Author: Mai Hoang Bien
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Introduction

Classically, modules were used as a method in the study of representation theory. Since the 1950s, the scope of module theory has become much broader. An effective way to understand the behavior of an arbitrary module is to study its endomorphism ring. The goal of studying endomorphism rings, in our case, is to consider the decompositions of a module, so that the following examples concern direct sums and direct summands.

Let M_R be a right module over an arbitrary ring R and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. There is a classical fact that if S is a local ring, that is, the set of all non-invertible elements of S is a two-sided ideal of S, then M_R is indecomposable. In [15], Facchini collected several properties of M_R that hold when the endomorphism ring S of M_R is semilocal, that is, $S/J(S)$ is a semisimple artinian ring where $J(S)$ is the Jacobson radical of S. In fact, if S is semilocal, then:

1. M_R is a direct sum of finitely many indecomposable modules.
2. M_R is directly finite. That is, if $M_R \oplus N_R \cong M_R \oplus N'_R$, then $N_R \cong N'_R$.
3. If n is a positive integer and $M_R^n \cong N_R^n$, then $M_R \cong N_R$. This property is called *n-th root property.*
4. If N_R is a module isomorphic to a direct summand of M_R, then the endomorphism ring $\mathrm{End}(N_R)$ of N_R is also semilocal.
5. If N_R is an R-module with semilocal endomorphism ring, then the endomorphism ring $\mathrm{End}(M_R \oplus N_R)$ is also semilocal.

Moreover, the study of rings can be aided by the machinery provided by endomorphism rings.

It has now become difficult to establish the border between module theory and the theory of endomorphism rings.

The aim of this thesis is to study some classes of modules such as injective modules, Loewy modules and max modules, and their endomorphism rings. In fact, in Chapter 2, we consider the endomorphism ring of a square-free injective module. Here, an injective module is called *square-free* if M_R has no direct summand isomorphic to $N \oplus N$ for some non-zero direct summand N of M_R. It is well known that an injective module is indecomposable if and only if its endomorphism ring is local [33]. For convenience, some basic notions concerning injective modules are presented in Chapter 1. The main result of Chapter 2 is to prove that an injective module is square-free if and only if its endomorphism ring is quasi-duo.

In Chapter 3, we describe all maximal right (left, two-sided) ideals of the endomorphism ring of an injective module. All maximal right (left, two-sided) ideals of the endomorphism ring of a vector space over a division ring were described completely in [35].

In Chapter 4, we first consider the class of Loewy modules with finite Loewy invariants and their endomorphism rings. It is trivial that every artinian module is a Loewy module with finite Loewy invariants. Facchini proved that if the base ring is commutative then the class of Loewy modules with finite Loewy invariants coincides with the class of artinian modules [12]. In this Chapter, we present an example to show that this is not true for modules over non-commutative rings. In 1993, Camps and Dicks proved that the endomorphism ring of an artinian module is semilocal [9]. We prove that every Loewy module with finite Loewy invariants also has a semilocal endomorphism ring. We then obtain similar results for the dual class of max modules over a semilocal ring. Here, a module M_R is called a *max module* if every non-zero submodule of M_R has a superfluous radical.

Chapter 5 is devoted to considering two questions concerning maximal subfields of a division algebra proposed in [32]. In fact, we prove that for any division algebra D with center F, there exist x, y, a, b in the multiplicative group D^* of D such that $F(xy - yx)$

and $F(aba^{-1}b^{-1})$ are maximal subfields of D.