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Part II Long–Range Percolation

4 Introduction to Part II

This chapter serves as an introduction to Chapters 5 and 6, where two long–range percolation models are studied. In Section 4.1 we give a short introduction to Bernoulli bond percolation, which is the most standard of all percolation models. In Section 4.2 the first model is presented and results of Chapter 5 are listed. Section 4.3 contains a short introduction to the second model, random interlacements, and moreover the result of Chapter 6 is listed.

4.1 Bernoulli bond percolation

This section is based on Chapter 1 of Grimmett [G00].

The standard introductory example of percolation theory is the following: Suppose that a large porous stone is immersed into a bucket of water. What is the probability that the center of the stone gets wet? This can be modelled in terms of a simple stochastic model, nowadays known as percolation model, due to Broadbent and Hammersley [BH57]. Let $p \in [0,1]$ and declare each edge of $\mathbb{Z}^d$ to be open, independently of all other edges, with probability p and closed otherwise. The edges of $\mathbb{Z}^d$ represent the passageways of the stone, and the parameter p is the proportion of passages that are broad enough to allow water to pass along them. The stone is modelled as a large finite subset of $\mathbb{Z}^d$ and a vertex x inside the stone is wet when there is a path of open edges from x to the boundary of the stone. Percolation theory is mainly concerned with the study of such open paths. Questions of interest are for instance: (1) *Is there an infinite cluster of open paths?* (2) *How many of such infinite clusters are there?* (3) *Consider the graph whose edges are precisely the open edges and whose vertices are the endpoints of those edges. What geometric properties does this graph have?*
The above described model is referred to as Bernoulli bond percolation and it is the most studied of all percolation models. The motivations are manyfold. (1) The model is easy to formulate but not unrealistic in qualitative predictions for random media. (2) It is a simple model in which the phenomenon of a phase transition may be observed, i.e., when $d \geq 2$ it can be shown that there is a $p_c \in (0,1)$ such that for all $p < p_c$ all clusters of open edges are finite, whereas for all $p > p_c$ there is with probability one a (unique) infinite cluster of open edges. (3) It serves as a jumpboard to develop techniques for more complicated models and even leads to new branches of mathematics. A beautiful example is the theory of the Schramm-Loewner evolution, which arises as the scaling limit of general two-dimensional percolation models in statistical physics at criticality including Bernoulli percolation itself. (4) It leads to many beautiful conjectures that are

easy to state but hard to prove. (5) Partial results for models with a complex dependency structure can be obtained by comparing them with Bernoulli bond percolation. Grimmett [G00] states the following example. Consider a physical model having a parameter T, that is interpreted as the temperature of the system. It may be expected that there exists a critical value T_c marking a phase transition. While this fact may itself be unproven, it may be possible to prove by comparison with Bernoulli bond percolation that the behaviour of the process for small T is qualitatively different from that for large T.

4.2 Brownian paths homogeneously distributed in space

The model

For $\lambda > 0$, let $(\Omega, \mathcal{A}, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \mathrm{Leb}_d$ is defined, where Leb_d is the d-dimensional Lebesgue measure. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\ x \in \mathcal{E}\}$ such that $B_0^x = x$ for each $x \in \mathcal{E}$ and such that $(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$. For $t, r \geq 0$ we study the *occupied* set

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r), \tag{4.2.1}$$

where $\mathcal{B}(y, r)$ denotes the ball with respect to the euclidean norm around $y \in \mathbb{R}^d$ with radius r. If $d \leq 3$, then we put $r = 0$. The reason for that will become clearer when discussing the results. In the remainder of this section we write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.
We are interested in the percolative properties of $\mathcal{O}_{t,r}$: Is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: Is there a percolation transition in t?

Motivation and related models

The model described above fits into the class of continuum percolation models, which have been studied intensively by both mathematicians and physicists. Their first appearance can be traced back (at least) to Gilbert [G61] under the name of random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $R > 0$. He did this by connecting each two points of a Poisson point process on $\mathbb{R}^2$ whenever their distance is less than R. Another application, mentioned in his work is the modeling of a contagious infection. Here, each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: Consider a point process (e.g. a Poisson point process) and attach to each of its points a geometric object, like a disk of random radius (Boolean model) or a segment of random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: to each point of a Poisson point process we attach a Brownian path (a path of a Wiener sausage when $d \geq 4$). This can be seen as a model of defects that are randomly distributed in a material and are propagating at random. We can think for example of an (infinite) piece of wood containing (homogeneously distributed) worms, where each worm eats its way through the piece of wood at random (see Menshikov, Molchanov and Sidorenko [MMS88] for other physical motivations of continuum percolation). The informal description above is reminiscent of (and actually borrowed from) the problem of the disconnection of a cylinder by a random walk, which itself is linked to interlacement percolation [S10]. The latter is defined as the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, starting from the uniform distribution and running up to time uN^d in the limit as $N \uparrow \infty$. Here, u plays the role of an intensity parameter for the interlacement set. However, even though the model of random interlacements and our model seem to share some similarities, there is an important difference: in the interlacement model the number of trajectories that enter a ball of radius R scales like cR^{d-2} for some $c > 0$, whereas in our model it is at least of order R^d.

Another motivation for studying our model is that it should arise as the scaling limit of a class of discrete dependent percolation models, namely a system of independent finite-time random walks homogeneously distributed on $\mathbb{Z}^d$. The latter can also be seen as a system of non-interacting ideal polymer chains.

Results

Fix $\lambda > 0$.

Theorem 4.2.1. ***[No percolation for*** $d = 1$*] Let $d = 1$. Then, for all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*

Theorem 4.2.2. ***[Percolation phase transition and uniqueness for*** $d = 2, 3$*] Let $d = 2, 3$. Then there exists a $t_c = t_c(\lambda, d) > 0$ such that, for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster whereas, for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*

Let $d \geq 4$ and $r > 0$. We denote by $\lambda_c(r)$ the critical value such that for all $\lambda < \lambda_c(r)$ the set $\mathcal{O}_{0,r}$ almost surely does not contain an unbounded cluster, whereas for $\lambda > \lambda_c(r)$ it does. Gouéré [G08] showed that $\lambda_c(r) > 0$ for $r > 0$ and $\lim_{r \to 0} \lambda_c(r) = \infty$.

Theorem 4.2.3. ***[Percolation phase transition and uniqueness for*** $d \geq 4$*] Let $d \geq 4$, and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then there exists a $t_c = t_c(\lambda, d, r) > 0$ such that, for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster whereas, for $t > t_c$, $\mathcal{O}_{t,r}$ has almost surely a unique unbounded cluster.*

Comments on the results

Theorems 4.2.1–4.2.3 describe a phase transition in t. It would be possible to play with the intensity λ instead. Indeed, when we multiply the intensity λ by a factor η

we change the typical distance between two Poisson points by a factor $\eta^{-1/d}$. By scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead. Hence, tuning λ boils down to tuning t.

It is worthwhile to mention that Theorem 4.2.2 is stated only in the case $r = 0$, which is the case of interest to us. The result is the same when $r > 0$, up to minor modifications. However, if $d \geq 4$, then the paths of two independent d-dimensional Brownian motions starting at different points do not intersect and r has to be chosen positive, otherwise no percolation phase transition occurs.

To sum up, the above settle the first questions typically asked when studying a new percolation model. Many challenges are open. One may wonder, for instance, how fast is the decay of the probability (in the supercritical regime) that a ball of a certain size, centered at the origin, is contained in the vacant set. Moreover, it would be interesting to investigate the scaling behaviour of t_c in dimension $d \geq 4$ as r tends to zero. One could ask for sharp upper and lower bounds on t_c. Finally, it is not clear whether percolation occurs at t_c or not.

4.3 Random interlacements

Introduction to the model

The model of random interlacements has been introduced by Sznitman [S10] as a family of random subsets of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$, $u \geq 0$, where u plays the role of an intensity parameter. $\mathcal{I}^u$ locally "looks like" the trace of a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ run up to time uN^d (see Windisch [W08], Teixeira and Windisch [TW11]). With the help of the inclusion-exclusion formula the distribution of the set $\mathcal{I}^u$ can also be characterized as

$$\mathbb{P}[K \cap \mathcal{I}^u = \emptyset] = e^{-u\,\mathrm{cap}(K)}, \qquad K \subset\subset \mathbb{Z}^d.$$

Here, $\mathrm{cap}(K)$ denotes the capacity of the compact set K, defined as

$$\mathrm{cap}(K) := \sum_{x \in K} e_K(x), \quad \text{with} \quad \mathrm{e}_K(x) := P_x\big[\widetilde{H}_K = \infty\big]\mathbb{1}_{\{x \in K\}}, \tag{4.3.1}$$

where P_x denotes the law of simple random walk $w = (w(n))_{n \in \mathbb{N}_0}$ started at $x \in \mathbb{Z}^d$ and $\widetilde{H}_K$ denotes the first hitting time of the set K by the random walk:

$$\widetilde{H}_K(w) = \inf\{n \in \mathbb{N}\colon\ w(n) \in K\}. \tag{4.3.2}$$

In a more constructive fashion, random interlacements at level u can also be obtained by considering the trace of the elements in the support of a Poisson point process with intensity parameter u, taking values in the space of locally finite measures on doubly infinite simple random walk trajectories modulo time shifts. This constructive definition suggests that the model exhibits long–range dependence. Indeed the asymptotics

$$\mathrm{Cov}(\mathbb{1}_{x\in\mathcal{I}^u}, \mathbb{1}_{y\in\mathcal{I}^u}) \sim c(u)|x-y|_2^{-(d-2)}, \tag{4.3.3}$$

where $|\cdot|_2$ denotes the euclidean norm on $\mathbb{Z}^d$ (and similarly for $\mathcal{I}^u$ replaced by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$) holds for $|x-y|_2 \to \infty$, as can be deduced from (0.11) in Sznitman [S10]. As a consequence, standard techniques from Bernoulli percolation do not apply. For example, due to (4.3.3) the Peierls argument and the van den Berg-Kesten inequality break down. The long–range dependence also entails that random interlacements neither stochastically dominate nor can be dominated by Bernoulli percolation (see Remark 1.6 (1) of [S10]). Moreover random interlacements do not fulfill the finite energy property (see Remark 2.2 (3) of [S10]). These features make the model more appealing, and at the same time more complicated to investigate.

State of the art and motivation

During the past couple of years there has been intensive research on random interlacements. Basic properties such as the shift-invariance, ergodicity and connectedness of $\mathcal{I}^u$ have been established in Sznitman [S10]. Since then, a deeper understanding has been obtained of the geometry of random interlacements. Ráth and Sapozhnikov [RS11] have shown transience for random interlacements $\mathcal{I}^u$ throughout the whole range of parameters $u \in (0,\infty)$. Ráth and Sapozhnikov [RS10] and Procaccia and Tykesson [PT11] have shown that any two points of the set $\mathcal{I}^u$ can be connected by using at most $\lceil d/2 \rceil$ trajectories from the constructive definition described above. Recently, with the help of extensions of the techniques in [RS10], this result has been generalized to an arbitrary number of points by Lacoin and Tykesson [LT12]. Another step in showing that the geometry of random interlacements resembles that of $\mathbb{Z}^d$ has been undertaken by Černý and Popov [CP12], who prove that the chemical distance (also called graph distance or internal distance) in the set $\mathcal{I}^u$ is comparable to that of $\mathbb{Z}^d$. Using this result, they prove a shape theorem for balls in $\mathcal{I}^u$ with respect to the metric induced by the chemical distance.

It is particularly interesting to obtain a deeper understanding of the vacant set $\mathcal{V}^u$ and its geometry. On the one hand, this is more challenging than the investigation of $\mathcal{I}^u$, in the sense that one cannot directly take advantage of the many tools available for simple random walk, that have proven to be very helpful in understanding the set $\mathcal{I}^u$. On the other hand, it has been shown by Sznitman [S10] and Sidoravicius and Sznitman [SS09] that there exists a non-trivial percolation phase transition for $\mathcal{V}^u$ at some $u_*(d) \in (0,\infty)$ in the following sense: For $u > u_*(d)$ the vacant set $\mathcal{V}^u$ as a subgraph of $\mathbb{Z}^d$ contains only finite connected components (subcritical phase), whereas for $u \in [0, u_*(d))$ it contains an infinite connected component almost surely (supercritical phase). Using the techniques of Burton and Keane [BK89], and taking care of the difficulties that arise from the lack of the finite energy property for random interlacements, Teixeira [T09] has shown uniqueness of the infinite connected component of $\mathcal{V}^u$ (denoted by $\mathcal{V}^u_\infty$) in the supercritical

phase.
To the best of our knowledge, our result is the first that is valid throughout most of the supercritical phase.

The result

Recall that a connected graph $G = (V, E)$ with finite degree, with vertex set V and edge set E, is called transient if simple random walk on G is transient.

Theorem 4.3.1. *Let $\varepsilon \in (0, 1)$. There is a $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that, for all $d \geq d_0$ and all $u \leq (1 - \varepsilon)u_*(d)$, the unique infinite connected component $\mathcal{V}^u_\infty$ of the vacant set $\mathcal{V}^u$ of random interlacements in $\mathbb{Z}^d$ is transient $\mathbb{P}$-a.s.*

Comments on the result

Theorem 4.3.1 provides a rough geometric description of the infinite connected component of the vacant set that is valid throughout most of the supercritical phase when d is large enough.

Besides the challenge to extend our result to the entire supercritical phase, it would be interesting to obtain a more precise understanding of $\mathcal{V}^u_\infty$. Results in this direction have been obtained in Drewitz, Ráth, Sapozhnikov and Procaccia, Rosenthal, Sapozhnikov [DRS12b, PRS13]. A key assumption in these papers is a local uniqueness property (in our context of $\mathcal{V}^u_\infty$), which roughly states that with high probability the second largest component in a macroscopic box is small compared to the largest connected component in the same box. This local uniqueness property has so far only been established for a non-degenerate part of the supercricital phase, and obtaining its validity throughout the whole supercritical phase would be an interesting topic for further investigation.