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The Parabolic Anderson Model and Long-Range Percolation

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Preface

This thesis has two parts.

Part I deals with the parabolic Anderson model. This is the partial differential equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a dynamic random environment that drives the equation. We take the initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. The question of interest is how the exponential growth rate of u depends on the diffusion constant κ. This can be monitored via the annealed and quenched Lyapunov exponent. We focus on the latter.

Part II deals with two different percolation models. The occupied set of the first percolation model is obtained by taking the union of a collection of independent Brownian motions running up to time $t \geq 0$, whose initial positions are distributed according to a Poisson point process. The question we investigate is whether the occupied set undergoes a non-trivial percolation phase transition in t or not. We further investigate the uniqueness of the unbounded components in the supercritical regime. The occupied set of the second percolation model is given by the random interlacement set. This is a family of random subsets $\mathcal{I}^u$, $u \geq 0$, on $\mathbb{Z}^d$, $d \geq 3$, that locally describes the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$ running up to time uN^d. It has been shown that the vacant set $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ undergoes a non-trivial percolation phase transition in u. We describe the geometry of the vacant set $\mathcal{V}^u$ in the supercritical regime for intensities u that are close to the criticial percolation parameter.

Part I (Chapters $1-3$) deals with the parabolic Anderson model and is based on the articles [EdHM14a] and [EdHM14b]. **Part II** (Chapters $4-6$) deals with the two percolation models and is based on the articles [EMP14] and [DE14].