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Part I The Parabolic Anderson Model

1 Introduction to Part I

1.1 The parabolic Anderson model

This section borrows parts of Section 1 in Erhard, den Hollander, Maillard [EdHM14a].

The Model

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\, t \geq 0. \tag{1.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi(x,t)\colon\, x \in \mathbb{Z}^d\} \tag{1.1.3}$$

is an $\mathbb{R}$-valued random field playing the role a of *dynamic random environment* that drives the equation. As initial condition for (1.1.1) we take

- $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, with u_0 non-negative and bounded. (1.1.4)

The Feynman-Kac formula

A formal solution of (1.1.1) and (1.1.4) is given by the Feynman-Kac formula

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\,ds\right\} u_0(X^\kappa(t))\right). \tag{1.1.5}$$

Here, $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$, with law P_x and expectation E_x when $X^\kappa(0) = x$. The representation in (1.1.5) may, on a heuristic level, be explained as follows. Consider the equations

$$\frac{\partial}{\partial t}v(x,t) = \kappa\Delta v(x,t) \quad \text{and} \quad \frac{\partial}{\partial t}w(x,t) = \xi(x,t)w(x,t) \tag{1.1.6}$$

with initial conditions v_0 and w_0, respectively. Note that $\kappa\Delta$ is the generator of X^κ. Therefore, the solution to the first equation is given by $v(x,t) = E_x(v_0(X^\kappa(t)))$. The solution to the second equation is given by $w(x,t) = e^{\int_0^t \xi(x,s)\,ds} w_0(x)$. Thus, (1.1.5) may be interpreted as a combination of these two solutions. The Laplacian in the first equation has the tendency to make the solution flat (e.g. $v_0 \equiv 1$ implies $v \equiv 1$). On the other hand, in the second equation there is no such smoothing and w is irregular. In (1.1.1), these two effects compete with each other, and it is this competition that makes the model more appealing but also more complicated to study.

Two interpretations of (1.1.1)

• One interpretation comes from *population dynamics.* Consider the special case where $\xi(x,t) = \gamma\bar{\xi}(x,t) - \delta$ with $\delta, \gamma \in (0,\infty)$ and $\bar{\xi}$ an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed dynamics with $\bar{\xi}(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;
- B-particles die at rate δ;
- the average number of B-particles at site x at time 0 is $u_0(x)$.

Then

$$u(x,t) = \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t\\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{1.1.7}$$

• Another interpretation comes from *random walk moving through a random field of sinks and sources.* Here, sites $(x,t) \in \mathbb{Z}^d \times [0,\infty)$ with $\xi(x,t) < 0$ are interpreted as sinks and sites $(x,t) \in \mathbb{Z}^d \times [0,\infty)$ with $\xi(x,t) > 0$ are interpreted as sources. The case in which ξ does not depend on time and is such that $\xi \in \{-\infty, 0\}$ has a particularly nice interpretation in terms of *survival probabilities,* and is sometimes referred to as *random walk among Bernoulli traps.* More precisely, assume that $u_0 \equiv 1$ and let

$$\mathcal{O} = \{z \in \mathbb{Z}^d : \xi(z) = -\infty\} \tag{1.1.8}$$

be the set of traps. Then the Feynman-Kac formula (1.1.5) reads

$$u(x,t) = P_x(X^\kappa(s) \subset \mathcal{O}^c \text{ for all } s \in [0,t]), \tag{1.1.9}$$

i.e., $u(x,t)$ equals the probability that the random walk X^κ starting at x does not get killed by any of the traps in $\mathcal{O}$ until time t.

Three problems related to (1.1.1)

• Burger's equation

Burger's equation is a fundamental equation in hydrodynamics (see [CM94]), and reads

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) + v(x,t) \cdot \nabla v(x,t) = \kappa \Delta v(x,t) + f(x,t), \\ v(x,0) = v_0(x), \end{cases} \qquad x \in \mathbb{R}^d. \tag{1.1.10}$$

Here, Δ denotes the usual Laplacian in $\mathbb{R}^d$, ∇ denotes the gradient, the v-field is $\mathbb{R}^d$-valued and f is an external force. In the case where v_0 and f can be written as gradients, the solution of (1.1.10) may be obtained via the substitution

$$v(x,t) = -2\kappa \nabla \log \varphi(x,t). \tag{1.1.11}$$

Substituting (1.1.11) into (1.1.10), we get

$$\begin{cases} \frac{\partial}{\partial t} \varphi(x,t) = \kappa \Delta \varphi(x,t) + F(x,t)\varphi(x,t), \\ -2\kappa \nabla \varphi_0(x) = v_0(x), \\ \nabla F(x,t) = f(x,t). \end{cases} \tag{1.1.12}$$

Thus, (1.1.10) is transformed into the continuum version of the parabolic Anderson equation (1.1.1) and it is therefore enough to study the behaviour of φ.

• Advection-Convection equation for a temperature field

Consider the following equation for the scalar temperature field T:

$$\begin{cases} \frac{\partial}{\partial t} T(x,t) + v(x,t) \cdot \nabla T(x,t) = \kappa \Delta T(x,t), \\ T(x,0) = T_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1.13}$$

This equation is used for the analysis of turbulent diffusions. It is argued in [CM94] (see also [AM90]), that if the dimension is two and the velocity field v has the form

$$v(x_1, x_2) = \begin{bmatrix} 0 \\ v(x_1) \end{bmatrix}, \tag{1.1.14}$$

then, under some additional assumptions, the Laplace transform T_λ, $\lambda \in \mathbb{R}$, of T with respect to the second space coordinate, satisfies the equation

$$\frac{\partial T_\lambda(t,x_1)}{\partial t} = \kappa \frac{\partial^2 T_\lambda(t,x_1)}{\partial x_1^2} + (\lambda^2 - \lambda v(x_1)) T_\lambda(t,x_1) \tag{1.1.15}$$

with initial condition

$$T_\lambda(0,x_1) = \int_{-\infty}^{\infty} T(0,x_1,x_2) e^{\lambda x_2}\, dx_2. \tag{1.1.16}$$

This is a space continuum version of equation (1.1.1).

• Random motion in random media

The parabolic Anderson model is an example of a class of models for random motion in random media. Indeed, from the Feynman-Kac formula (1.1.5) we see that, in order to understand the solution of (1.1.1), we have to understand how X^κ sees ξ. This immediately links (1.1.1) to the model of *random walk in random scenery,* where the typical behaviour of $\int_0^t \xi(X^\kappa(s), s)\, ds$ is studied rather than the large deviation behaviour of $\int_0^t \xi(X^\kappa(s), t-s)\, ds$. Equation (1.1.1) is also deeply connected to random polymer models, since $\int_0^t \xi(X^\kappa(s), t-s)\, ds$ may be considered as a Hamiltonian and the Feynman-Kac formula is the corresponding partition function. *Random walks in random environment* are in the same spirit. Here, instead of the random branching rates as explained in the lines preceding (1.1.7), the transition probabilities of the random walk are random. All these models have in common that two types of randomness interact with each other and it is the goal to understand this interaction.

1.2 The parabolic Anderson model in a static random environment

In this section we give an overview of the state of the art of the parabolic Anderson equation when ξ does not depend on time, i.e., when (1.1.1) is of the form

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \xi(x) u(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{1.2.1}$$

Here, $\xi = \{\xi(x), x \in \mathbb{Z}^d\}$ is a collection of i.i.d. random variables with law $\mathbb{P}$. We refer to (1.2.1) as the *static PAM.* What follows is based on the overview articles of Gärtner and König [GK05] and König and Wolff [KW13], and we refer the reader to these sources for a more detailed presentation.

The first rigorous mathematics on the static PAM can be traced back to the works of Gärtner and Molchanov [GM90, GM98] in 1990 and 1998, respectively. In these impressive papers a complete answer to the questions of existence and uniqueness of solutions to (1.2.1) is provided. Moreover, some geometric properties of the u-field are derived. In particular, it is shown that, under a mild assumption on the moments of ξ and under a weak condition on the negative tails of ξ, the unique non-negative solution of (1.2.1) is given by the Feynman-Kac formula

$$u(x,t) = E_x \left(\exp\left\{ \int_0^t \xi(X^\kappa(s))\, ds \right\} u_0(X^\kappa(t)) \right). \tag{1.2.2}$$

Three main questions have guided the research on the static PAM.

- What is the asymptotic behaviour of $u(\cdot,t)$ as $t\to\infty$?
- Where does the main mass of $u(\cdot,t)$ come from? Which regions contribute most to $u(\cdot,t)$? What determines these regions? How many are there and how far are they from each other?
- What do the typical shapes of the potential ξ and the solution $u(\cdot,t)$ look like?

These three questions may also be characterized by the behaviour of the typical paths $\{X^\kappa(s)\colon\ 0\le s\le t\}$ that contribute most to the Feynman-Kac formula in (1.2.2). On the one hand, the typical paths should aim at finding spots where ξ obtains high values, to make the integral in (1.2.2) large. On the other hand, the probability for far away excursions is small, so that X^κ has to find a compromise between moving very quickly towards a region with exceptionally high values of ξ and performing typical excursions. The first order approximation to u comes from those paths that find such a good compromise. The second order approximation to u comes from the precise manner in which the paths move, i.e., from the geometry of the ξ-field, see [GM98].

The notion of *intermittency* has played a major role in the investigation of the above mentioned questions. Intermittency means that the main contribution to the solution comes from small islands that have large distances to each other and make the u-field look irregular. This irregularity can be quantified by looking at the moments of u in the following way.

Definition 1.2.1. *Fix $p\in\mathbb{N}$ and let*

$$\Lambda_p(\kappa,t)=\frac{1}{p}\log\mathbb{E}[u(0,t)^p]. \tag{1.2.3}$$

The solution of (1.2.1) *is called p-intermittent for $p\ge 2$, when* $\lim_{t\to\infty}[\Lambda_p(\kappa,t)-\Lambda_{p-1}(\kappa,t)]=\infty$.

In [GK05] the following argument was given to explain why Definition 1.2.1 indeed pertains to the geometric picture given above. Suppose that Definition 1.2.1 is fullfilled for some $p\in\mathbb{N}\setminus\{1\}$ and let l_p be such that $\Lambda_{p-1}(\kappa,\cdot)\ll l_p(\cdot)\ll\Lambda_p(\kappa,\cdot)$. Then

$$\begin{aligned}\mathbb{P}\Big(u(0,t)>e^{l_p(t)}\Big)&\le e^{-(p-1)l_p(t)}\mathbb{E}\Big(u(0,t)^{p-1}\Big)\\&=\exp\Big\{(p-1)\Lambda_{p-1}(\kappa,t)-(p-1)l_p(t)\Big\}\to 0.\end{aligned} \tag{1.2.4}$$

Hence, by the stationarity of $u(\cdot,t)$, the density of the point process

$$\Gamma(t)=\{x\in\mathbb{Z}^d:\ u(x,t)>e^{l_p(t)}\} \tag{1.2.5}$$

vanishes as $t \to \infty$. However,

$$\mathbb{E}\Big(u(0,t)^p \mathbb{1}\{u(0,t) \le e^{l_p(t)}\}\Big) \le e^{pl_p(t)} = e^{pl_p(t)-p\Lambda_p(\kappa,t)}\mathbb{E}(u(0,t)^p) = o(\mathbb{E}(u(0,t)^p)), \tag{1.2.6}$$

so that

$$\mathbb{E}\Big(u(0,t)^p\Big) \sim \mathbb{E}\Big(u(0,t)^p \mathbb{1}\{u(0,t) > e^{l_p(t)}\}\Big). \tag{1.2.7}$$

Birkhoff's ergodic theorem yields for a large centered box $B \subseteq \mathbb{Z}^d$,

$$|B|^{-1} \sum_{x \in B} u(x,t)^p \approx |B|^{-1} \sum_{x \in B \cap \Gamma(t)} u(x,t)^p. \tag{1.2.8}$$

Hence Definition 1.2.1 means that the p-th moment of $u(0,t)$ is generated by the high values of $u(\cdot,t)$ on the thin set $\Gamma(t)$. The drawback however, is that the above approach does not yield any information about the geometric structure of $\Gamma(t)$. In what follows we refer to the connected components of $\Gamma(t)$ as relevant islands. Theorem 3.2 of [GM90] reads as follows.

Theorem 1.2.2. *Let ξ be an i.i.d. field of real-valued random variables. Assume that $\xi(0)$ has finite exponential moments of all positive orders. Then u is p-intermittent for all $p \in \mathbb{N} \setminus \{1\}$ when ess sup $[\xi(0)] = \infty$.*

Since [GM90, GM98] many more results were found. In particular, van der Hofstad, König and Mörters [HKM06] proved that exactly four qualitatively different types of asymptotic behaviour of u can occur. These four universality classes depend on the upper tail of the distribution of ξ. It turned out that the double-exponential distribution

$$\mathbb{P}(\xi(0) > r) = \exp\{-e^{r/\rho}\}, \quad r \in \mathbb{R}, \tag{1.2.9}$$

where $\rho \in (0,\infty)$ is a parameter, plays an important role. In terms of this distribution the four universality classes of the static PAM may be described as follows.

1. **The single peak case:** This is the boundary case with $\rho = \infty$, corresponding to cases beyond the double exponential distribution. Here the relevant islands shrink to single sites as $t \to \infty$.
2. **The double-exponential case:** This is the case with $\rho \in (0,\infty)$. Here the relevant islands stay bounded as $t \to \infty$.
3. **The almost bounded case:** Let ξ be bounded from above and in the vicinity of the distribution

$$\mathbb{P}(\xi(0) > -r) = \exp\{-Cr^{-\gamma/(1-\gamma)}\}, \quad \gamma \in (0,1), \tag{1.2.10}$$

as $r \to 0$. The almost bounded case is an interpolation between the double-exponential distribution with $\rho = 0$ and the distribution (1.2.10) with $\gamma = 1$. Here the sizes of the relevant islands grow slower than any power of t to infinity.
4. **The bounded case:** Here the sizes of the relevant islands grow at least as fast as some power of t as $t \to \infty$.

In some cases more is known. König, Lacoin, Mörters and Sidorova [KLMS09] showed that if ξ is Pareto-distributed, then with high probability there is only one relevant island. A similar result was later obtained by Lacoin and Mörters [LM12] for the exponential distribution, and was extended by Sidorova and Twarovski [ST14] and Fiodorov and Muirhead [FM14] to the case of Weibull-distributed potentials. These potentials fall into the first universality class. For the double-exponential distribution, which falls into the second universality class, the concentration phenomenon on a single island has been explored by Biskup and König [BK14].

1.3 The parabolic Anderson model in a dynamic random environment

In this section we focus on equation (1.1.1) when ξ depends on time. This will be referred to as *dynamic PAM*. Unlike the static case, which was treated in the previous section, much less is known for the dynamic case. Most studies were concerned with the exponential growth rate, namely, the p-th *annealed* Lyapunov exponent

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}[u^p(0,t)], \qquad p \in \mathbb{N} \tag{1.3.1}$$

and the *quenched* Lyapunov exponent, which is the almost sure limit

$$\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t). \tag{1.3.2}$$

These were investigated as a function of the diffusion constant κ.

In the next two sections we summarize the literature on the dynamic PAM.

1.3.1 White noise

This section is a copy of Section 1.3.1 in Gärtner, den Hollander, Maillard [GdHM12].

Carmona and Molchanov [CM94] obtained a qualitative description of both the *quenched* and the *annealed* Lyapunov exponents when ξ is white noise, i.e.,

$$\xi(x,t) = \frac{\partial}{\partial t} W(x,t), \tag{1.3.3}$$

where $W = (W_t)_{t\geq 0}$ with $W_t = \{W(x,t)\colon\ x \in \mathbb{Z}^d\}$ is a space-time field of independent Brownian motions. This choice is special because the increments of ξ are *independent in space and time.* They showed that if $u(\cdot,0)$ has compact support (e.g. $u(\cdot,0) = \delta_0(\cdot)$ as in (1.1.4)), then the quenched Lyapunov exponent $\lambda_0^{u_0}(\kappa)$ defined in (1.3.2) exists and is constant ξ-a.s., and is independent of $u(\cdot,0)$. Moreover, they found that the asymptotics of $\lambda_0^{u_0}(\kappa)$ as $\kappa \downarrow 0$ is singular, namely, there are constants $C_1, C_2 \in (0,\infty)$ and $\kappa_0 \in (0,\infty)$ such that

$$C_1 \frac{1}{\log(1/\kappa)} \leq \lambda_0^{u_0}(\kappa) \leq C_2 \frac{\log\log(1/\kappa)}{\log(1/\kappa)} \qquad \forall\, 0 < \kappa \leq \kappa_0. \tag{1.3.4}$$

Subsequently, Carmona, Molchanov and Viens [CMV96], Carmona, Koralov and Molchanov [CKM01], and Cranston, Mountford and Shiga [CMS02], proved the existence of $\lambda_0^{u_0}$ when $u(\cdot,0)$ has non-compact support (e.g. $u(\cdot,0) \equiv 1$), showed that there is a constant $C \in (0,\infty)$ such that

$$\lim_{\kappa \downarrow 0} \log(1/\kappa)\, \lambda_0^{u_0}(\kappa) = C, \tag{1.3.5}$$

and proved that

$$\lim_{p \downarrow 0} \lambda_p(\kappa) = \lambda_0^{u_0}(\kappa) \quad \forall\, \kappa \in [0,\infty). \tag{1.3.6}$$

(These results were later extended to Lévy white noise by Cranston, Mountford and Shiga [CMS05], and to colored noise by Kim, Viens and Vizcarra [KVV08].) Further refinements on the behavior of the Lyapunov exponents were proven in Carmona and Molchanov [CM94] and Greven and den Hollander [GdH07]. In particular, it was shown that $\lambda_1(\kappa) = \frac{1}{2}$ for all $\kappa \in [0,\infty)$, while for the other Lyapunov exponents the following dichotomy holds (see Figs. 1.1–1.2):

- $d = 1, 2$: $\lambda_0^{u_0}(\kappa) < \frac{1}{2}$, $\lambda_p(\kappa) > \frac{1}{2}$ for $p \in \mathbb{N}\backslash\{1\}$, for $\kappa \in [0,\infty)$;
- $d \geq 3$: there exist $0 < \kappa_0 \leq \kappa_2 \leq \kappa_3 \leq \ldots < \infty$ such that

$$\lambda_0^{u_0}(\kappa) - \tfrac{1}{2} \begin{cases} < 0, & \text{for } \kappa \in [0,\kappa_0), \\ = 0, & \text{for } \kappa \in [\kappa_0,\infty), \end{cases} \tag{1.3.7}$$

  and

$$\lambda_p(\kappa) - \tfrac{1}{2} \begin{cases} > 0, & \text{for } \kappa \in [0,\kappa_p), \\ = 0, & \text{for } \kappa \in [\kappa_p,\infty), \end{cases} \qquad p \in \mathbb{N}\backslash\{1\}. \tag{1.3.8}$$

It was further shown in [CM94] that $\lambda_p(\kappa) > 1/2$ already implies the chain of inequalities $\lambda_p(\kappa) < \lambda_{p+1}(\kappa) < \cdots$, which yields p-intermittency, see (1.2.3). Moreover, variational formulas for κ_p were derived, which in turn led to upper and lower bounds on κ_p, and to the identification of the asymptotics of κ_p for $p \to \infty$ (κ_p grows linearly with p). In addition, it was shown that for every $p \in \mathbb{N}\backslash\{1\}$ there exists a $d(p) < \infty$ such that $\kappa_p < \kappa_{p+1}$ for $d \geq d(p)$. Moreover, it was shown that $\kappa_0 < \kappa_2$ in Birkner, Greven and den Hollander [BGdH08] ($d \geq 5$), Birkner and Sun [BS10] ($d = 4$), Berger and Toninelli [BT09], Birkner and Sun [BS11] ($d = 3$). Note that, by Hölder's inequality, all curves in Figs. 1.1–1.2 are distinct whenever they are different from $\frac{1}{2}$.

1.3.2 Interacting particle systems

This section is largely a copy of Sections 1.1 and 1.3.2 in Gärtner, den Hollander, Maillard [GdHM12].

Three examples for ξ which is dependent in *space and time* have received a special attention in recent years.

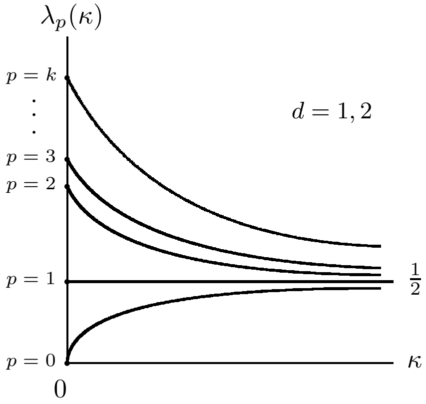

Figure 1.1: Quenched and annealed Lyapunov exponents when $d = 1, 2$ for white noise, with $\lambda_0(\kappa) = \lambda_0^{u_0}(\kappa)$.

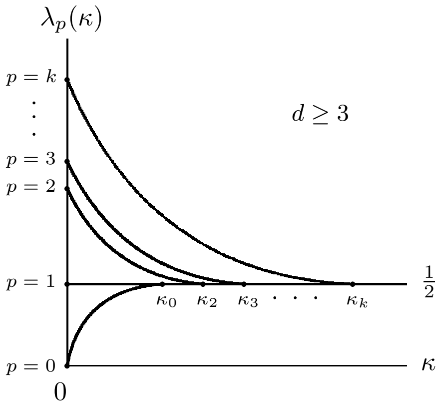

Figure 1.2: Quenched and annealed Lyapunov exponents when $d \geq 3$ for white noise, with $\lambda_0(\kappa) = \lambda_0^{u_0}(\kappa)$.

(1) *Independent Simple Random Walks* (ISRW) [Kipnis and Landim [KL99], Chapter 1]. Here, $\xi_t \in \Omega = (\mathbb{N} \cup \{0\})^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the number of particles at site x at time t. Under the ISRW-dynamics particles move around independently as simple random walks stepping at rate 1. ξ_0 is drawn according to the equilibrium ν_ρ with density $\rho \in (0,\infty)$, which is a Poisson product measure.

(2) *Symmetric Exclusion Process* (SEP) [Liggett [L85], Chapter VIII]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the presence ($\xi(x,t) = 1$) or absence ($\xi(x,t) = 0$) of a particle at site x at time t. Under the SEP-dynamics particles move around independently according to an irreducible symmetric random walk transition kernel at rate 1, but subject to the restriction that no two particles can occupy the same site. ξ_0 is drawn according to the equilibrium ν_ρ with density $\rho \in (0,1)$, which is a Bernoulli product measure.

(3) *Symmetric Voter Model* (SVM) [Liggett [L85], Chapter V]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the opinion of a voter at site x at time t. Under the SVM-dynamics each voter imposes its opinion on another voter according to an irreducible symmetric random walk transition kernel at rate 1. ξ_0 is either drawn according to the equilibrium distribution ν_ρ with density $\rho \in (0,1)$, which is not a product measure or according to a Bernoulli product measure.

Let $\gamma > 0$. Kesten and Sidoravicius [KS03], and Gärtner and den Hollander [GdH06], considered the case where ξ is γ times the number of particles in the ISRW dynamics. The survival versus extinction pattern [KS03] and the annealed Lyapunov exponents [GdH06] were analyzed, in particular, their dependence on d, κ, γ and ρ. The case where ξ is a single random walk was studied by Gärtner and Heydenreich [GH06]. Gärtner, den Hollander and Maillard [GdHM07], [GdHM09], [GdHM10] subsequently considered the cases where ξ is γ times an exclusion process, respectively, γ times a voter model. In each of these cases, a fairly complete picture of the behavior of the annealed Lyapunov exponents was obtained, including the presence or absence of *intermittency*, i.e., $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for some or all values of $p \in \mathbb{N}\backslash\{1\}$ and $\kappa \in [0,\infty)$. Several conjectures were formulated as well. In what follows we describe these results in some more detail. We refer the reader to Gärtner, den Hollander and Maillard [GdHM08] for an overview.

It was shown in Gärtner and den Hollander [GdH06], and Gärtner, den Hollander and Maillard [GdHM07], [GdHM09], [GdHM10] that for ISRW, SEP and SVM in equilibrium the function $\kappa \mapsto \lambda_p(\kappa)$ satisfies:

- If $d \geq 1$ and $p \in \mathbb{N}$, then the limit in (1.3.1) exists for all $\kappa \in [0,\infty)$. Moreover, if $\lambda_p(0) < \infty$, then $\kappa \mapsto \lambda_p(\kappa)$ is finite, continuous, strictly decreasing and convex on $[0,\infty)$.
- There are two regimes (we summarize results only for the case where the random walk transition kernel has finite second moment and we recall that ρ describes the density of particles):
  - *Strongly catalytic regime* (see Fig. 1.3):
    - ISRW: $d = 1,2$, $p \in \mathbb{N}$ or $d \geq 3$, $p \geq 1/\gamma G_d$: $\lambda_p \equiv \infty$ on $[0,\infty)$. (G_d is the Green function at the origin of simple random walk.)

    - SEP: $d = 1, 2$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0, \infty)$.
    - SVM: $d = 1, 2, 3, 4$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0, \infty)$.
  - *Weakly catalytic regime* (see Fig. 1.4–1.5):
    - ISRW: $d \geq 3$, $p < 1/\gamma G_d$: $\rho\gamma < \lambda_p < \infty$ on $[0, \infty)$.
    - SEP: $d \geq 3$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0, \infty)$.
    - SVM: $d \geq 5$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0, \infty)$.
- For all three dynamics, in the weakly catalytic regime $\lim_{\kappa\to\infty} \kappa[\lambda_p(\kappa) - \rho\gamma] = C_1 + C_2 p^2 1_{\{d=d_c\}}$ with $C_1, C_2 \in (0, \infty)$ and d_c a critical dimension: $d_c = 3$ for ISRW, SEP and $d_c = 5$ for SVM.
- Intermittent behavior:
  - In the strongly catalytic regime, there is no intermittency for all three dynamics.
  - In the weakly catalytic regime, there is full intermittency for:
    - all three dynamics when $0 \leq \kappa \ll 1$.
    - ISRW and SEP in $d = 3$ when $\kappa \gg 1$.
    - SVM in $d = 5$ when $\kappa \gg 1$.

Note: For SVM the convexity of $\kappa \mapsto \lambda_p(\kappa)$ and its scaling behavior for $\kappa \to \infty$ have not actually been proved, but have been argued on heuristic grounds.

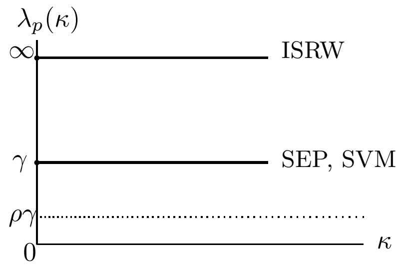

Figure 1.3: Triviality of the annealed Lyapunov exponents for ISRW, SEP, SVM in the strongly catalytic regime; i.e. below the critical dimension

Recently, there has been further progress for the case where ξ consists of 1 random walk (Schnitzler and Wolff [SW12]) or n independent random walks (Castell, Gün and Maillard [CGM12]), ξ is the SVM (Maillard, Mountford and Schöpfer [MMS12]), and for the trapping version of the PAM with $\gamma \in (-\infty, 0)$ (Drewitz, Gärtner, Ramírez and Sun [DGRS12]).

The first attempt to analyze the quenched Lyapunov exponent (1.3.2) for a dynamic ξ that has correlations in space and time was made in [GdHM12]. Several properties, such as (1) the existence of the quenched Lyapunov exponent for initial conditions u_0

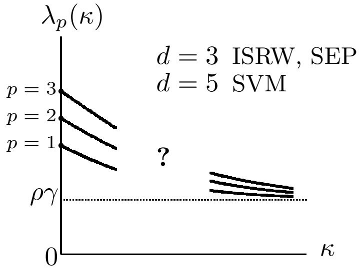

Figure 1.4: Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime at the critical dimension.

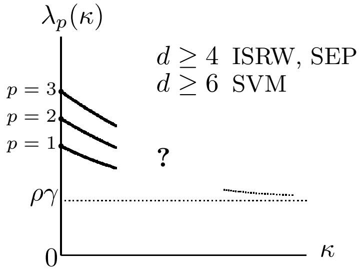

Figure 1.5: Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime above the critical dimension.

with compact support; (2) the Lipschitz continuity of the map $\kappa \mapsto \lambda_0(\kappa)$ outside any neighborhood of zero; (3) the non-Lipschitz continuity of $\kappa \mapsto \lambda_0(\kappa)$ at $\kappa = 0$; (4) the strict lower bound $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$, were derived under weak assumptions on ξ. Moreover assume that ξ satisfies a strong assumption on its occupation times, i.e., ξ is a Markov process such that uniformly in its initial configuration η there is a $c > 0$ such that for all $\mu, t > 0$

$$\mathbb{E}_\eta\left(\exp\left\{\mu\int_0^t (\xi(0,s) - \mathbb{E}(\xi(0,0)))\,ds\right\}\right) \leq \exp\{c\mu^2 t\}, \tag{1.3.9}$$

where $\mathbb{E}_\eta$ is the expectation of ξ when started at η, then even more can be said. Namely, the following asymptotic behaviour holds:

$$\limsup_{\kappa\to 0} \frac{\log(1/\kappa)}{\log\log(1/\kappa)}[\lambda_0(\kappa) - \mathbb{E}(\xi(0,0)] < \infty. \tag{1.3.10}$$

• *The goal of the work in this thesis is to broaden the understanding of the quenched Lyapunov exponent under assumptions on ξ that are as weak as possible.*

1.4 Overview of the results

In this section we give an overview over the main results, that will be presented in Chapter 2 and Chapter 3. Most of the results require space-time mixing conditions, which we call Gärtner-mixing conditions, whose precise definition can be found in the respective chapters.

1.4.1 Results of Chapter 2: basic properties of the quenched Lyapunov exponent

In Chapter 2 we derive basic properties of the solution of equation (1.1.1), such as existence and uniqueness, and of the quenched Lyapunov exponent (1.3.2) such as finiteness, independence on the initial condition and the non Lipschitz continuity in $\kappa = 0$.

Definition 1.4.1. *A field $q = \{q(x)\colon\ x \in \mathbb{Z}^d\}$ is said to be percolating from below if for every $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{Z}^d\colon\ q(x) \le \alpha\}$ contains an infinite connected component. Otherwise q is said to be non-percolating from below.*

It was shown in [GM90] that if q is non-percolating from below, then (1.2.1) has at most one non-negative solution. We will show that a similar condition suffices for *dynamic* ξ, namely, (1.1.1) has at most one non-negative solution when there is a $T > 0$ such that

$$q^T = \{q^T(x)\colon\ x \in \mathbb{Z}^d\} \quad \text{with} \quad q^T(x) = \sup_{0 \le t \le T} q(x,t). \tag{1.4.1}$$

Theorem 1.4.2. ***[Uniqueness]*** *Consider a deterministic $q\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ such that:*
(1) There is a $T > 0$ such that q^T is non-percolating from below.
(2) $q^T(x) < \infty$ for all $T > 0$ and $x \in \mathbb{Z}^d$.
Then the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + q(x,t) u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \ge 0, \tag{1.4.2}$$

has at most one non-negative solution.

Theorem 1.4.3. ***[Existence]*** *Suppose that:*
(1) $s \mapsto \xi(x,s)$ is locally integrable for every x, ξ-a.s., i.e., for every compact subset $K \subset [0,\infty)$ and every x the map $s \mapsto \mathbb{1}\{s \in K\}\xi(x,s)$ is integrable ξ-a.s.
(2) $\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \ge 0$.
Then the function defined by the Feynman-Kac formula

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\, ds\right\} u_0(X^\kappa(t))\right) \tag{1.4.3}$$

solves (1.1.1) with initial condition u_0.

From now on we assume that ξ satisfies the conditions of Theorems 1.4.2–1.4.3 (where q is replaced by ξ in Theorem 1.4.2).

The following two results concern the finiteness and existence of $\lambda_0^{u_0}(\kappa)$. See Corollary 1.4.8 for examples of ξ, that satisfy the assumptions of Theorems 1.4.4 and 1.4.5.

Theorem 1.4.4. ***[Finiteness]*** *If ξ is Gärtner-positive-hyper-mixing, then $\lambda_0^{\delta_0}(\kappa) < \infty$.*

From now on we also assume that ξ satisfies the conditions of Theorem 1.4.4. The following result extends Gärtner, den Hollander and Maillard [GdHM12], Theorem 1.1, in which it was shown that for the initial condition $u_0 = \delta_0$ the quenched Lyapunov exponent exists and is constant ξ-a.s.

Theorem 1.4.5. ***[Initial Condition]*** *If ξ is reversible in time or symmetric in space, type-II Gärtner-mixing and Gärtner-negative-hyper-mixing, then $\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, is constant ξ-a.s., and is independent of u_0.*

The next results concern the dependence on κ of $\lambda_0(\kappa)$ and are valid under certain conditions on the occupation times of ξ, which are similar to (1.3.9) but still weaker than (1.3.9).

Theorem 1.4.6. ***[Continuity at $\kappa = 0$]*** *If ξ is Gärtner-regular, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is continuous at zero.*

Theorem 1.4.7. ***[Not Lipschitz at $\kappa = 0$]*** *If ξ is Gärtner-volatile, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is not Lipschitz continuous in zero.*

1.4.1.1 Examples

Corollary 1.4.8. ***[Examples for Theorems 1.4.4–1.4.5]***
(1) Let $X = (X_t)_{t\geq 0}$ be a stationary and ergodic $\mathbb{R}$-valued Markov process. Let $(X_\cdot(x))_{x\in\mathbb{Z}^d}$ be independent copies of X. Define ξ by $\xi(x,t) = X_t(x)$. If

$$\mathbb{E}\left[e^{q \sup_{s\in[0,1]} X_s}\right] < \infty \qquad \forall\, q \geq 0, \tag{1.4.4}$$

then ξ fulfills the conditions of Theorem 1.4.4. If, moreover, the left-hand side of (1.4.4) is finite for all $q \leq 0$, then ξ satisfies the conditions of Theorem 1.4.5.
(2) Let ξ be the zero-range process with rate function $g\colon \mathbb{N}_0 \to (0,\infty)$, $g(k) = k^\beta$, $\beta \in (0,1]$, and transition probabilities given by a simple random walk on $\mathbb{Z}^d$. If ξ starts from the product measure π_ρ, $\rho \in (0,\infty)$, with marginals

$$\pi_\rho\big\{\eta \in \mathbb{N}_0^{\mathbb{Z}^d} \colon \eta(x) = k\big\} = \begin{cases} \gamma\, \frac{\rho^k}{g(1)\times\cdots\times g(k)}, & \text{if } k > 0, \\ \gamma, & \text{if } k = 0, \end{cases} \tag{1.4.5}$$

where $\gamma \in (0,\infty)$ is a normalization constant, then ξ satisfies the conditions of Theorems 1.4.4–1.4.5.

Corollary 1.4.9. ***[Examples for Theorem 1.4.6]***
(1) If ξ is a bounded interacting particle system in the so-called $M < \varepsilon$ regime (see Liggett [L85]), then the conditions of Theorem 1.4.6 are satisfied.
(2) If ξ is the exclusion process with an irreducible, symmetric and transient random walk transition kernel, then the conditions of Theorem 1.4.6 are satisfied.
(3) If ξ is the dynamics defined by

$$\xi(x,t) = \sum_{y\in\mathbb{Z}^d} \sum_{j=1}^{N_y} \delta_{Y_j^y(t)}(x), \tag{1.4.6}$$

where $\{Y_j^y\colon\, y \in \mathbb{Z}^d, 1 \le j \le N_y, Y_j^y(0) = y\}$ is a collection of independent continuous-time simple random walks jumping at rate one, and $(N_y)_{y\in\mathbb{Z}^d}$ is a Poisson random field with intensity ν for some $\nu \in (0,\infty)$. If $d \ge 3$, then the conditions of Theorem 1.4.6 are satisfied.

Remark 1.4.10. *Corollaries 1.4.8–1.4.9 list only a few examples that match the conditions. It is a separate problem to verify these conditions for as broad a class of interacting particle systems as possible.*

1.4.2 Result of Chapter 3: space-time ergodicity for the quenched Lyapunov exponent

In Chapter 3 we prove the following.

Theorem 1.4.11. *If $u_0 = \delta_0$ and ξ is Gärtner-hyper-mixing, then*

$$\lim_{\kappa\to\infty} \lambda_0^{\delta_0}(\kappa) = \mathbb{E}(\xi(0,0)). \tag{1.4.7}$$

• **Examples:** The two examples listed in Corollary 1.4.8 are examples of fields that are Gärtner-hyper-mixing.

Theorem 1.4.11 yields a partial answer to the question: Which random walk paths give the main contribution to the Feynman-Kac formula in (1.4.3)? Indeed, Theorem 1.4.11 shows that, for large κ and any *dynamic* ξ that is Gärtner-hyper-mixing, the main contribution comes from those paths that spend most of their time in regions where ξ looks typical. This is in sharp contrast with what is known for the parabolic Anderson model with a *static* i.i.d. random environment $\xi = \{\xi(x)\colon\, x \in \mathbb{Z}^d\}$. In this case the main contribution to the Feynman-Kac formula in (1.4.3) comes from those paths that are localized, in the sense that they spend almost all of their time in regions where ξ is large. The latter implies that for bounded ξ the quenched Lyapunov exponent equals $\operatorname{ess\,sup} \xi(0)$ instead of $\mathbb{E}(\xi(0))$.

Theorem 1.4.11, jointly with the results in Section 1.4.1, suggest the picture of $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ in Figure 1.6.

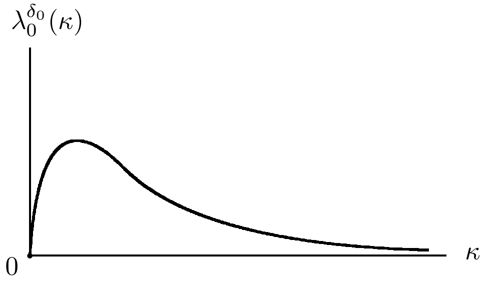

Figure 1.6: Qualitative behavior of $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$.

1.5 Open problems

We close this introduction by listing some open problems. We hope to address some of these in future works.

- Show that the graph of the quenched Lyapunov exponent indeed looks as indicated in Figure 1.6, i.e., show that $\kappa \mapsto \lambda_0(\kappa)$ has a unique maximum.
- Investigate (1.1.1) when the discrete Laplacian is replaced by a random discrete Laplacian. This amounts to replacing the simple random walk in the Feynman-Kac formula (1.1.5) by a random walk in a random environment.
- Determine the rate at which $\lambda_0(\kappa)$ converges to $\mathbb{E}(\xi(0,0))$ as $\kappa \to \infty$, i.e., determine a function $f : [0,\infty) \to [0,\infty)$ such that $\lim_{\kappa\to\infty} f(\kappa) = \infty$ and such that $f(\kappa)[\lambda_0(\kappa) - \mathbb{E}(\xi(0,0)]$ is bounded from above and below as $\kappa \to \infty$.
- Investigate the fluctuations of $\lambda_0(\kappa)$ as $\kappa \to \infty$.
- Investigate whether or not the parabolic Anderson model falls in the same universality class as the KPZ-equation.