset$ (both of them contain $\widetilde{y}_0$).

3. Since every weakly special subset of $\mathcal{X}^+$ is algebraic by Lemma 1.3.8, $\widetilde{F}$ is also the smallest weakly special subset which contains $\widetilde{Y}$. Therefore F is the smallest weakly special subvariety of S which contains Y.

□

Corollary 2.3.3. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ *be the uniformization map. Let Y be an irreducible subvariety of S, then Y is weakly special if and only if one (equivalently any) irreducible component of* $\mathrm{unif}^{-1}(Y)$ *is algebraic.*

If Y is weakly special, then $Y = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y})$ where N is the connected algebraic monodromy group associated with Y^{sm}, $U_N := U \cap N$ and $\widetilde{y}$ is any point of $\mathrm{unif}^{-1}(Y)$.

Proof. The "only if" part is immediate by Lemma 1.3.8. Now we prove the "if" part.

We first of all quickly show that if one irreducible component of $\mathrm{unif}^{-1}(Y)$ is algebraic, so are the others. The proof is the same as [65, first paragraph of the proof of Theorem 4.1]. Suppose that $\widetilde{Y}$ is an irreducible component of $\mathrm{unif}^{-1}(Y)$ which is algebraic, i.e. $\widetilde{Y}$ is an irreducible component of $\mathcal{X}^+ \cap Z$ for some algebraic subvariety Z of $\mathcal{X}^\vee$. Then for any $\gamma \in \Gamma \subset P(\mathbb{R})U(\mathbb{C})$,

$$\gamma\widetilde{Y} = \gamma(\mathcal{X}^+ \cap Z) \subset \mathcal{X}^+ \cap \gamma Z = \gamma\gamma^{-1}(\mathcal{X}^+ \cap \gamma Z) \subset \gamma\widetilde{Y}.$$

Hence it follows that $\gamma\widetilde{Y} = \mathcal{X}^+ \cap \gamma Z$ is algebraic.

Next under the notation of Theorem 2.3.1, $\widetilde{Y} = \overline{\widetilde{Y}} = \widetilde{F}$ since $\widetilde{Y}$ is algebraic. Hence $\widetilde{Y}$ is weakly special, and so is Y.

Finally if Y is weakly special, then for any $\widetilde{y} \in \mathrm{unif}^{-1}(Y)$ and $\widetilde{Y}$ the irreducible component of $\mathrm{unif}^{-1}(Y)$ which contains $\widetilde{y}$, $\widetilde{Y} = \widetilde{F} = N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y}$ by Theorem 2.3.1, and hence $Y = \mathrm{unif}(N(\mathbb{R})^+U_N(\mathbb{C})\widetilde{y})$. □