Author: Gao, Ziyang
Title: The mixed Ax-Lindemann theorem and its applications to the Zilber-Pink conjecture
Issue Date: 2014-11-24

Chapter 1

Preliminaries

1.1 Mixed Shimura varieties

1.1.1 Mixed Hodge structure

In this section we recall some background knowledge about rational mixed Hodge structures. Most of this section is taken from [53, Chapter 1].

1.1.1.1 Definitions about mixed Hodge structures

We start by collecting some basic notions about Hodge structures. This subsection is taken from [53, 1.1 and 1.2]. In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Let M be a free R-module of finite rank. A **pure Hodge structure of weight** $n \in \mathbb{Z}$ on M is a decomposition $M_{\mathbb{C}} = \oplus_{p+q=n} M^{p,q}$ into $\mathbb{C}$-vector spaces such that for all $p, q \in \mathbb{Z}$ with $p + q = n$ one has $\overline{M^{q,p}} = M^{p,q}$. The associated (descending) **Hodge filtration** on $M_{\mathbb{C}}$ is defined by $F^p M_{\mathbb{C}} := \oplus_{p' \geqslant p} M^{p',q}$. It determines the Hodge structure uniquely, because $M^{p,q} = F^p M_{\mathbb{C}} \cap \overline{F^q M_{\mathbb{C}}}$.

A **mixed** R**-Hodge structure** on M is a triple $(M, \{W_n M\}_{n\in\mathbb{Z}}, \{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}})$ consisting of an ascending exhausting separated filtration $\{W_n M\}_{n\in\mathbb{Z}}$ of M by R-modules of finite rank with each $M/W_n M$ free, called **weight filtration**, together with a descending exhausting separated filtration $\{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}}$ of $M_{\mathbb{C}}$, called **Hodge filtration**, such that for all $n \in \mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $\mathrm{Gr}_n^W M := W_n M/W_{n-1} M$. A pure Hodge structure of weight n is considered a special case of a mixed Hodge structure by defining the weight filtration as $W_{n'} M = M$ for $n' \geqslant n$ and $W_{n'} M = 0$ for $n' < n$.

The **Hodge numbers** are defined as $h^{p,q} := \dim_{\mathbb{C}}(\mathrm{Gr}_{p+q}^W M)^{p,q}$. They satisfy $h^{q,p} = h^{p,q}$, almost all $h^{p,q}$ are zero, and their sum is equal to the dimension of M. If $A \subset \mathbb{Z} \oplus \mathbb{Z}$ is an arbitrary subset, then we say that the Hodge structure $(M, \{W_n M\}_{n\in\mathbb{Z}}, \{F^p M_{\mathbb{C}}\}_{p\in\mathbb{Z}})$ is **of type** A, if $h^{p,q} = 0$ for all $(p, q) \notin A$. The weights that occur in a mixed Hodge structure are the numbers $p + q$ for all pairs (p, q), for which $h^{p,q} \neq 0$. The notions **of weight** $\leqslant n$ and **of weight** $\geqslant n$ are defined in the obvious way.

A **morphism of mixed** R**-Hodge structures** is a homomorphism $f : M \to M'$ such that $f(W_n M) \subset W_n M'$ and $f(F^p M_{\mathbb{C}}) \subset F^p M'_{\mathbb{C}}$ for all $n, p \in \mathbb{Z}$. The rational mixed Hodge structures form an abelian category with these morphisms. Given mixed R-Hodge structures on M_1 and M_2, there are canonical rational mixed Hodge structures on $M_1 \oplus M_2$, on the dual $M_1^\vee$ and on $\mathrm{Hom}(M_1, M_2)$.

A mixed Hodge structure on M **splits over** $\mathbb{R}$, if there exists a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$ such that $W_n M_{\mathbb{C}} = \oplus_{p+q \leqslant n} M^{p,q}$, $F^p M_{\mathbb{C}} = \oplus_{p' \geqslant p} M^{p',q}$ and $\overline{M^{q,p}} = M^{p,q}$. This decomposition is then uniquely determined by these properties. Every pure Hodge structure splits over $\mathbb{R}$, but not every mixed Hodge structure does. If one weakens the requirements, however, one can still associate to every mixed Hodge structure a canonical decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$, as in the following proposition.

Proposition 1.1.1 (Deligne)**.** *Fix a mixed R-Hodge structure on M.*

1. *There exists a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$ such that $W_n M_{\mathbb{C}} = \oplus_{p+q \leqslant n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \oplus_{p' \geqslant p} M^{p',q}$.*
2. *The Hodge structure is uniquely determined by any such decomposition.*
3. *There exists a unique decomposition as in (1) which also satisfies*

$$\overline{M^{q,p}} \equiv M^{p,q} \mod \oplus_{p'<p, q'<q} M^{p',q'}.$$

Proof. [53, 1.2]. □

1.1.1.2 Equivariant families of mixed Hodge structures

The reference for this subsection is [53, 1.3-1.7]. In this section, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. The torus $\mathbb{S}$ is called the **Deligne-torus**. Over $\mathbb{C}$ it is canonically isomorphic to $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, but the action of complex conjugation is twisted by the automorphism c that interchanges the two factors. In particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ corresponds to the points of the form $(z, \overline{z})$ with $z \in \mathbb{C}^*$. While the character group of $\mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{Z}$ in the standard way, we identify the character group of $\mathbb{S}$ with $\mathbb{Z} \oplus \mathbb{Z}$ such that the character (p, q) maps $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ to $z^{-p}\overline{z}^{-q} \in \mathbb{C}^*$. Under this identification the complex conjugation operates on $\mathbb{Z} \oplus \mathbb{Z}$ by interchanging the two factors. The following homomorphisms are important in the theory:

- the weight $\omega \colon \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ induced by $\mathbb{R}^* \subset \mathbb{C}^*$;
- $\mu \colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ sending $z \in \mathbb{C}^* \mapsto (z, 1) \in \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$;
- the norm $N \colon \mathbb{S} \twoheadrightarrow \mathbb{G}_{m,\mathbb{R}}$ sending $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^* \mapsto z\overline{z} \in \mathbb{R}^*$. The kernel $\mathbb{S}^1$ of N is anisotropic over $\mathbb{R}$, and we have a short exact sequence $1 \to \mathbb{S}^1 \to \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}} \to 1$.

Let M be a free R-module of finite rank. The choice of a representation $k \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ is equivalent to the choice of a decomposition $M_{\mathbb{C}} = \oplus_{p,q} M^{p,q}$, where $M^{p,q}$ is the eigenspace in $M_{\mathbb{C}}$ to the character (p, q). As in the last subsection we call $W_n M_{\mathbb{C}} = \oplus_{p+q \leqslant n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \oplus_{p' \geqslant p} M^{p',q}$ the associated weight filtration, respectively Hodge filtration, and define the notions "of type A", pure, etc. in the same way. These notions coincide with those of the last subsection, if the filtrations are those of a mixed R-Hodge structure on M. The following two propositions will tell us under which condition on k this is the case for $R = \mathbb{Q}$.

Proposition 1.1.2. *Let P be a connected $\mathbb{Q}$-linear algebraic group. Let $W := \mathcal{R}_u(P)$ be its unipotent radical, let $G := P/W$ and let $\pi\colon P \to G$ be the quotient map. Let $h\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ be a homomorphism such that the following conditions holds:*

- *$\pi \circ h\colon \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$ is defined over $\mathbb{R}$;*
- *$\pi \circ h \circ \omega\colon \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter of the center of G, which is defined over $\mathbb{Q}$;*
- *Under the weight filtration on $(\mathrm{Lie}\, P)_{\mathbb{C}}$ defined by $\mathrm{Ad}_P \circ h$ we have $W_{-1}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W$.*

Then

1. *For every ($\mathbb{Q}$-)representation $\rho\colon P \to \mathrm{GL}(M)$, the homomorphism $\rho \circ h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ induces a rational mixed Hodge structure on M.*
2. *The weight filtration on M is stable under P.*
3. *For any $p \in P(\mathbb{R})W(\mathbb{C})$, the assertions (1) and (2) also hold for $\mathrm{int}(p) \circ h$ in place of h. The weight filtration and the Hodge numbers in any representation are the same for $\mathrm{int}(p) \circ h$ and for h.*

Proof. [53, 1.4]. □

Proposition 1.1.3. *Let M be a finite dimensional $\mathbb{Q}$-vector space. A representation $k\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ defines a rational mixed Hodge structure on M iff there exist a connected $\mathbb{Q}$-linear algebraic group P, a representation $\rho\colon P \to \mathrm{GL}(M)$ and a homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ such that $k = \rho \circ h$ and the conditions in Proposition 1.1.2 are satisfied. Moreover, every rational mixed Hodge structure on M is obtained by a unique representation $k\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ with the property above.*

Proof. This is [53, 1.5] except the "Moreover" part, where the existence of k has been explained in the paragraph before Proposition 1.1.2 and the uniqueness of k follows from Proposition 1.1.1(3). □

Now we are ready to discuss equivariant families of Hodge structures, or more precisely homogeneous spaces parametrizing certain rational mixed Hodge structures.

Proposition 1.1.4. *Let P be a $\mathbb{Q}$-linear algebraic group and let $W := \mathcal{R}_u(P)$ be its unipotent radical. Let $\mathcal{X}_W$ be a $P(\mathbb{R})W(\mathbb{C})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. Assume that for one (and hence for all by Proposition 1.1.2(3)) $h \in \mathcal{X}_W$, the* *conditions in Proposition 1.1.2 holds. Let M be any faithful representation of P and let φ be the obvious map*

$$\mathcal{X}_W \to \{\textit{rational mixed Hodge structures on } M\}.$$

Then:

1. *There exists a unique structure on $\varphi(\mathcal{X}_W)$ as a complex manifold such that the Hodge filtration on $M_{\mathbb{C}}$ depends analytically on $\varphi(h) \in \varphi(\mathcal{X}_W)$. This structure is $P(\mathbb{R})W(\mathbb{C})$-invariant and $W(\mathbb{C})$ acts analytically on $\varphi(\mathcal{X}_W)$.*

2. *For any other representation M' of P the analogous map*

$$\varphi' \colon \mathcal{X}_W \to \{\textit{rational mixed Hodge structures on } M'\}$$

factors through $\varphi(\mathcal{X}_W)$. The Hodge filtration on M' varies analytically with $\varphi(h) \in \varphi(\mathcal{X}_W)$.

3. *If in addition M' is faithful, then $\varphi(\mathcal{X}_W)$ and $\varphi'(\mathcal{X}_W)$ are canonically isomorphic and the isomorphism is compatible with the complex structure.*

Proof. [53, 1.7]. □

1.1.1.3 Mumford-Tate group and polarizations

In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$. Also M will be a free R-module of finite rank equipped with a mixed R-Hodge structure $(M, \{W_nM\}_{n\in\mathbb{Z}}, \{F^pM_{\mathbb{C}}\}_{p\in\mathbb{Z}})$. By Proposition 1.1.3, the corresponding rational mixed Hodge structure on $M_{\mathbb{Q}}$ gives rises to a representation $k \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$.

Definition 1.1.5. *The **Mumford-Tate group** of this mixed R-Hodge structure is defined to be the smallest $\mathbb{Q}$-subgroup P of $\mathrm{GL}(M_{\mathbb{Q}})$ such that $k(\mathbb{S}_{\mathbb{C}}) \subset P_{\mathbb{C}}$.*

Before defining the polarizations of pure Hodge structures, we introduce the **Tate Hodge structure**, which is defined to be the free R-module of rank 1 $R(1) := 2\pi\sqrt{-1}R$ with the pure R-Hodge structure of type $(-1,-1)$. For every $n \in \mathbb{Z}$, we get a pure R-Hodge structure of type $(-n,-n)$ on $R(n) := R(1)^{\otimes n}$.

Definition 1.1.6. *Suppose that the R-Hodge structure on M is pure of weight n. A **polarization** of this Hodge structure is a homomorphism of Hodge structures*

$$Q \colon M \otimes M \to R(-n)$$

which is $(-1)^n$-symmetric and such that the real-valued symmetric bilinear form

$$Q'(u,v) := (2\pi\sqrt{-1})^n Q(Cu, v)$$

is positive-definite on $M_{\mathbb{R}}$, where C acts on $M^{p,q}$ by $C|_{M^{p,q}} = (\sqrt{-1})^{p-q}$.

1.1.1.4 Variation of mixed Hodge structures

The reference for this subsection is [53, 1.9-1.13]. In this subsection, $R = \mathbb{Z}$ or $\mathbb{Q}$.

Definition 1.1.7. *([45, Definition 14.44]) Let S be a complex manifold. A* ***variation of mixed R-Hodge structures over*** *S is a triple $(\mathbb{V}, W_{\cdot}, \mathcal{F}^{\cdot})$ with*

1. *a local system $\mathbb{V}$ of free R-modules of finite rank on S;*
2. *a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}$ by local subsystems with torsion free $\mathrm{Gr}_n^W \mathbb{V}$ for each n (this is called the weight filtration);*
3. *a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{R_S} \mathcal{O}_S$, where R_S is the constant sheaf over S, by holomorphic subbundles (this is called the Hodge filtration).*

such that

1. *for each $s \in S$, the filtrations $\{\mathcal{F}^p(s)\}$ and $\{W_m\}$ of $\mathbb{V}(s) \simeq \mathbb{V}_s \otimes_R \mathbb{C}$ define a mixed Hodge structure on the R-module of finite rank $\mathbb{V}_s$;*
2. *the connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S^1$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$ satisfies the Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_S^1.$$

Definition 1.1.8. *A variation of mixed Hodge structures over S is said to be* ***graded-polarizable*** *if the induced variations of pure Hodge structure $\mathrm{Gr}_n^W \mathbb{V}$ are all polarizable, i.e. for each n, there exists a flat morphism of variations*

$$Q_n \colon \mathrm{Gr}_n^W \mathbb{V} \otimes \mathrm{Gr}_n^W \mathbb{V} \to R(-n)_S$$

which induces on each fibre a polarization of the corresponding Hodge structure of weight n.

Proposition 1.1.9. *Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. Then we have a variation of rational mixed Hodge structures on M over $\varphi(\mathcal{X}_W)$ iff for one (and hence for all) $h \in \mathcal{X}_W$ the Hodge structure on $\mathrm{Lie}\, P$ is of type*

$$\{(-1,1),(0,0),(1,-1),(-1,0),(0,-1),(-1,-1)\}.$$

Proof. [53, 1.10]. □

Proposition 1.1.10. *Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. Assume*

- *for one (and hence all) $h \in \mathcal{X}_W$, the conjugation by $h \circ \pi(\sqrt{-1})$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$ where $G := P/W$ and G^{ad} possesses no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact;*
- *$P/P^{\mathrm{der}} = Z(G)$ is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$;*
- *M is an irreducible representation of P and the Hodge structure on M induced by one (and hence all) $h \in \mathcal{X}_W$ is pure of weight n.*

Then there exist a one dimensional representation of P on $\mathbb{Q}(-n)$ and a P-equivariant bilinear form $\Psi\colon M \times M \to \mathbb{Q}(-n)$ such that for all $h \in \mathcal{X}_W$ either Ψ or $-\Psi$ is a polarization of the corresponding Hodge structure on M.

Proof. [53, 1.12 and 1.13]. □

1.1.1.5 Replace $\mathcal{X}_W$ by a smaller orbit

The reference for this subsection is [53, 1.15 and 1.16].

Let P, $\mathcal{X}_W$, M and φ be as in Proposition 1.1.4. The aim of this subsection is to find a subgroup U of W such that the image of an orbit under $P(\mathbb{R})U(\mathbb{C})$ under φ is the same as $\varphi(\mathcal{X}_W)$.

Let $U < W$ be the unique connected subgroup such that $\mathrm{Lie}\, U = W_{-2}(\mathrm{Lie}\, W)$. By Proposition 1.1.2(3), it does not depend on $h \in \mathcal{X}_W$. Let π' be the quotient $P \to P/U$.

Proposition 1.1.11. *Under the notation as above. Let*

$$\mathcal{X} := \{h \in \mathcal{X}_W |\ \pi' \circ h\colon \mathbb{S}_{\mathbb{C}} \to (P/U)_{\mathbb{C}} \text{ is defined over } \mathbb{R}\}.$$

Then

1. *$\mathcal{X}$ is a non-empty $P(\mathbb{R})U(\mathbb{C})$-orbit in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$;*
2. *$\varphi(\mathcal{X}) = \varphi(\mathcal{X}_W)$;*
3. *If $F^0(\mathrm{Lie}\, U)_{\mathbb{C}} = 0$, then $\varphi(\mathcal{X}) \simeq \mathcal{X}$.*

Proof. [53, 1.16]. □

1.1.2 Mixed Shimura data and mixed Shimura varieties

1.1.2.1 Definitions and basic properties

Definition 1.1.12. *A **mixed Shimura datum** $(P, \mathcal{X})$ is a pair where*

- *P is a connected linear algebraic group over $\mathbb{Q}$ with unipotent radical W and with another algebraic subgroup $U \subset W$ which is normal in P and uniquely determined by $\mathcal{X}$ using condition (3) below;*

- *$\mathcal{X}$ is a left homogeneous space under the subgroup $P(\mathbb{R})U(\mathbb{C}) \subset P(\mathbb{C})$, and $\mathcal{X} \xrightarrow{h} \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is a $P(\mathbb{R})U(\mathbb{C})$-equivariant map such that every fibre of h consists of at most finitely many points,*

such that for some (equivalently for all) $x \in \mathcal{X}$,

1. *the composite homomorphism $\mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$,*
2. *the adjoint representation induces on $\mathrm{Lie}\,P$ a rational mixed Hodge structure of type*
$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\},$$
3. *the weight filtration on $\mathrm{Lie}\,P$ is given by*
$$W_n(\mathrm{Lie}\,P) = \begin{cases} 0 & \text{if } n < -2 \\ \mathrm{Lie}\,U & \text{if } n = -2 \\ \mathrm{Lie}\,W & \text{if } n = -1 \\ \mathrm{Lie}\,P & \text{if } n \geqslant 0 \end{cases},$$
4. *the conjugation by $h_x(\sqrt{-1})$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$ where $G := P/W$, and G^{ad} possesses no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact,*
5. *$P/P^{\mathrm{der}} = Z(G)$ is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.*

*If in addition P is reductive (resp. U is trivial), then $(P,\mathcal{X})$ is called a **pure Shimura datum** (resp. a **mixed Shimura datum of Kuga type**).*

Remark 1.1.13. *1. Let $\omega : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ be $t \in \mathbb{R}^* \mapsto t \in \mathbb{C}^*$. Conditions (2) and (3) together imply that the composite homomorphism $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\omega} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is a co-character of the center of P/W defined over $\mathbb{R}$. This map is called the weight. Furthermore, condition (5) implies that the weight is defined over $\mathbb{Q}$.*

2. *Condition (5) also implies that every sufficiently small congruence subgroup Γ of $P(\mathbb{Q})$ is contained in $P^{\mathrm{der}}(\mathbb{Q})$ (cf [53, the proof of 3.3(a)]). Fix a Levi decomposition $P = W \rtimes G$ ([55, Theorem 2.3]), then $P^{\mathrm{der}} = W \rtimes G^{\mathrm{der}}$, and hence for any congruence subgroup $\Gamma < P^{\mathrm{der}}(\mathbb{Q})$, Γ is Zariski dense in P^{der} by condition (4) ([55, Theorem 4.10]).*
3. *Condition (5) in the definition is not too strict because we are only interested in a connected component of $\mathcal{X}$ ([53, 1.29]).*

Theorem 1.1.14. *Let $(P,\mathcal{X})$ be a mixed Shimura datum. Then $\mathcal{X}$ possesses a canonical $P(\mathbb{R})U(\mathbb{C})$-invariant complex structure and every connected component of $\mathcal{X}$ is isomorphic to a holomorphic vector bundle on a hermitian symmetric domain.*

Proof. The existence of the complex structure follows from Proposition 1.1.4 and Proposition 1.1.11. We will give the construction of this complex structure at the beginning of §1.3.1.

The second claim is [53, 2.19]. □

Definition 1.1.15. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let K be an open compact subgroup of $P(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adèle of $\mathbb{Q}$. The corresponding **mixed Shimura variety** is defined as*

$$M_K(P, \mathcal{X}) := P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K,$$

*where $P(\mathbb{Q})$ acts diagonally on both factors on the left and K acts on $P(\mathbb{A}_f)$ on the right. The mixed Shimura variety $M_K(P, \mathcal{X})$ is said to be **pure (resp. of Kuga type)** if $(P, \mathcal{X})$ is pure (resp. of Kuga type).*

In this article, we only consider connected mixed Shimura data and connected mixed Shimura varieties except in §4.2.

Definition 1.1.16.

1. *A **connected mixed Shimura datum** is a pair $(P, \mathcal{X}^+)$, where P is as in Definition 1.1.12, $\mathcal{X}^+ \stackrel{h}{\hookrightarrow} \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$ is an orbit under the subgroup $P(\mathbb{R})^+U(\mathbb{C}) \subset P(\mathbb{C})$ such that for one (and hence for all) $x \in \mathcal{X}^+$ the conditions (1)-(5) in Definition 1.1.12 are satisfied.*

2. *A **connected mixed Shimura variety** S associated with $(P, \mathcal{X}^+)$ is of the form $\Gamma\backslash\mathcal{X}^+$ for some congruence subgroup $\Gamma \subset P(\mathbb{Q})_+ := P(\mathbb{Q}) \cap P(\mathbb{R})_+$, where $P(\mathbb{R})_+$ is the stabilizer in $P(\mathbb{R})$ of $\mathcal{X}^+ \subset \mathrm{Hom}_\mathbb{C}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$.*

Mixed Shimura varieties and connected mixed Shimura varieties are closely related. Their relationship is summarized in the following proposition.

Proposition 1.1.17. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let K be an open compact subgroup of $P(\mathbb{A}_f)$. Let $\mathcal{X}^+$ be a connected component of $\mathcal{X}$.*

1. *The pair $(P, \mathcal{X}^+)$ is a connected mixed Shimura datum.*

2. *The set $P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K$ is a finite set.*

3. *For any $p_f \in P(\mathbb{A}_f)$, $\Gamma(p_f) := P(\mathbb{Q})_+ \cap p_f K p_f^{-1}$ is a congruence subgroup of $P(\mathbb{Q})_+$ depending only on $[p_f] \in P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K$ and K.*

4. $$M_K(P, \mathcal{X}) = \coprod_{[p_f] \in P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K} \Gamma(p_f)\backslash\mathcal{X}^+.$$

Proof. [53, 3.2] and [55, Theorem 8.1]. □

This proposition allows us to consider only connected mixed Shimura data and connected mixed Shimura varieties in this dissertation. One advantage of doing this is because of the notion which we introduce now: recall the following definition, which Pink calls "irreducible" in [53, 2.13].

Definition 1.1.18. *A connected mixed Shimura datum $(P, \mathcal{X}^+)$ is said to **have generic Mumford-Tate group** if P possesses no proper normal subgroup P' such that for one (equivalently all) $x \in \mathcal{X}^+$, h_x factors through $P'_{\mathbb{C}} \subset P_{\mathbb{C}}$. We shall denote this case by $P = \mathrm{MT}(\mathcal{X}^+)$. (This terminology will be explained in Remark 2.2.6).*

Proposition 1.1.19. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum, then*

1. *there exists a connected mixed Shimura datum $(P', \mathcal{X}'^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $P' = \mathrm{MT}(\mathcal{X}'^+)$ and $\mathcal{X}'^+ = \mathcal{X}^+$;*
2. *if $(P, \mathcal{X}^+)$ has generic Mumford-Tate group, then P acts on U via a character. In particular, any subgroup of U is normal in P.*

Proof. [53, 2.13, 2.14]. □

Definition 1.1.20. *A **(Shimura) morphism of connected mixed Shimura data** $(Q, \mathcal{Y}^+) \to (P, \mathcal{X}^+)$ is a homomorphism $\varphi \colon Q \to P$ of algebraic groups over $\mathbb{Q}$ which induces a map $\mathcal{Y}^+ \to \mathcal{X}^+$, $y \mapsto \varphi \circ y$. A **Shimura morphism of connected mixed Shimura varieties** is a morphism of varieties induced by a Shimura morphism of connected mixed Shimura data.*

A very important result of the theory of Shimura varieties is that the category of connected mixed Shimura varieties is a subcategory of the category of algebraic varieties. More precisely,

Theorem 1.1.21.
1. *Let S be a connected mixed Shimura variety associated with $(P, \mathcal{X}^+)$ and let $\mathrm{unif} \colon \mathcal{X}^+ \to S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Then there is a canonical structure of a normal complex quasi-projective algebraic variety on S (the complex structure comes from the $P(\mathbb{R})^+U(\mathbb{C})$-invariant complex structure of $\mathcal{X}^+$ given in Theorem 1.1.14). Moreover if Γ is neat, then S is smooth.*
2. *Every Shimura morphism between connected mixed Shimura varieties is algebraic.*

Proof. [53, 3.3 and 9.24]. □

1.1.2.2 Construction of new mixed Shimura data from a given one

Given a (connected) mixed Shimura datum $(P, \mathcal{X})$, we define in this section its quotient mixed Shimura data and its unipotent extensions.

Proposition 1.1.22 (Quotient mixed Shimura datum)**.** *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let P_0 be a normal subgroup of P. Then there exist a quotient mixed Shimura datum $(P, \mathcal{X})/P_0$ and a morphism $(P, \mathcal{X}) \to (P, \mathcal{X})/P_0$, unique up to isomorphism, such that every Shimura morphism $(P, \mathcal{X}) \to (P', \mathcal{X}')$, where the homomorphism $P \to P'$ factors through P/P_0, factors in a unique way through $(P, \mathcal{X})/P_0$. In fact the underlying group for $(P, \mathcal{X})/P_0$ is P/P_0.*

Proof. This is [53, 2.9] except the "In fact" part, which is clear by the proof. ☐

Proposition 1.1.23 (Unipotent extension of a mixed Shimura datum)**.** *Let $(P, \mathcal{X})$ be a mixed Shimura datum and let $1 \to W_0 \to P_1 \to P \to 1$ be an extension of P by a unipotent group W_0. Let $G := P/\mathcal{R}_u(P)$. Assume that the Lie algebra of every irreducible subquotient of* $\mathrm{Lie}\, W_0$ *is of Hodge type $\{(-1,0),(0,-1),(-1,-1)\}$ as representation of G, and that the center of G acts on it through a torus that is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$. Then:*

1. *There exist a mixed Shimura datum $(P_1, \mathcal{X}_1)$ and a morphism $(P_1, \mathcal{X}_1) \to (P, \mathcal{X})$ that extends the given homomorphism $P_1 \to P$, with the property $(P_1, \mathcal{X}_1)/W_0 \simeq (P, \mathcal{X})$. They are uniquely determined up to isomorphism.*
2. *For every morphism $(P', \mathcal{X}') \to (P, \mathcal{X})$ and every factorization $P' \to P_1 \to P$, there exists exactly one extension $(P', \mathcal{X}') \to (P_1, \mathcal{X}_1) \to (P, \mathcal{X})$.*

Proof. This is [53, 2.17]. ☐

Example 1.1.24. *Let us see a particular example of the unipotent extensions of a given connected mixed Shimura datum. This is [54, Construction 2.9].*

Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum and let V' be a finite dimensional representation of P. Then we can define the $\mathbb{Q}$-linear algebraic group $V' \rtimes P$. Assume that for one (and hence for all) $x \in \mathcal{X}^+$, the induced rational mixed Hodge structure on V' has type $\{(-1,0),(0,-1)\}$. Let

$$V'(\mathbb{R}) \rtimes \mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, (V' \rtimes P)_\mathbb{C})$$

denote the conjugacy class under $V'(\mathbb{R}) \rtimes (P(\mathbb{R})^+ U(\mathbb{C})) = (V' \rtimes P)(\mathbb{R})^+ U(\mathbb{C})$ generated by $\mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$. There is a natural bijection

$$V'(\mathbb{R}) \times \mathcal{X}^+ \xrightarrow{\sim} V'(\mathbb{R}) \rtimes \mathcal{X}^+, \qquad (v', x) \mapsto \mathrm{int}(v') \circ x.$$

Under this bejection the action of $(v,p) \in V'(\mathbb{R}) \rtimes (P(\mathbb{R})^+ U(\mathbb{C}))$ corresponds to the twisted action $(v,p) \cdot (v', x) = (pv' + v, px)$. The complex structure of the fiber over $x \in \mathcal{X}^+$ of the projection

$$V'(\mathbb{R}) \rtimes \mathcal{X}^+ \to \mathcal{X}^+$$

is given by $V'(\mathbb{R}) \simeq V'(\mathbb{C})/F_x^0 V'(\mathbb{C})$.

The pair $(V' \rtimes P, V'(\mathbb{R}) \rtimes \mathcal{X}^+)$ is the extension of $(P, \mathcal{X}^+)$ by V'.

Notation 1.1.25. *For convenience, we fix some notation here. Given a connected mixed Shimura datum $(P, \mathcal{X}^+)$, we always denote by $W = \mathcal{R}_u(P)$ the unipotent radical of P, $G := P/W$ the reductive part, $U \lhd P$ the weight -2 part, $V := W/U$ the weight -1 part and $(P/U, \mathcal{X}^+_{P/U}) := (P, \mathcal{X}^+)/U$ (resp. $(G, \mathcal{X}^+_G) := (P, \mathcal{X}^+)/W$) the corresponding connected mixed Shimura datum* *whose weight* -2 *part is trivial (resp. pure Shimura datum). If we have several connected mixed Shimura data, say* $(P, \mathcal{X}^+)$ *and* $(Q, \mathcal{Y}^+)$*, then we distinguish the different parts associated with them by adding subscript* W_P, W_Q, G_P, G_Q, *etc. For a connected mixed Shimura variety* S*, we denote by* $S_{P/U}$ *(resp.* S_G*) its image under the Shimura morphism induced by* $(P, \mathcal{X}^+) \to (P/U, \mathcal{X}^+_{P/U})$ *(resp.* $(P, \mathcal{X}^+) \to (G, \mathcal{X}^+_G)$*). The pure Shimura datum* $(G, \mathcal{X}^+_G)$ *will be called the* ***pure part of*** $(P, \mathcal{X}^+)$ *and* S_G *will be called the* ***pure part of*** S.

1.1.2.3 Examples of Shimura morphisms

In this subsection, we discuss some Shimura morphisms. The first corresponds to families of abelian varieties. Then we define Shimura immersions, Shimura submersions and Shimura coverings.

Proposition 1.1.26. *Let* $S = \Gamma\backslash\mathcal{X}^+$ *be a connected mixed Shimura variety of Kuga type associated with* $(P, \mathcal{X}^+)$ *and let* S_G *be its pure part. Assume that* $\Gamma = \Gamma_V \rtimes \Gamma_G$ *and that* Γ_G *is neat. Then* $S \to S_G$ *is an abelian scheme.*

Proof. [53, 3.12(a) and 3.22(a)]. □

Proposition 1.1.27. *Let* $\varphi\colon (P, \mathcal{X}^+) \to (P', \mathcal{X}'^+)$ *be a Shimura morphism and let* $\Gamma \subset P(\mathbb{Q})_+$ *and* $\Gamma' \subset P'(\mathbb{Q})_+$ *be congruence subgroups such that* $\varphi(\Gamma) \subset \Gamma'$*. Then the map*

$$[\varphi]\colon \Gamma\backslash\mathcal{X}^+ \to \Gamma'\backslash\mathcal{X}'^+, \qquad [x] \mapsto [\varphi \circ x]$$

is well-defined and algebraic. Moreover, $[\varphi]$ *is*

1. *a finite morphism if* $\mathrm{Ker}(\varphi)^\circ$ *is a torus. In this case* $[\varphi]$ *is called a* ***Shimura immersion****.*

2. *surjective if Im*(φ) *contains* P'^{der}*. In this case* $[\varphi]$ *is called a* ***Shimura submersion****.*

3. *a (possibly ramified) covering if the conditions in (1) and (2) both hold. In this case* $[\varphi]$ *is called a* ***Shimura covering****.*

Proof. [53, 3.4 and 9.24]. □

At the end of this subsection, we state the following property for Shimura morphisms.

Proposition 1.1.28. *Let* $(Q, \mathcal{Y}) \xrightarrow{f} (P, \mathcal{X})$ *be a Shimura morphism, then* $f(W_Q) \subset W_P$ *(resp.* $f(U_Q) \subset f(U_P)$*), and hence* f *induces*

$$\overline{f} : (G_Q, \mathcal{Y}_{G_Q}) \to (G_P, \mathcal{X}_{G_P}) \qquad \textit{(resp. } \overline{f}' : (Q/Q_U, \mathcal{Y}_{Q/U_Q}) \to (P/U_P, \mathcal{X}_{P/U_P})\textit{)}.$$

Furthermore, if the underlying homomorphism of algebraic groups f *is injective, then so are* $\overline{f}$ *and* $\overline{f}'$.

Proof. Since

$$\operatorname{Lie} W_P = W_{-1}(\operatorname{Lie} P) \qquad \text{and} \qquad \operatorname{Lie} W_Q = W_{-1}(\operatorname{Lie} Q),$$

by the following commutative diagram

$$\begin{array}{ccc} \operatorname{Lie} W_Q & \longrightarrow & \operatorname{Lie} W_P \\ \wr\downarrow \exp & & \downarrow \exp \\ W_Q & \xrightarrow{f} & P \end{array}$$

(here exp is algebraic and is an isomorphism as a morphism between algebraic varieties because W_Q is unipotent), $f(W_Q) \subset W_P$.

Hence f induces a map $G_Q \to G_P$. Now the existence of $\overline{f}$ follows from the universal property of the quotient Shimura datum (Proposition 1.1.22).

Furthermore, suppose now that f is injective. By Levi decomposition, the exact sequence

$$1 \to W_Q \to Q \xrightarrow{\pi_Q} G_Q \to 1$$

splits. Choose a splitting $s_Q \colon G_Q \to Q$, then we have the following diagram whose solid arrows commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_Q & \longrightarrow & Q & \underset{\pi_Q}{\overset{s_Q}{\rightleftarrows}} & G_Q & \longrightarrow & 1 \\ & & \downarrow & & f\downarrow & \swarrow_{\lambda} & \downarrow \overline{f} & & \\ 1 & \longrightarrow & W_P & \longrightarrow & P & \xrightarrow[\pi_P]{} & G_P & \longrightarrow & 1 \end{array},$$

where $\lambda := f \circ s_Q$. Then λ is injective since f, s_Q are. And $\pi_P \circ \lambda = \pi_P \circ f \circ s_Q = \overline{f} \circ \pi_Q \circ s_Q = \overline{f}$, so we have

$$\operatorname{Ker}(\overline{f}) = G_Q \cap W_P$$

where the intersection is taken in P. $(G_Q \cap W_P)^\circ$ is smooth (since we are in the characteristic 0), connected unipotent (since it is in W_P) and normal in G_Q (since W_P is normal in P), so it is trivial since G_Q is reductive. So $G_Q \cap W_P$ is finite, hence trivial because W_P is unipotent over $\mathbb{Q}$. To sum it up, $\overline{f}$ is injective.

The proof for the statements with U's is similar. □

1.1.2.4 Generalized Hecke orbits

The reference for this subsection is [54, Section 3]. Let $S = \Gamma \backslash \mathcal{X}^+$ be a connected mixed Shimura variety associated with $(P, \mathcal{X}^+)$ and let $\operatorname{unif} \colon \mathcal{X}^+ \to S$ be the uniformization.

Definition 1.1.29. *1. For any $\varphi \in \operatorname{Aut}\big((P, \mathcal{X}^+)\big)$, the diagram of Shimura coverings*

$$S = \Gamma \backslash \mathcal{X}^+ \xleftarrow{[\mathrm{id}]} \big(\Gamma \cap \varphi^{-1}(\Gamma)\big) \backslash \mathcal{X}^+ \xrightarrow{[\varphi]} \Gamma \backslash \mathcal{X}^+ = S$$

is called a ***generalized Hecke correspondence*** *on* S *and is denoted by* T_φ. *For any subset* $Z \subset S$, *the subset*

$$T_\varphi(Z) := [\varphi]\left([\mathrm{id}]^{-1}(Z)\right)$$

is called the ***translate of*** Z ***under*** T_φ. *We also abbreviate* $T_\varphi(s) := T_\varphi(\{s\})$.

2. *The generalized Hecke correspondence associated with an inner automorphism* $\mathrm{int}(p)\colon p' \mapsto pp'p^{-1}$ *for an element* $p \in P(\mathbb{Q})_+$ *is called a* ***(usual) Hecke correspondence*** *on* S *and is denoted by* T_p.

Definition 1.1.30. *Fix a point* $s \in S$.

1. *The union of* $T_\varphi(s)$ *for all* $\varphi \in \mathrm{Aut}\left((P, \mathcal{X}^+)\right)$ *is called the* ***generalized Hecke orbit of*** s.

2. *The union of* $T_p(s)$ *for all* $p \in P(\mathbb{Q})_+$ *is called the* ***(usual) Hecke orbit of*** s.

The following proposition, whose proof we omit, is very easy to check by definition.

Proposition 1.1.31. *Let* s *be a point of* S. *Let* $\widetilde{s} \in \mathcal{X}^+$ *be such that* $\mathrm{unif}(\widetilde{s}) = s$. *Then the generalized Hecke orbit of* s *equals*

$$\mathrm{unif}\Big(\mathrm{Aut}\left((P, \mathcal{X}^+)\right) \cdot \widetilde{s}\Big).$$

The generalized Hecke orbits in a particular connected mixed Shimura variety (the universal family of principally polarized abelian varieties) will be computed in the last chapter of this dissertation (5.1.1).

1.1.2.5 Structure of the underlying group

The reference for this subsection is [53, 2.15].

For a given connected mixed Shimura datum $(P, \mathcal{X}^+)$, we can associate to P a 4-tuple (G, V, U, Ψ) which is defined as follows:

- $G := P/\mathcal{R}_u(P)$ is the reductive part of P;

- U is the normal subgroup of P as in Definition 1.1.12 and $V := \mathcal{R}_u(P)/U$. Both of them are vector groups with an action of G induced by conjugation in P (which factors through G for reason of weight);

- The commutator on $W := \mathcal{R}_u(P)$ induces a G-equivariant alternating form $\Psi\colon V \times V \to U$ by reason of weight as explained by Pink in [53, 2.15]. Moreover, Ψ is given by a polynomial with coefficients in $\mathbb{Q}$.

On the other hand, P is uniquely determined up to isomorphism by this 4-tuple in the following sense:

- let W be the central extension of V by U defined by Ψ. More concretely, $W = U \times V$ as a $\mathbb{Q}$-variety and the group law on W is (this can be proved using the Baker-Campbell-Hausdorff formula)

$$(u, v)(u', v') = (u + u' + \frac{1}{2}\Psi(v, v'), v + v');$$

- define the action of G on W by $g((u, v)) := (gu, gv)$;
- define $P := W \rtimes G$.

1.1.3 Mixed Shimura varieties of Siegel type and the reduction lemma

The reference for this subsection is [53, 2.7, 2.25, 10.1-10.14].

Let $g \in \mathbb{N}_{>0}$. Let V_{2g} be a $\mathbb{Q}$-vector space of dimension $2g$ and let

$$\Psi\colon V_{2g} \times V_{2g} \to U_{2g} := \mathbb{G}_{a,\mathbb{Q}}$$

be a non-degenerate alternating form. Define

$$\mathrm{GSp}_{2g} := \{h \in \mathrm{GL}(V_{2g}) | \Psi(hv, hv') = \nu(h)\Psi(v, v') \text{ with } \nu(h) \in \mathbb{G}_m\},$$

and $\mathbb{H}_g$ the set of all homomorphisms

$$\mathbb{S} \to \mathrm{GSp}_{2g,\mathbb{R}}$$

which induce a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on V_{2g} and for which either Ψ or $-\Psi$ defines a polarization. Let $\mathbb{H}_g^+$ be the set of all such homomorphisms such that Ψ defines a polarization.

GSp_{2g} acts on U_{2g} by the scalar ν, which induces a pure Hodge structure of type $(-1, -1)$ on U_{2g}. Let W_{2g} be the central extension of V_{2g} by U_{2g} defined by Ψ, then the action of GSp_{2g} on W_{2g} induces a Hodge structure of type $\{(-1, 0), (0, -1), (-1, -1)\}$ on $\mathrm{Lie}\, W_{2g}$.

By Proposition 1.1.23, there are connected mixed Shimura data $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$ and $(P_{2g}, \mathcal{X}_{2g}^+)$, where $P_{2g,\mathrm{a}} := V_{2g} \rtimes \mathrm{GSp}_{2g}$ and $P_{2g} := W_{2g} \rtimes \mathrm{GSp}_{2g}$.

Definition 1.1.32. *The connected mixed Shimura data* $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$, $(P_{2g,\mathrm{a}}, \mathcal{X}_{2g,\mathrm{a}}^+)$ *and* $(P_{2g}, \mathcal{X}_{2g}^+)$ *are called* ***of Siegel type*** *(of genus g).*

Next we introduce connected mixed Shimura varieties of Siegel type. They have very good modular interpretation ([53, 10.8-10.14]).

For $M \geqslant 4$ and even, define

$$\Gamma_{\mathrm{GSp}}(M) := \{h \in \mathrm{GSp}_{2g}(\mathbb{Z}) | h \equiv 1 \bmod M\} \tag{1.1.1}$$

and

$$\Gamma_W(M) := (M \cdot U_{2g}(\mathbb{Z})) \times (M \cdot V_{2g}(\mathbb{Z}))$$

under the identification $W \simeq U \times V$ in §1.1.2.5. $\Gamma_W(M)$ is indeed a subgroup of $W(\mathbb{Z})$ by the group operation (defined by Ψ). Let $\Gamma_V(M) := M \cdot V_{2g}(\mathbb{Z})$, and write

$$\mathcal{A}_g(M) := \Gamma_{\mathrm{GSp}}(M)\backslash\mathbb{H}_g^+ \tag{1.1.2}$$

$$\mathfrak{A}_g(M) := (\Gamma_V(M) \rtimes \Gamma_{\mathrm{GSp}}(M))\backslash\mathcal{X}_{2g,\mathrm{a}}^+ \tag{1.1.3}$$

$$\mathfrak{L}_g(M) := (\Gamma_W(M) \rtimes \Gamma_{\mathrm{GSp}}(M))\backslash\mathcal{X}_{2g}^+, \tag{1.1.4}$$

Definition 1.1.33. *The connected mixed Shimura varieties $\mathcal{A}_g(M)$, $\mathfrak{A}_g(M)$ and $\mathfrak{L}_g(M)$ are called of Siegel type of level M (and of genus g).*

Connected mixed Shimura varieties of Siegel type have very good moduli interpretation:

Theorem 1.1.34.

1. *$\mathfrak{A}_g(M)$ is the universal family of principally polarized abelian varieties of dimension g with a level-M-structure over the fine moduli space $\mathcal{A}_g(M)$.*
2. *$\mathfrak{L}_g(M) \to \mathfrak{A}_g(M)$ is a $\mathbb{G}_m$-torsor which is totally symmetric. Its inverse $\mathbb{G}_m$-torsor, i.e. replace the $\mathbb{G}_m$-action by its inverse, is relatively ample w.r.t. $\mathfrak{A}_g(M) \to \mathcal{A}_g(M)$. From now on, we replace the $\mathbb{G}_m$-torsor $\mathfrak{L}_g(M) \to \mathfrak{A}_g(M)$ by its inverse, but hence as a variety the "new" $\mathfrak{L}_g(M)$ is still equal to the "old" one.*
3. *Any point $a \in \mathcal{A}_g(M)$ represents the principally polarized abelian variety $(\mathfrak{A}_g(M)_a, \mathfrak{L}_g(M)_a)$ with some level-M-structure.*
4. *The varieties $\mathfrak{L}_g(M)$, $\mathfrak{A}_g(M)$ and $\mathcal{A}_g(M)$ are all canonically defined over $\overline{\mathbb{Q}}$.*
5. *$\mathfrak{A}_g(M) \to \mathcal{A}_g(M)$ can be compactified over $\overline{\mathbb{Q}}$ to smooth varieties $\overline{\mathfrak{A}_g(M)} \to \overline{\mathcal{A}_g(M)}$ such that any multiplication $[n]\colon \mathfrak{A}_g(M) \to \mathcal{A}_g(M)$ extends to the compactification.*
6. *$\mathfrak{L}_g(M)$ extends to an ample $\mathbb{G}_m$-torsor $\overline{\mathfrak{L}_g(M)} \to \overline{\mathfrak{A}_g(M)}$ over $\overline{\mathbb{Q}}$.*

Proof. See [53, 10.5, 10.9, 10.10, 11.16] for the first four assertions. For (5) see [53, 6.25, 9.24, 12.4]. For (6) see [53, 8.6, 8.13, 9.13, 9.16, 12.4]. □

Denote by $\mathrm{GSp}_0 := \mathbb{G}_m$ and $P_0 := \mathbb{G}_a \rtimes \mathbb{G}_m$ with the standard action of $\mathbb{G}_m$ on $\mathbb{G}_a$. Pink proved the following lemma ([53, 2.26])

Lemma 1.1.35 (Reduction Lemma)**.** *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum with generic Mumford-Tate group.*

1. *If V is trivial, then there exist a connected pure Shimura datum $(G_0, \mathcal{D}^+)$ and an embedding*

$$(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_0, \mathcal{X}_0^+)$$

where $r = \dim(U)$ *(see [53, 2.8, 2.14] for definition of* $(P_0, \mathcal{X}_0^+)$*);*

2. *If* V *is not trivial, then there exist a connected pure Shimura datum* $(G_0, \mathcal{D}^+)$ *and Shimura morphisms*

$$(P', \mathcal{X}'^+) \twoheadrightarrow (P, \mathcal{X}^+)$$

$$\text{and } (P', \mathcal{X}'^+) \overset{\lambda}{\hookrightarrow} (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r} (P_{2g}, \mathcal{X}_{2g}^+)$$

such that $\mathrm{Ker}(P' \to P)$ *is of dimension 1 and of weight -2. Moreover* $\lambda|_V : V \simeq V_{2g} \to \oplus_{i=1}^r V_{2g}$ *is the diagonal map,* $\lambda|_{U'} : U' \simeq \oplus_{i=1}^r U_{2g}$ *and* $G \xrightarrow{\lambda|_G} G_0 \times \prod_{i=1}^r \mathrm{GSp}_{2g} \to \mathrm{GSp}_{2g}$ *is non-trivial for each projection.*

Proof. The statement except the last claim of the "Moreover" part is [53, 2.26 statement & pp 45]. For the last part, call $p_i : G \to \mathrm{GSp}_{2g}$ the composite with the i-th projection. If p_i is trivial, then $p_i(P', \mathcal{X}'^+)$ is trivial since a connected mixed Shimura datum is trivial if its pure part is trivial. This contradicts the dimension of V. □

1.1.4 A group theoretical proposition

Proposition 1.1.36. *Let* $1 \to N \to Q \xrightarrow{\varphi} Q' \to 1$ *be an exact sequence of algebraic groups over* $\mathbb{Q}$*. Then the following diagram with solid arrows is commutative and all the lines and columns are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_N := \mathcal{R}_u(N) & \longrightarrow & N & \underset{\pi_N}{\overset{s_N}{\rightleftarrows}} & G_N := N/W_N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_Q := \mathcal{R}_u(Q) & \longrightarrow & Q & \underset{\pi_Q}{\overset{s_Q}{\rightleftarrows}} & G_Q := Q/W_Q & \longrightarrow & 1. \\
 & & \downarrow & & \downarrow \varphi & & \downarrow \overline{\varphi} & & \\
1 & \longrightarrow & W_{Q'} := \mathcal{R}_u(Q') & \longrightarrow & Q' & \underset{\pi_{Q'}}{\overset{s_{Q'}}{\rightleftarrows}} & G_{Q'} := Q'/W_{Q'} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Moreover, if we fix a morphism s_Q *which splits the middle line (such an* s_Q *exists by Levi decomposition), then we can deduce* s_N *and* $s_{Q'}$ *which split the other two lines. Note that in this case, the action of* G_N *on* $W_{Q'}$ *induced by* s_Q *is trivial.*

Proof. The two bottom lines are already exact. By group theory, we know $\varphi(W_Q(\overline{\mathbb{Q}})) = W_{Q'}(\overline{\mathbb{Q}})$ ([13, Corollary 14.11]), and since the set of closed points of W_Q (resp. $W_{Q'}$) is dense on W_Q (resp. $W_{Q'}$), we have $\varphi(W_Q) = W_{Q'}$. In consequence, we have the map $\overline{\varphi}$, which is surjective since φ is. Now we get the solid diagram with exact lines and columns but with W_N replaced by $N \cap W_Q$ and G_N replaced by $N/(N \cap W_Q)$. But $N/(N \cap W_Q)$, being normal in G_Q, is reductive ([13, 14.2 Corollary(b)]). Hence $N \cap W_Q = \mathcal{R}_u(N) = W_N$ and we get the desired solid diagram.

If we have an s_Q, then to get a desired $s_{Q'}$ (and s_N) is equivalent to prove that $\varphi \circ s_Q(G_N)$ is trivial, i.e. the intersection of this image with $W_{Q'}$ (in Q') is trivial and the projection of this image to $G_{Q'}$ (under $\pi_{Q'}$) is trivial. The projection is trivial by a simple diagram-chasing. The neutral component of the intersection is trivial since it is reductive and unipotent, and hence the intersection is trivial since $W_{Q'}$ is unipotent over $\mathbb{Q}$. Now the triviality of the action of G_N on $W_{Q'}$ induced by s_Q is automatic. □

Corollary 1.1.37. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum. Suppose $N \lhd P$. Then there are decompositions*

$$V = V_N \oplus V_N^{\perp} \qquad \textit{(resp. } U = U_N \oplus U_N^{\perp}\textit{)}$$

as G-modules, where $V_N := V \cap N$ (resp. $U_N := U \cap N$), such that the action of $G_N := N/V_N$ on $V_N^{\perp}$ (resp. $U_N^{\perp}$) is trivial.

Proof. To prove the decomposition of V, apply Proposition 1.1.36 to the exact sequence

$$1 \to V_N \rtimes G_N \to V \rtimes G \to (V/V_N) \rtimes (G/G_N) \to 1,$$

then since G is reductive, the vertical line on the left (in the diagram of the proposition) splits. The conjugation by P on V factors through G by reason of weights, and hence equals to the action of G on V induced by any Levi decomposition s_P. So the action of G_N on $V_N^{\perp}$ is trivial by the last assertion of Proposition 1.1.36.

To prove the decomposition of U, it suffices to apply Proposition 1.1.36 to the exact sequence

$$1 \to U_N \rtimes G_N \to U \rtimes G \to (U/U_N) \rtimes (G/G_N) \to 1.$$

□

In fact we have a better result if $(P, \mathcal{X}^+)$ is with generic Mumford-Tate group.

Proposition 1.1.38. *Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum such that $P = \mathrm{MT}(\mathcal{X}^+)$. Suppose $N \lhd P$ such that N possesses no non-trivial torus quotient. Then G_N acts trivially on U.*

Proof. By Reduction Lemma (Lemma 1.1.35), we may assume that $(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^{r}(P_{2g}, \mathcal{X}_{2g}^+)$ $(g \geqslant 0)$. Since N possesses no non-trivial torus quotient, G_N is semi-simple (the last line of the proof of Proposition 1.2.4). So

$$G_N = G_N^{\mathrm{der}} < G^{\mathrm{der}} < (G_0 \times \prod_{i=1}^{r} \mathrm{GSp}_{2g})^{\mathrm{der}} = G_0^{\mathrm{der}} \times \prod_{i=1}^{r} \mathrm{Sp}_{2g}$$

where $\mathrm{Sp}_0 := 1$. Hence G_N acts trivially on U since $G_0^{\mathrm{der}} \times \prod_{i=1}^{r} \mathrm{Sp}_{2g}$ acts trivially on $\oplus_{i=1}^{r} U_{2g}$. □

1.2 Weakly special subvarieties

1.2.1 Definition and basic properties

Definition 1.2.1. *(Pink, [54, Definition 4.1(b)]) Let S be a connected mixed Shimura variety. Consider any Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ and any point $t' \in T'$. Then any irreducible component of $[i]([\varphi]^{-1}(t'))$ is called a **weakly special subvariety** of S. We will prove later in Remark 1.2.5 that weakly special subvarieties of S are indeed closed subvarieties.*

Since any Shimura morphism is related to a Shimura morphism between Shimura data, we will try to rephrase this definition in the context of Shimura data:

Definition 1.2.2. *Given a connected mixed Shimura datum $(P, \mathcal{X}^+)$, a **weakly special subset** of $\mathcal{X}^+$ is a connected component of $i(\varphi^{-1}(y')) \subset \mathcal{X}^+$ for a point $y' \in \mathcal{Y}'^+$, where i, φ, $\mathcal{Y}'^+$ are in the following diagram of Shimura morphisms*

$$\begin{array}{ccc} & (Q, \mathcal{Y}^+) & \\ {}^{\varphi}\swarrow & & \searrow^{i} \\ (Q', \mathcal{Y}'^+) & & (P, \mathcal{X}^+) \end{array} .$$

Remark 1.2.3. *1. In the definition above, let $N := \mathrm{Ker}(Q \to Q')$ and let $U_N := U_Q \cap N$, then $i(\varphi^{-1}(y'))$ is a connected component of $N(\mathbb{R})U_N(\mathbb{C})y$ where $\varphi(y) = y'$. So $i(\varphi^{-1}(y'))$ is smooth as an analytic variety. In particular, its connected components and complex analytic irreducible components coincide. As a result, we can replace "a connected component" by "a complex analytic irreducible component" in Definition 1.2.2.*

2. If furthermore N is connected, then $i(\varphi^{-1}(y'))$ itself is connected (hence also complex analytic irreducible). The proof is as follows: Consider the image of $\varphi^{-1}(y')$ under the projection $(Q, \mathcal{Y}^+) \xrightarrow{\pi} (G_Q, \mathcal{Y}_{G_Q}^+) := (Q, \mathcal{Y}^+)/W_Q$. By the decomposition ([39, 3.6])

$$(G_Q^{\mathrm{ad}}, \mathcal{Y}_{G_Q}^+) = (G_N^{\mathrm{ad}}, \mathcal{Y}_1^+) \times (G_2, \mathcal{Y}_2^+)$$

where $G_N := N/W \cap N$, we have $\pi(\varphi^{-1}(y')) = \mathcal{Y}_1^+ \times \{y_2\}$. So $\pi(\varphi^{-1}(y')) = G_N(\mathbb{R})^+\pi(y)$. But $W_N(\mathbb{R})U_N(\mathbb{C})$ ($W_N := W \cap N$) is connected, hence $\varphi^{-1}(y') = N(\mathbb{R})^+U_N(\mathbb{C})y$, which is connected. In consequence, $i(\varphi^{-1}(y'))$ also is connected.

Proposition 1.2.4. *For any weakly special subvariety of S (resp. weakly special subset of $\mathcal{X}^+$), the Shimura morphisms in Definition 1.2.1 (resp. Definition 1.2.2) can be chosen such that*

- *the underlying homomorphism of algebraic groups i is injective, and hence i is an embedding in the sense of [53, 2.3];*
- *the underlying homomorphism of algebraic groups φ is surjective, and its kernel N is connected. Moreover, N possesses no non-trivial torus quotient (or equivalently, $G_N := N/(W \cap N)$ is semi-simple);*
- *φ is a quotient Shimura morphism.*

Proof. If $P = \mathrm{MT}(\mathcal{X}^+)$, then the first two points except the statement in the bracket are proved by [54, Proposition 4.4]. The general cases follow directly from Proposition 1.1.19(1). The third assertion can be proved by the universal property of quotient Shimura data given in Proposition 1.1.22. Now we are left to prove the statement in the bracket.

$G_N \lhd G$ since $G_N = N/(W \cap N) \hookrightarrow G = P/W$ and $N \lhd P$, and hence G_N is reductive ([13, 14.2, Corollary(b)]). By [13, 14.2 Proposition(2)], G_N is the almost-product of G_N^{der} and $Z(G_N)^\circ$, and $Z(G_N)^\circ$ equals the radical of G_N which is a torus. So N possesses no non-trivial torus quotient iff G_N possesses no non-trivial torus quotient iff G_N is semi-simple. □

Remark 1.2.5. *We can now prove that weakly special subvarieties of S are closed. By the proposition above, we can choose i to be injective. Then $[i]$ is finite by Proposition 1.1.27(1). Hence $[i]([\varphi]^{-1}(t'))$ is closed.*

Lemma 1.2.6. *Suppose that the Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ are associated with the morphisms of mixed Shimura data*

$$(Q', \mathcal{Y}'^+) \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+)$$

so that we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Y}'^+ & \xleftarrow{\quad\varphi\quad} & \mathcal{Y}^+ & \xrightarrow{\quad i\quad} & \mathcal{X}^+ \\ \Big\downarrow{\scriptstyle \mathrm{unif}_{\mathcal{Y}'^+}} & & \Big\downarrow{\scriptstyle \mathrm{unif}_{\mathcal{Y}^+}} & & \Big\downarrow{\scriptstyle \mathrm{unif}_{\mathcal{X}^+}} \\ T' = \Delta'\backslash\mathcal{Y}'^+ & \xleftarrow{[\varphi]} & T = \Delta\backslash\mathcal{Y}^+ & \xrightarrow{[i]} & S = \Gamma\backslash\mathcal{X}^+ \end{array},$$

then for any point $y' \in \mathcal{Y}'^+$, any irreducible component of $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is also an irreducible component of $[i]([\varphi]^{-1}(\mathrm{unif}_{\mathcal{Y}'^+}(y')))$.

Proof. Let $N := \mathrm{Ker}(\varphi)$ and let U_Q be the weight -2 part of Q, then we have

$$\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y'))) \subset [i]([\varphi]^{-1}(\mathrm{unif}_{\mathcal{Y}'^+}(y'))),$$

and both of them are of constant dimension d, where d is the dimension of any orbit of $N(\mathbb{R})^+(U_Q \cap N)(\mathbb{C})$. This allows us to conclude. □

The following Proposition tells us that the two definitions of weak specialness are compatible.

Proposition 1.2.7. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S = \Gamma\backslash\mathcal{X}^+$ *be the uniformization. Then a subvariety Z of S is weakly special if and only if Z is the image of some weakly special subset of $\mathcal{X}^+$.*

Proof. The "if" part is immediate by Lemma 1.2.6. We prove the "only if" part. We assume that i, φ are as in Proposition 1.2.4. For any weakly special subvariety $Z \subset S$, suppose that we have a diagram as in Lemma 1.2.6 and that Z is an irreducible component of $[i]([\varphi]^{-1}(t'))$. Since

$$[i]([\varphi]^{-1}(t')) \subset \bigcup_{y' \in \mathrm{unif}_{\mathcal{Y}^+}^{-1}(t')} \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y'))) = \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(\mathrm{unif}_{\mathcal{Y}^+}^{-1}(t')))),$$

there exists a $y' \in \mathcal{Y}'^+$ lying over t' such that Z is an irreducible component of $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ by Lemma 1.2.6. By Remark 1.2.3.2, $i(\varphi^{-1}(y'))$ is complex analytic irreducible, so $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is also complex analytic irreducible when S is regarded as an analytic variety. Hence $\mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is irreducible as an algebraic variety. So $Z = \mathrm{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$. □

Next we come to special subvarieties of connected mixed Shimura varieties.

Definition 1.2.8. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$.*

1. *A **special subvariety** of S is the image of any Shimura morphism $T \to S$ of connected mixed Shimura varieties;*
2. *A point $x \in \mathcal{X}^+$ and its image in S are called **special** if the homomorphism $x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ factors through $T_{\mathbb{C}}$ for a torus $T \subset P$.*

Remark 1.2.9. *By definition, $x \in \mathcal{X}^+$ is special if and only if it is the image of a Shimura morphism $(T, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$. Hence a special point is just a special subvariety of dimension* 0.

The following result is easy to prove. It tells us that special subvarieties of S are precisely connected mixed Shimura subvarieties of S.

Lemma 1.2.10. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ and let* $\mathrm{unif}\colon \mathcal{X}^+ \to S$ *be the uniformizing map, then a subvariety of S is special if and only if it is of the form* $\mathrm{unif}(\mathcal{Y}^+)$ *for some $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$.*

Proposition 1.2.11. *Every special subvariety of S contains a Zariski dense subset of special points.*

Proof. [54, Proposition 4.14]. □

The relation between special and weakly special subvarieties is:

Proposition 1.2.12. *A subvariety of S is special if and only if it is weakly special and contains a special point.*

Proof. [54, Proposition 4.2, Proposition 4.15]. □

We close this section by proving that this definition of weakly special subvarieties is compatible with the one (which is already known) for pure Shimura varieties.

Proposition 1.2.13. *A weakly special subvariety of a pure Shimura variety S is a subvariety of the same form as in [65, Definition 2.1].*

Proof. This is pointed out in [54, Remark 4.5]. We give a (relatively) detailed proof here. We prove the result for weakly special subsets. Assume that S is associated with the connected pure Shimura datum $(P, \mathcal{X}^+)$. For a subset of the same form as in [65, Definition 2.1], take $(Q, \mathcal{Y}^+) = (H, X_H^+)$ and $(Q', \mathcal{Y}'^+) = (H_1, X_1^+)$ (same notation as [65, Definition 2.1]). Then by definition such a subset is weakly special (as in Definition 1.2.2).

On the other hand, suppose that we have a weakly special subset $\widetilde{F}$ defined by a diagram as in Definition 1.2.2 satisfying Proposition 1.2.4. Let $N := \mathrm{Ker}(\varphi)$, then the homogeneous spaces of the connected pure Shimura data $(Q', \mathcal{Y}'^+) = (Q, \mathcal{Y}^+)/N$ and $(Q, \mathcal{Y}^+)/Z(Q)N = (Q^{\mathrm{ad}}, \mathcal{Y}^{\mathrm{ad}+})/N^{\mathrm{ad}}$ are canonically isomorphic to each other ([38, Proposition 5.7]). Hence we may replace $(Q', \mathcal{Y}'^+)$ by $(Q^{\mathrm{ad}}, \mathcal{Y}^{\mathrm{ad}+})/N^{\mathrm{ad}}$. But by [39, 3.6, 3.7], $(Q^{\mathrm{ad}}, \mathcal{Y}^{\mathrm{ad}+}) = (N^{\mathrm{ad}}, \mathcal{Y}_1^+) \times (Q_2, \mathcal{Y}_2^+)$. So $\widetilde{F}$ is of the same form as in [65, Definition 2.1]. □

1.2.2 Weakly special subvarieties in Kuga varieties

In this section, we consider only connected mixed Shimura varieties of Kuga type. Through the whole section, $S = \Gamma \backslash \mathcal{X}^+$ will be a connected mixed Shimura variety of Kuga type which is associated with the connected mixed Shimura datum $(P, \mathcal{X}^+)$ with $\Gamma = \Gamma_V \rtimes \Gamma_G$ neat. Then $W_{-2}(P)$ is trivial by definition. Denote by $V = \mathcal{R}_u(P)$ and

$$\begin{array}{ccc} (P, \mathcal{X}^+) & \xrightarrow{\pi} & (G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/V \\ \downarrow{\scriptstyle \mathrm{unif}} & & \downarrow{\scriptstyle \mathrm{unif}_{\mathcal{X}_G^+}} \\ S & \xrightarrow{[\pi]} & S_G \end{array} .$$

By Example 1.1.24, there is a natural bijection $V(\mathbb{R}) \times \mathcal{X}_G^+ \simeq \mathcal{X}^+$. By Proposition 1.1.26, $S \xrightarrow{[\pi]} S_G$ is a family of abelian varieties. Let $[\varepsilon]\colon S_G \to S$ be the zero-section of $[\pi]$. Then $[\varepsilon]$ corresponds to $\varepsilon\colon (G, \mathcal{X}_G^+) \hookrightarrow (P, \mathcal{X}^+)$. The Shimura morphism ε is a section of π and determines a Levi-decomposition of $P = V \rtimes^{\varepsilon} G$. A particular example is $\mathfrak{A}_g \to \mathcal{A}_g$, where ε is the natural inclusion $\mathrm{GSp}_{2g} = \{0\} \times \mathrm{GSp}_{2g} < V_{2g} \rtimes \mathrm{GSp}_{2g} = P_{2g,\mathrm{a}}$.

The goal of this section is to prove the following proposition:

Proposition 1.2.14. *Let B be an irreducible subvariety of S_G and $X := [\pi]^{-1}(B)$. Define $\mathcal{C}$ to be the isotrivial part of $X \to B$, i.e. the largest isotrivial abelian subscheme of X over B. Then*

$$\{\textit{translates of abelian subscheme of } X \to B \textit{ by a torsion section and then by a constant section of } \mathcal{C} \to B\} = \{X \cap E \mid E \textit{ weakly special in } S\}.$$

Let us define constant sections of $\mathcal{C} \to B$. By definition of isotriviality, there exists a finite cover $B' \to B$ such that $\mathcal{C} \times_B B' \simeq \mathcal{C}_{b_0} \times B'$ for any $b_0 \in B$. A **constant section of** $\mathcal{C} \to B$ is then defined to be the image of the graph of a constant morphism $B' \to \mathcal{C}_{b_0}$ in $\mathcal{C} \times_B B'$ under the projection $\mathcal{C} \times_B B' \to \mathcal{C}$.

Proposition 1.2.14 has the following corollary, which describes weakly special subvarieties of connected mixed Shimura varieties of Kuga type in geometric terms.

Corollary 1.2.15. *An irreducible subvariety Y of S is weakly special iff the followings hold:*

1. *$[\pi]Y$ is a totally geodesic subvariety of S_G;*
2. *Y is the translate of an abelian subscheme of $[\pi]^{-1}([\pi]Y)$ (over $[\pi]Y$) by a torsion section and then by a constant section of the isotrivial part of $[\pi]^{-1}[\pi]Y \to [\pi]Y$.*

Proof. This follows directly from [39, 4.3] and Proposition 1.2.14. □

We start from the following proposition which is not hard to prove using Levi decomposition [55, Theorem 2.3]. Another proof can be found in [33, Section 5.1].

Proposition 1.2.16. *To give a Shimura subdatum $(Q, \mathcal{Y}^+)$ of $(P, \mathcal{X}^+)$ is equivalent to give:*

- *a pure Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$ of $(G, \mathcal{X}_G^+)$;*
- *a G_Q-submodule V_Q of V (V is a G-module, and therefore a G_Q-module);*
- *an element $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$.*

Proof. We only give the constructions here.

1. Given $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$, we have $V_Q := \mathcal{R}_u(Q) < \mathcal{R}_u(P) = V$. Therefore the inclusion $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$ induces

   $$(G_Q, \mathcal{Y}^+_{G_Q}) := (Q, \mathcal{Y}^+)/V_Q \subset (G, \mathcal{X}^+_G) = (P, \mathcal{X}^+)/V.$$

   The fact that V_Q is a G_Q-submodule of V is clear. Now it suffices to find $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$.

   Consider the group $Q^\natural := (V/V_Q) \rtimes G_Q$, where the action is induced by the natural one of G_Q on V. By definition, $Q^\natural = \pi^{-1}(G_Q)/V_Q$. Now the inclusion $(Q, \mathcal{Y}^+) \subset (P, \mathcal{X}^+)$ induces another inclusion (which we call i')

   $$G_Q = Q/V_Q \subset \pi^{-1}(G_Q)/V_Q = Q^\natural.$$

   We have the following diagram, whose solide arrows commute:

   $$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 & \longrightarrow & G_Q & \xrightarrow{=} & G_Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle i'} & & \downarrow & & \\ 1 & \longrightarrow & V/V_Q & \longrightarrow & Q^\natural & \underset{s_Q}{\overset{\dashleftarrow}{\longrightarrow}} & G_Q & \longrightarrow & 1 \end{array}$$

   where s_Q is the homomorphism $G_Q = \{0\} \rtimes G_Q < (V/V_Q) \rtimes G_Q = Q^\natural$. Now i' and s_Q are two Levi-decompositions for $Q^\natural$. By [55, Theorem 2.3], s_Q equals the conjugation of i' by an element $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$. Moreover, the choice of $\overline{v}_0$ is unique.

2. Conversely, given the three data as in the Proposition, the underlying group Q is the conjugate of $V_Q \rtimes G_Q < V \rtimes G$ (compatible Levi-decompositions) by $(v_0, 1)$ in P. The space

   $$\mathcal{Y}^+ = \big(v_0 + V_Q(\mathbb{R})\big) \times \mathcal{Y}^+_{G_Q} \subset V(\mathbb{R}) \times \mathcal{X}^+_G \simeq \mathcal{X}^+$$

   where v_0 is any lift of $\overline{v}_0$ to $V(\mathbb{Q})$.

□

Proposition 1.2.17. *A subvariety Y of S is weakly special iff there exist*

- *a pure Shimura subdatum $(G_Q, \mathcal{Y}^+_{G_Q})$ of $(G, \mathcal{X}^+_G)$;*
- *a point $v_0 \in V(\mathbb{Q})$;*
- *a normal semi-simple connected subgroup G_N of G_Q and a point $\widetilde{y}_G \in \mathcal{Y}^+_{G_Q}$;*
- *a G_Q-submodule V_N of V;*
- *a G_Q-submodule $V_N^\perp$ of V on which G_N acts trivially, and a point $v \in V_N^\perp(\mathbb{R})$*

such that

$$Y = \mathrm{unif}\Big(\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+ \widetilde{y}_G\Big).$$

Here $\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+ \widetilde{y}_G \subset V(\mathbb{R}) \times \mathcal{X}_G^+ \simeq \mathcal{X}^+$.

Proof. 1. Given a weakly special subvariety Y of S, let $(Q, \mathcal{Y}^+)$, N and $\widetilde{y}$ be as in Definition 1.2.2 and Proposition 1.2.4. By Proposition 1.2.16, $(Q, \mathcal{Y}^+)$ corresponds to a Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$ of $(G, \mathcal{X}_G^+)$, a G_Q-submodule V_Q of V and a point $\overline{v}_0 \in (V/V_Q)(\mathbb{Q})$. Let v_0 be any lift of $\overline{v}_0$ to $V(\mathbb{Q})$. Let $G_N := N/(V_Q \cap N)$, then G_N is a connected nomral subgroup of G_Q, and hence is reductive. Since N possesses no non-trivial torus quotient, G_N is semi-simple. Let $\widetilde{y}_G := \pi(\widetilde{y})$.

Let $V_N := V_Q \cap N$, then V_N is a G_Q-submodule of V_Q since N is normal in Q. By Corollary 1.1.37, there exists a G_Q-submodule $V_N^\perp$ of V_Q such that $V_Q = V_N \oplus V_N^\perp$ and G_N acts trivially on $V_N^\perp$. Write $\widetilde{y} = (\widetilde{y}_V, \widetilde{y}_G) \in (v_0 + V_Q(\mathbb{R})) \times \mathcal{Y}_{G_Q}^+ = \mathcal{Y}^+ \subset \mathcal{X}^+$ (here we use the second part of the proof of Proposition 1.2.16).

To simplify the computation below, we introduce a new Shimura subdatum $(Q', \mathcal{Y}')$ of $(P, \mathcal{X}^+)$: $(Q', \mathcal{Y}')$ is defined to be the conjugate of $(Q, \mathcal{Y}^+)$ by $(-v_0, 1)$. By the second part of the proof of Proposition 1.2.16, $(Q', \mathcal{Y}') = (V_Q \rtimes G_Q, V_Q(\mathbb{R}) \times \mathcal{Y}_{G_Q}^+) \subset (V \rtimes \mathrm{GSp}_{2g}, \mathcal{X}^+)$. Let $N' := V_N \rtimes G_N < V \rtimes \mathrm{GSp}_{2g}$, then N' is the conjugate of N by $(-v_0, 1)$. Let $\widetilde{y}' := (\widetilde{y}_V - v_0, \widetilde{y}_G) \in \mathcal{Y}'^+$.

Let v be the $V_N^\perp(\mathbb{R})$-factor of $\widetilde{y}_V - v_0$ under $V_Q = V_N \oplus V_N^\perp$. Then since G_N acts trivially on $V_N^\perp$, we have

$$N'(\mathbb{R})^+ \widetilde{y}' = \big(v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+ \widetilde{y}_G \subset \mathcal{Y}'^+.$$

Hence $N(\mathbb{R})^+ \widetilde{y} = \big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+ \widetilde{y}_G$. Now the conclusion follows.

2. Conversely given all these data, let the Shimura subdatum $(Q, \mathcal{Y}^+)$ be the one obtained from $(G_Q, \mathcal{Y}_{G_Q}^+)$, $V_N \oplus V_N^\perp$ and v_0 by Proposition 1.2.16. Let N be the subgroup of Q which is defined to be $V_N \rtimes G_N$ conjugated by $(v_0, 1)$ in P. Then since G_N acts trivially on $V_N^\perp$, we have $N \lhd Q$. Let $\widetilde{y} := (v_0 + v, \widetilde{y}_G)$. Now we have

$$\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+ \widetilde{y}_G = N(\mathbb{R})^+ \widetilde{y}.$$

The group N is by definition connected and it possesses no non-trivial torus quotient since G_N is semi-simple. Hence Y is weakly special by definition.

□

Now we can prove Proposition 1.2.14:

Proof of Proposition 1.2.14. 1. Prove "$\supset$". For this it suffices to prove:

For any weakly special subvariety Y of S, Y is the translate of an abelian subscheme of $[\pi]^{-1}([\pi]Y)$ (over $[\pi]Y$) by a torsion section and then by a constant section of the isotrivial part of $[\pi]^{-1}[\pi]Y \to [\pi]Y$.

Let Y be a weakly special subvariety of S. Then associated to Y there are data as in Proposition 1.2.17 and

$$Y = \mathrm{unif}\Big(\big(v_0 + v + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G\Big).$$

Let $B' := [\pi]Y$ and $X' := [\pi]^{-1}(B')$.

Now $X' \to B'$ is an abelian scheme. Since V_N is a G_Q-submodule of V, $\mathrm{unif}\big(V_N(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\big)$ is an abelian subscheme of X' over B'. Therefore,

$$\mathrm{unif}\Big(\big(v_0 + V_N(\mathbb{R})\big) \times G_N(\mathbb{R})^+\widetilde{y}_G\Big)$$

is the translate of B' by a torsion section of $X' \to B'$. But $v \in V_N^{\perp}(\mathbb{R})$ and G_N acts trivially on $V_N^{\perp}$, so $\mathrm{unif}\big(V_N^{\perp}(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\big)$ is an isotrivial abelian scheme over B'. Therefore Y is the translate of an abelian subscheme of $X' \to B'$ by a torsion section and then by a constant section of the isotrivial part of $X' \to B'$.

2. Prove "$\subset$". Let Y be a subvariety of X such that Y is the translate of an abelian subscheme of $X \to B$ translated by a torsion section and then by a section of $\mathcal{C} \to B$, where $\mathcal{C} \to B$ is the isotrivial part of $X \to B$. Let us find a weakly special subvariety E of S associated with the data in Proposition 1.2.17 such that $Y = E \cap X$.

   Let B' be the smallest weakly special subvariety of S_G containing B. Then by definition there exist a Shimura subdatum $(G_Q, \mathcal{Y}_{G_Q}^+)$, a connected semi-simple normal subgroup G_N of G_Q and a point $\widetilde{y}_G \in \mathcal{Y}_{G_Q}^+$ such that $B' = \mathrm{unif}_G\big(G_N(\mathbb{R})^+\widetilde{y}_G\big)$. Moreover by [39, 3.6, 3.7], G_N can be taken to be the connected algebraic monodromy group of $(B')^{\mathrm{sm}}$, i.e. the neutral component of the Zariski closure of $\Gamma_{B'^{\mathrm{sm}}}$:=the image of $\pi_1((B')^{\mathrm{sm}}) \to \pi_1(S_G) = \Gamma_G$.

   Let $X' := [\pi]^{-1}(B')$. Then the isotrivial part $\mathcal{C}'$ of $X' \to B'$ is

$$\mathrm{unif}\big(V'(\mathbb{R}) \times G_N(\mathbb{R})^+\widetilde{y}_G\big),$$

   where V' is the largest G_Q-submodule of V on which G_N acts trivially. This V' is the $V_N^{\perp}$ we want in Proposition 1.2.17.

   A key step is to prove that as subvarieties of S, we have

$$\mathcal{C} = \mathcal{C}' \cap X \tag{1.2.1}$$

   It is clear that $\mathcal{C}' \cap X \subset \mathcal{C}$. For the other inclusion, suppose that $\mathcal{C}$ is defined by the G_Q-submodule V'' of V (i.e. $\mathcal{C} = \mathrm{unif}(V''(\mathbb{R}) \times \widetilde{B})$ for

$\widetilde{B} := \mathrm{unif}_G^{-1}(B))$, then $\Gamma_{B'^{\mathrm{sm}}}$ acts trivially on V''. However the action of G on V is algebraic, therefore $\overline{\Gamma_{B'^{\mathrm{sm}}}}^{\mathrm{Zar}}$ acts trivially on V''. So G_N acts trivially on V''. By the maximality of V', $V'' \subset V'$. So $\mathcal{C} \subset \mathcal{C}'$. Now (1.2.1) follows.

Now since Y is the translate of an abelian subscheme by a torsion section and then by a constant section of $\mathcal{C} \to B$, there exists, by (1.2.1), a G_Q-submodule V_N of V such that

$$Y = \mathrm{unif}\left(\left(v_0 + v + V_N(\mathbb{R})\right) \times \widetilde{B}\right)$$

where $v_0 \in V(\mathbb{Q})$ corresponds to the torsion section and $v \in V'(\mathbb{R})$ corresponds to the constant section of $\mathcal{C} \to B$. In other words,

$$Y = E \cap X, \text{ where } E = \mathrm{unif}\left(\left(v_0 + v + V_N(\mathbb{R})\right) \times G_N(\mathbb{R})^+ \widetilde{y}_G\right)$$

and E is the weakly special subvariety of S we desire.

□

1.3 The bi-algebraic setting

1.3.1 Realization of the uniformizing space

Let $(P, \mathcal{X}^+)$ be a connected mixed Shimura datum. We first define the dual $\mathcal{X}^\vee$ of $\mathcal{X}^+$ (see [53, 1.7(a)] or [37, Chapter VI, Proposition 1.3]):

Let M be a faithful representation of P and take any $x_0 \in \mathcal{X}^+$. The weight filtration on M is constant, so the Hodge filtration $x \mapsto \mathrm{Fil}_x^{\cdot}(M_{\mathbb{C}})$ gives an injective map $\mathcal{X}^+ \hookrightarrow \mathrm{Grass}(M)(\mathbb{C})$ to a certain flag variety. In fact, this injective map factors through

$$\mathcal{X}^+ = P(\mathbb{R})^+ U(\mathbb{C})/C(x_0) \hookrightarrow P(\mathbb{C})/F_{x_0}^0 P(\mathbb{C}) \hookrightarrow \mathrm{Grass}(M)(\mathbb{C})$$

where $C(x_0)$ is the stabilizer of x_0 in $P(\mathbb{R})^+ U(\mathbb{C})$. The first injection is an open immersion ([53, 1.7(a)] or [37, Chapter VI, (1.2.1)]). We define the dual $\mathcal{X}^\vee$ of $\mathcal{X}^+$ to be

$$\mathcal{X}^\vee := P(\mathbb{C})/F_{x_0}^0 P(\mathbb{C}).$$

$\mathcal{X}^\vee$ is a connected smooth complex algebraic variety.

Proposition 1.3.1. *Under the open immersion $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$, $\mathcal{X}^+$ is realized as a semi-algebraic set which is also a complex manifold.*

Proof. $\mathcal{X}^+$ is smooth since it is a homogeneous space, and the open immersion endows it with a complex structure. For semi-algebraicity, consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^\vee} \\ \mathcal{X}_G^+ & \hookrightarrow & \mathcal{X}_G^\vee \end{array}.$$

As $\pi^\vee$ is algebraic, the conclusion follows from [64, Lemme 2.1]. □

Remark 1.3.2. *It is not hard to see that $\mathcal{X}^\vee$ is a projective variety if and only if $(P, \mathcal{X}^+)$ is pure. The argument is as follows: $\mathcal{X}^\vee$ is a holomorphic vector bundle over $\mathcal{X}_G^\vee$ where the fibre is homeomorphism to $W(\mathbb{R})U(\mathbb{C})$. $\mathcal{X}_G^\vee$ is projective, so $\mathcal{X}^\vee$ is projective if and only if it is a trivial vector bundle over $\mathcal{X}_G^\vee$, i.e. if and only if W is trivial.*

Let us take a closer look at the semi-algebraic structrue of $\mathcal{X}^+$. By [71, pp 6], there exists a Shimura morphism $i\colon (G, \mathcal{X}_G^+) \to (P, \mathcal{X}^+)$ such that $\pi \circ i = \mathrm{id}$. The morphism i defines a Levi decomposition of $P = W \rtimes G$. By definition $\mathcal{X}^+ \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$. Define a bijective map

$$\begin{array}{ccc} W(\mathbb{R})U(\mathbb{C}) \times \mathcal{X}_G^+ & \longrightarrow & \mathcal{X}^+ \\ (w, x) & \mapsto & \mathrm{int}(w) \circ i(x) \end{array}.$$

Identify P with the 4-tuple (G, V, U, Ψ) as in §1.1.2.5. Since $W \simeq U \times V$ as $\mathbb{Q}$-varieties, we can define a bijection induced by the one above

$$\rho : U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+ \xrightarrow{\sim} \mathcal{X}^+ \tag{1.3.1}$$

$P(\mathbb{R})^+U(\mathbb{C})$ acts on $\mathcal{X}^+$ by definition. There is also a natural action of $P(\mathbb{R})^+U(\mathbb{C})$ on $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$ which is defined as follows. Under the notation of §1.1.2.5, for any $(u, v, g) \in P(\mathbb{R})^+U(\mathbb{C})$ and $(u', v', x) \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$,

$$(u, v, g) \cdot (u', v', x) := (u + gu' + \frac{1}{2}\Psi(v, v'), v + gv', gx). \tag{1.3.2}$$

This action is algebraic since Ψ is a polynomial over $\mathbb{Q}$ (see §2.2). The map ρ is $P(\mathbb{R})^+U(\mathbb{C})$-equivariant by an easy calculation.

Proposition 1.3.3. *The map ρ is semi-algebraic.*

Proof. It is enough to prove that the graph of ρ is semi-algebraic. This is true since ρ is $P(\mathbb{R})^+U(\mathbb{C})$-equivariant and the actions of $P(\mathbb{R})^+U(\mathbb{C})$ on both sides are algebraic and transitive. Explicitly, fix a point $x_0 \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$, the graph of ρ

$$\begin{aligned} Gr(\rho) &= \{(gx_0, \rho(gx_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ |\ g \in P(\mathbb{R})^+U(\mathbb{C})\} \text{ (transitivity)} \\ &= \{(gx_0, g\rho(x_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ |\ g \in P(\mathbb{R})^+U(\mathbb{C})\} \text{ (equivariance)} \\ &= P(\mathbb{R})^+U(\mathbb{C}) \cdot (x_0, \rho(x_0)) \end{aligned}$$

is semi-algebraic since the action of $P(\mathbb{R})^+U(\mathbb{C})$ on $(U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+$ is algebraic. □

Remark 1.3.4. *If U is trivial, then $\mathcal{X}^+ = V(\mathbb{R}) \rtimes \mathcal{X}_G^+$ under the notation of Example 1.1.24. In this case, the complex structure of $\mathcal{X}^+$ given via $\mathcal{X}^\vee$ is the same as the one given in Example 1.1.24 since for the projection $\mathcal{X}^+ \xrightarrow{\pi} \mathcal{X}_G^+$, the complex structure of any fibre $\mathcal{X}_{x_G}^+$ ($x_G \in \mathcal{X}_G^+$) given by $\mathcal{X}^\vee$ is the same as the one obtained from $\mathcal{X}_{x_G}^+ \simeq V(\mathbb{C})/F_{x_G}^0 V(\mathbb{C})$ (see [53, 3.13, 3.14]). In particular this holds for $\mathcal{X}_{2g,\mathrm{a}}^+$ (see §1.1.3 for notation). Therefore for any $\mathfrak{A}_g(M)$, the fundamental set $[0,N)^{2g} \times \mathcal{F}_G \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,\mathrm{a}}^+$ is the one considered in [47].*

1.3.2 Algebraicity in the uniformizing space

Definition 1.3.5. *Let $\widetilde{Y}$ be an analytic subset of $\mathcal{X}^+$, then*

1. *$\widetilde{Y}$ is called an **irreducible algebraic subset** of $\mathcal{X}^+$ if it is a complex analytic irreducible component of the intersection of its Zariski closure in $\mathcal{X}^\vee$ and $\mathcal{X}^+$;*
2. *$\widetilde{Y}$ is called **algebraic** if it is a finite union of irreducible algebraic subsets of $\mathcal{X}^+$.*

In view of Definition 1.3.5, we are in the following bi-algebraic situation: both $\mathcal{X}^+$ are S are algebraic, but $\mathrm{unif}\colon \mathcal{X}^+ \to S$ is transcendental. Hence a priori there is no relation between the algebraic structures on $\mathcal{X}^+$ and on S. Therefore a natural question arises: what are the bi-algebraic objects? This question will be answered in the following sections. We state the result here:

Theorem 1.3.6. *A subset $Y \subset S$ is weakly special iff $\widetilde{Y}$ (a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$) is algebraic in $\mathcal{X}^+$ and Y is an irreducible subvariety of S.*

Remark 1.3.7. *Recall the following result of Pila-Tsimerman [49, Lemma 4.1]: maximal connected irreducible semi-algebraic subsets of $\mathcal{X}^+$ which are contained in a complex analytic subset of $\mathcal{X}^+$ are all algebraic (see the paragraph before Theorem 3.1.2 for the definition of "connected irreducible semi-algebraic subsets"). Hence an equivalent way to restate Theorem 1.3.6 is to replace "$\widetilde{Y}$ is algebraic in $\mathcal{X}^+$" by "$\widetilde{Y}$ is a semi-algebraic subset of $\mathcal{X}^+$".*

A more refined version as well as the proof of this theorem will be given in Corollary 2.3.3. Here we only prove the easy part of the theorem, which is:

Lemma 1.3.8. *Any weakly special subset of $\mathcal{X}^+$ is irreducible algebraic.*

Proof. Suppose that $\widetilde{Z}$ is a weakly special subset of $\mathcal{X}^+$. Use the notation of Definition 1.2.2 and assume that i and φ satisfy the properties in Proposition 1.2.4. Let $N := \mathrm{Ker}(Q \to Q')$ and let y be a point of the weakly special subset, then $\widetilde{Z} = N(\mathbb{R})^+ U_N(\mathbb{C}) y$ is complex analytic irreducible by Remark 1.2.3.2. But $N(\mathbb{R})^+ U_N(\mathbb{C}) y = N(\mathbb{C}) y \cap \mathcal{X}^+$ and $N(\mathbb{C}) y$ is algebraic, so $\widetilde{Z}$ is irreducible algebraic by definition. □

We finish this section by the functoriality of algebraicity:

Lemma 1.3.9 (functoriality of algebraicity)**.** *Let $f\colon (Q,\mathcal{Y}^+) \to (P,\mathcal{X}^+)$ be a Shimura morphism. Then there exists a unique morphism $f^\vee\colon \mathcal{Y}^\vee \to \mathcal{X}^\vee$ of algebraic varieties such that the diagram commutes:*

$$\begin{array}{ccc} \mathcal{Y}^+ & \hookrightarrow & \mathcal{Y}^\vee \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\vee} \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \end{array}.$$

Furthermore, for any irreducible algebraic subset $\widetilde{Z}$ of $\mathcal{Y}^+$, the closure in the archimedean topology of $f(\widetilde{Z})$ is irreducible algebraic in $\mathcal{X}^+$ and $f(\widetilde{Z})$ contains a dense open subset of this closure.

In particular, if f is an embedding, then an irreducible algebraic subset of $\mathcal{Y}^+$ is an irreducible component of the intersection of an irreducible algebraic subset of $\mathcal{X}^+$ with $\mathcal{Y}^+$.

Proof. Fix a point $x_0 \in \mathcal{Y}^+$, then we have

$$\begin{array}{ccc} \mathcal{Y}^+ = Q(\mathbb{R})^+U_Q(\mathbb{C})/C(x_0) & \hookrightarrow & \mathcal{Y}^\vee = Q(\mathbb{C})/F^0_{x_0}Q(\mathbb{C}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\vee} \\ \mathcal{X}^+ = P(\mathbb{R})^+U_P(\mathbb{C})/C(f(x_0)) & \hookrightarrow & \mathcal{X}^\vee = P(\mathbb{C})/F^0_{f(x_0)}P(\mathbb{C}) \end{array},$$

where $C(x_0)$ (resp. $C(f(x_0))$) denotes the stabilizer of x_0 (resp. $f(x_0)$) in $Q(\mathbb{R})U_Q(\mathbb{C})$ (resp. $P(\mathbb{R})U_P(\mathbb{C})$). The map $f^\vee$ is unique since $Q(\mathbb{R})U_Q(\mathbb{C})/C(x_0)$ is dense in $\mathcal{Y}^\vee$.

To prove the second statement, it is enough to prove the result for $f^\vee(\overline{\widetilde{Z}}^{\mathrm{Zar}}) \subset \mathcal{X}^\vee$ where $\overline{\widetilde{Z}}^{\mathrm{Zar}}$ is the Zariski closure of $\widetilde{Z}$ in $\mathcal{Y}^\vee$. This is then an algebro-geometric result, which follows easily from Chevalley's Theorem ([22, Chapitre IV, 1.8.4]) and [41, I.10, Theorem 1]. □