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Samenstelling van de promotiecommissie:

Promotor: Prof. Dr. W. Th. F. den Hollander (Universiteit Leiden)
Second promotor: Dr. G. Maillard (Université d’Aix-Marseille)
Overige leden: Prof. Dr. W. König (Technische Universität Berlin)
               Prof. Dr. P. Mörters (University of Bath)
               Prof. Dr. R. van der Hofstad (Technische Universiteit Eindhoven)
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Preface

This thesis has two parts.

**Part I** deals with the parabolic Anderson model. This is the partial differential equation
\[ \frac{\partial u(x, t)}{\partial t} = \kappa \Delta u(x, t) + \xi(x, t) u(x, t), \quad x \in \mathbb{Z}^d, \ t \geq 0, \]
where the $u$-field and the $\xi$-field are $\mathbb{R}$-valued, $\kappa \in [0, \infty)$ is the diffusion constant, and $\Delta$ is the discrete Laplacian. The $\xi$-field plays the role of a dynamic random environment that drives the equation. We take the initial condition $u(x, 0) = u_0(x), \ x \in \mathbb{Z}^d,$ to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: $p$ articles jump at rate $2d \kappa,$ split into two at rate $\xi \vee 0,$ and die at rate $(-\xi) \vee 0.$ The question of interest is how the exponential growth rate of $u$ depends on the diffusion constant $\kappa.$ This can be monitored via the annealed and quenched Lyapunov exponent. We focus on the latter.

**Part II** deals with two different percolation models. The occupied set of the first percolation model is obtained by taking the union of a collection of independent Brownian motions running up to time $t \geq 0,$ whose initial positions are distributed according to a Poisson point process. The question we investigate is whether the occupied set undergoes a non-trivial percolation phase transition in $t$ or not. We further investigate the uniqueness of the unbounded components in the supercritical regime. The occupied set of the second percolation model is given by the random interlacement set. This is a family of random subsets $I^u,$ $u \geq 0,$ on $\mathbb{Z}^d,$ $d \geq 3,$ that locally describes the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$ running up to time $uN^d.$ It has been shown that the vacant set $V^u = \mathbb{Z}^d \setminus I^u$ undergoes a non-trivial percolation phase transition in $u.$ We describe the geometry of the vacant set $V^u$ in the supercritical regime for intensities $u$ that are close to the critical percolation parameter.

**Part I** (Chapters 1 – 3) deals with the parabolic Anderson model and is based on the articles [EdHM14a] and [EdHM14b]. **Part II** (Chapters 4 – 6) deals with the two percolation models and is based on the articles [EMP14] and [DE14].