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Chapter 4

Loewy modules with finite Loewy invariants and max modules with finite radical invariants

For any left module ${}_RM$ over a ring R and any ordinal α, the Loewy series $\mathrm{soc}_\alpha({}_RM)$ of submodules of ${}_RM$ is defined setting

$$\mathrm{soc}_0({}_RM) := \ 0,$$

$$\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_\alpha({}_RM) := \mathrm{soc}(M/\mathrm{soc}_\alpha({}_RM))$$

for every ordinal α, and

$$\mathrm{soc}_\beta({}_RM) := \bigcup_{\alpha<\beta} \mathrm{soc}_\alpha({}_RM)$$

for every limit ordinal β (see [14, Section 2.11]). The members of the Loewy series of ${}_RM$ form a well ordered set of fully invariant submodules of ${}_RM$ and the Loewy series is stationary for every module ${}_RM$, that is, there exists an ordinal λ such that $\mathrm{soc}_\alpha({}_RM) = \mathrm{soc}_\lambda({}_RM)$ for $\alpha \geq \lambda$. For such an ordinal λ, the module $\delta({}_RM) := \mathrm{soc}_\lambda({}_RM)$ is called the *Loewy submodule* of ${}_RM$. If $\delta({}_RM) = {}_RM$, then ${}_RM$ is said to be a *Loewy* *module* or a *semiartinian module* or a *min module*. In this case, the least ordinal λ satisfying this property is called the *Loewy length*. For example, every artinian module is Loewy.

For each ordinal α, the Goldie dimension of $\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_\alpha({}_RM)$ is denoted by $d_\alpha({}_RM)$ and called the α-th *Loewy invariant* of ${}_RM$.

When the base ring R is commutative, Facchini showed in [12, Theorem 2.7] the following results: (1) Artinian R-modules are exactly Loewy modules with all Loewy invariants finite. (2) For every artinian module ${}_RM$ over R, one has that $d_r({}_RM) \leq \begin{pmatrix} d_1({}_RM)+r-1 \\ r \end{pmatrix}$ for every $r \geq 1$ and the Loewy length of ${}_RM$ is $\leq \omega$. Here, ω is the least infinite ordinal. (3) For every artinian module ${}_RM$, define $P({}_RM,t) = \sum_{n=0}^{\infty} d_n({}_RM)t^n \in \mathbb{Z}[[t]]$. Put $s = d_0({}_RM)d_1({}_RM)$. Then $P({}_RM,t)$ is a rational function in t of the form $f(t)/(1-t)^s$, where $f(t) \in \mathbb{Z}[t]$. If d is the order of the pole of $P({}_RM,t)$ at $t=1$, then, for all sufficiently large n, the n-th Loewy invariant $d_n({}_RM)$ and the length $l(\mathrm{soc}_n({}_RM))$ are polynomials in n with rational coefficients of degree $d-1$ and d respectively. We will recall these results in the first Section. In Section 4.2, we present examples that show that these properties, which hold for modules over a commutative ring, do not hold for modules over non-commutative rings, in general.

Camps and Dicks [9], answering a question posed by Menal [34], proved that artinian modules have semilocal endomorphism rings. In Section 4.3, it is proved, more generally, that Loewy modules ${}_RM$ with finite Loewy invariants have semilocal endomorphism rings. Some further properties of Loewy modules with finite Loewy invariants also are proved.

The dual notion of Loewy modules is that of max modules, which are defined in the first Section of this Chapter. For max modules, the best setting is that of a semilocal ring R as a base ring. In this case, max modules with finite radical invariants and semilocal endomorphism rings are considered in Section 4.4.

In Section 4.5, we apply our results to the study of modules over a perfect ring.

The main results in this Chapter appear in [17].

4.1 Loewy modules, max modules and semilocal rings

The aim of this Section is to recall some basic concepts of Loewy modules, max modules and perfect rings. For convenience, most modules used in this Chapter are left modules. Homomorphisms will be written on the right.

For Loewy modules and max modules, which are studied in this Chapter, we need the Goldie dimension and the dual Goldie dimension, so that we will review briefly these notions.

The Goldie dimension.

We begin with the Goldie dimension. The concept of "independence" of a set of submodules has been already mentioned in Chapter 3, but it is convenient to recall it again here. Let ${}_RM$ be a left R-module and $\{ M_i \mid i \in I\}$ be a non-empty set of submodules of ${}_RM$. One says that $\{ M_i \mid i \in I \}$ is *independent* if $M_i \neq 0$ and $M_i \cap \sum_{i\neq j\in I} M_j = 0$ for any $i \in I$, that is, $\sum_{i\in I} M_i = \bigoplus_{i\in I} M_i$. By convention, the sum of an empty set (of submodules) is 0. The following Lemma is basic.

Lemma 4.1.1. [14, Section 2.7] *If ${}_RM$ contains no infinite independent set of submodules, then there exists a positive integer n such that every independent set of submodules of ${}_RM$ has cardinality $\leq n$.*

The *Goldie dimension* of ${}_RM$, denoted by $\dim {}_RM$, is *infinite* if ${}_RM$ has an independent infinite set of submodules. Otherwise, $\dim {}_RM$ is the supremum of the set

$$\{ k \in \mathbb{N} \mid \text{ there exists an independent set of } k \text{ submodules of } {}_RM \}.$$

For a ring R, the *left Goldie dimension* of R is defined to be $\dim {}_RR$ and the *right Goldie dimension* of R is defined to be $\dim R_R$.

Recall that a non-zero module whose non-zero submodules are essential is called *uniform*. The following basic Proposition is from [30, 6.2 and Corollary 6.10].

Proposition 4.1.2. *Let ${}_RM, {}_RM'$ be left modules over a ring R such that* $\dim {}_RM$ *and* $\dim {}_RM'$ *are finite, and ${}_RN$ be a submodule of ${}_RM$. Then*

1. *If* $\dim {}_RM = n$*, then there exists an independent set* $\{ U_1, U_2, \cdots, U_n\}$ *of* n *uniform submodules of* ${}_RM$ *such that* $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ *is essential in* ${}_RM$*.*

2. $\dim {}_RN \leq \dim {}_RM$ *and the equality holds if and only if* ${}_RN$ *is essential in* ${}_RM$*.*

3. $\dim({}_RM \oplus {}_RM') = \dim {}_RM + \dim {}_RM'$*.*

The notion of Goldie dimension is "opposite" to that of dual Goldie dimension, which is defined as follows.

The dual Goldie dimension.

Let ${}_RM$ be a left R-module and $\{ M_i \mid i \in I \}$ be a non-empty set of submodules of ${}_RM$. One says that $\{ M_i \mid i \in I\}$ is *coindependent* if $M_i \neq {}_RM$ and $M_i + \bigcap_{j\in F} M_j = {}_RM$ for every $i \in I$ and non-empty finite set $F \subseteq I\backslash\{i\}$. By convention, the intersection of an empty set (of submodules) is ${}_RM$.

Lemma 4.1.3. [14, Theorem 2.40] *If* ${}_RM$ *contains no infinite coindependent set of submodules of* ${}_RM$*, then there exists a positive integer* n *such that every coindependent set of submodules of* ${}_RM$ *has cardinality* $\leq n$*.*

The *dual Goldie dimension* of ${}_RM$, denoted by $\text{codim}_R M$, is *infinite* if ${}_RM$ has a coindependent infinite set of submodules. Otherwise, $\text{codim}_R M$ is the supremum of the set

$$\{ k \in \mathbb{N} \mid \text{ there exists a coindependent set of } k \text{ submodules of } {}_RM \}.$$

The *left dual Goldie dimension* of a ring R is defined to be $\text{codim}_R R$ and the *right dual Goldie dimension* of a ring R is defined to be $\text{codim} R_R$.

Proposition 4.1.4. [14, Theorem 2.40] *Let* ${}_RM, {}_RM'$ *be left modules over a ring* R *such that* $\text{codim}_R M$ *and* $\text{codim}_R M$ *are finite, and* N *be a submodule of* ${}_RM$*. Then*

1. *If* $\text{codim}_R M = n$*, then there exists a coindependent set* $\{ N_1, N_2, \cdots, N_n\}$ *of* n *submodules of* ${}_RM$ *such that* $N_1 \cap N_2 \cap \cdots \cap N_n$ *is superfluous in* ${}_RM$ *and* ${}_RM/N_i$ *is couniform for any* $i \in I$*.*

2. $\text{codim}_R M/N \leq \text{codim}_R M$ *and the equality holds if and only if* N *is superfluous in* ${}_RM$*.*

3. $\operatorname{codim}({}_RM \oplus {}_RM') = \operatorname{codim}_R M + \operatorname{codim}_R M'$.

The Krull dimension.

We define a family of classes $\mathcal{K}_\alpha$ of modules for $\alpha = -1$ or an ordinal as follows. The class $\mathcal{K}_{-1}$ contains only the zero module. Assume that the class of $\mathcal{K}_\beta$ is defined for any $\beta < \alpha$. The class $\mathcal{K}_\alpha$ is defined as the class of all modules ${}_RM$ such that

1. ${}_RM \notin \bigcup_{\beta<\alpha} \mathcal{K}_\beta$.

2. For every countable descending chain $A_0 \geq A_1 \geq A_2 \geq \cdots$ of submodules of ${}_RM$, there exists a positive integer n such that the factors A_i/A_{i+1} are in $\bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for any $i \geq n$.

It is clear that all the classes $\mathcal{K}_\alpha$ are pairwise disjoint. If ${}_RM$ is contained in $\mathcal{K}_\alpha$ for some α, then ${}_RM$ is said to *have Krull dimension* α, denoted by $\operatorname{Kdim}_R M = \alpha$. Otherwise, we say that ${}_RM$ *fails to have Krull dimension.* By definition, every artinian module has Krull dimension 0. The following Proposition is from [14, Example 7.10, Propositions 7.11 and 7.13].

Proposition 4.1.5. *Let ${}_RM$ be a left module over a ring R and N be a submodule of ${}_RM$. Then*

1. *If ${}_RM$ is noetherian, then ${}_RM$ has Krull dimension.*

2. *If ${}_RM$ has Krull dimension, then the Goldie dimension* $\dim {}_RM$ *is finite.*

3. *${}_RM$ has Krull dimension if and only if ${}_RM/N$ and N have Krull dimension. In this case,* $\operatorname{Kdim}({}_RM) = \max\{\operatorname{Kdim}_R M/N, \operatorname{Kdim} N\}$.

Semilocal rings.

A *semilocal ring* is a ring R for which $R/J(R)$ is a semisimple artinian ring, that is, $R/J(R)$ is the direct product of finitely many matrix rings over division rings. Let R be a semilocal ring. The semilocal ring R is called *semiprimary* if the Jacobson radical $J(R)$ is nilpotent, that is, there exists a positive integer $n > 0$ such that $J(R)^n = 0$. The ring R is said to be *left perfect* (resp. *right perfect*) if the Jacobson radical $J(R)$ is left (resp. right) T-nilpotent, that is, for any sequence $\{a_1, a_2, \cdots\} \subseteq J(R)$, there exists a positive integer $n > 0$ such that $a_1 a_2 \cdots a_n = 0$ (resp. $a_n a_{n-1} \cdots a_1 = 0$). A ring which is both left and right perfect is called *perfect*. Hence, one has that

$$\text{one-sided artinian} \Rightarrow \text{semiprimary} \Rightarrow \text{perfect} \Rightarrow \text{semilocal}.$$

The following is the well known Theorem of Hopkins-Levitzki. It concerns artinian modules, noetherian modules and the length of a module, over a semiprimary ring. Let ${}_RM$ be a left module over a ring R. A chain of submodules $N_0 < N_1 < \cdots < N_n$ of ${}_RM$ is said to have *length* n. The *length* of ${}_RM$, denoted by $l({}_RM)$, is defined to be the supremum (eventually ∞) of the lengths of the chain of submodules of ${}_RM$.

Theorem 4.1.6. [29, Theorem 4.15] *For a left module ${}_RM$ over a semiprimary ring R, the following conditions are equivalent:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is artinian.*
3. *The length $l({}_RM)$ of ${}_RM$ is finite.*

For two arbitrary rings R and S, a ring homomorphism $f\colon R \to S$ is called *local* if, for any $r \in R \backslash R^*$, one has $f(r) \in S \backslash S^*$. Here, R^* and S^* are the sets of invertible elements of R and S respectively. Camps and Dicks proved in [9] the following Theorem.

Theorem 4.1.7. [9, Theorem 1] *For a ring R, the following statements are equivalent:*

1. *S is semilocal.*
2. *There exists a local homomorphism of R into a semilocal ring.*
3. *There exists a local homomorphism of R into a semisimple artinian ring.*

Theorem 4.1.8. [14, Page 59] *Let R be a ring.*

1. *If R is semisimple artinian ring, then* $\dim {}_RR = \dim R_R = \mathrm{codim}_R R = \mathrm{codim} R_R < \infty$. For any semisimple artinian ring R, we write $\dim(R)$ or $\mathrm{codim}(R)$ for these dimensions.

2. *If R is semilocal ring, then* $\mathrm{codim}({}_RR) = \mathrm{codim}(R_R) = \mathrm{codim}(R/J(R))$. For any semilocal ring R, we write $\mathrm{codim}(R)$ for these dimensions.

3. *If $R = M_n(D)$, the ring of $n \times n$ matrices over a division ring D, then* $\dim(R) = \mathrm{codim}(R) = n$.

Loewy modules.

The following properties are basic.

Proposition 4.1.9. [14, Lemma 5.28] *Let $_RM$ be a left module over a ring R. The following statements are equivalent:*

1. *$_RM$ is Loewy.*
2. *Every nonzero homomorphic image of $_RM$ has an essential socle.*

Corollary 4.1.10. *Factor modules of a Loewy module are Loewy.*

Proposition 4.1.11. [4, Theorem 28.4] *Every left module over a right perfect ring is Loewy.*

Proposition 4.1.12. *The Loewy invariants of an artinian module are finite.*

Max modules

Dually, the notion of *max modules* is "opposite" to that of Loewy modules. Recall that the *radical* of a module $_RM$, denoted by $\mathrm{rad}(_RM)$, is the intersection of all maximal submodules of $_RM$. In case $_RM$ has no maximal submodule, then $\mathrm{rad}(_RM) := {_RM}$. The descending chain of submodules

$$\mathrm{rad}_0(_RM) \geq \mathrm{rad}_1(_RM) \geq \cdots \geq \mathrm{rad}_\alpha(_RM) \geq \mathrm{rad}_{\alpha+1}(_RM) \geq \cdots$$

of $_RM$ defined by setting

$$\mathrm{rad}_0(_RM) := {_RM},$$

$$\mathrm{rad}_\alpha(_RM) := \mathrm{rad}(\mathrm{rad}_{\alpha-1}(_RM))$$

for any successor ordinal α, and

$$\mathrm{rad}_\beta(_RM) := \bigcap_{\alpha<\beta} \mathrm{rad}_\alpha(_RM)$$

for any limit ordinal β. This chain $\mathrm{rad}_\alpha(_RM)$ is called the *radical series* of $_RM$. If there exists an ordinal λ such that $\mathrm{rad}_\lambda(_RM) = 0$, then $_RM$ is said to be a *max module* or a *seminoetherian module*. In this case, the least ordinal λ satisfying this property is called the *max length* of ${}_RM$. Obviously, the class of max modules contains the class of noetherian ones.

Proposition 4.1.13. [38, Proposition 2.2] *Let ${}_RM$ be a left module over a ring R. The following statements are equivalent:*

1. *${}_RM$ is max.*
2. *Every non-zero submodule of ${}_RM$ has a superfluous radical.*

Corollary 4.1.14. *Submodules of a max module are max.*

Proposition 4.1.15. [4, Theorem 28.4] *Every left module over a left perfect ring is max.*

For any non-limit ordinal α, the Goldie dimension of $\text{rad}_\alpha({}_RM)/\text{rad}_{\alpha+1}({}_RM)$ is called the α-the *radical invariant* of ${}_RM$, denoted by $r_\alpha({}_RM)$.

Proposition 4.1.16. *In a noetherian module, every radical invariant is finite.*

Next, we summarize some important results about Loewy modules over a commutative ring. All of them are from [12] or [39].

Loewy modules over a commutative ring.

As we have already seen, from the definitions of Loewy modules and Loewy invariants, it follows that every artinian module is Loewy with finite invariants. Facchini proved that the converse is also true if the base ring is commutative.

Theorem 4.1.17. [12, Theorem 2.7] *Let R be a commutative ring and ${}_RM$ be a left R-module. The following statements are equivalent:*

1. *${}_RM$ is artinian.*
2. *${}_RM$ is Loewy and all its Loewy invariants are finite.*

If ${}_RM$ is a left R-module and S is the endomorphism ring of ${}_RM$, then ${}_RM$ has a natural structure of a right S-module with product given by $ms = (m)s$ for $s \in S, m \in M$. We emphasize this by writing M_S.

Theorem 4.1.18. [12, Theorem 2.8] *Let* ${}_RM$ *be an artinian left module with simple socle over a commutative ring* R*. Then* $S = \mathrm{End}({}_RM)$ *is a local Noetherian complete commutative ring and* M_S *is the injective envelope of the unique simple* S*-module.*

Theorem 4.1.19. *Let* ${}_RM$ *be a Loewy module with finite invariants* $d_\alpha({}_RM)$ *over a commutative ring* R*. If* $d_1({}_RM) = n$ *then* $d_r({}_RM) \leq \begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ *for every positive integer* $r \geq 1$*.*

Proof. This is a Corollary of [12, Theorem 3.1]. ∎

Theorem 4.1.20. [12, Theorem 3.2] *Let* ${}_RM$ *be a Loewy module with finite invariants* $d_\alpha({}_RM)$ *over a commutative ring* R*. Then* $P({}_RM, t) = \sum_{n=1}^{+\infty} d_n({}_RM)t^n$ *is a rational function in* t *of the form* $\frac{f(t)}{(1-t)^s}$*, where* $f(t) \in \mathbb{Z}[t]$ *and* $s = d_0({}_RM)d_1({}_RM)$*. Let* $d({}_RM)$ *be the order of the pole of* $P({}_RM, t)$ *at* $t = 1$*. Then, for all sufficiently large* n*,* $d_n({}_RM)$ *and the length* $l(\mathrm{soc}_n({}_RM))$ *of* $\mathrm{soc}_n({}_RM)$ *are polynomials in* n *with rational coefficients of degree* $d({}_RM) - 1$ *and* $d({}_RM)$ *respectively.*

Theorem 4.1.21. [39, Theorem 4.2] *Let* R *be a commutative ring,* ${}_RM$ *be a Loewy* R*-module,* α *be an ordinal and* r *be a positive integer. If the* α*-th and* $(\alpha + r)$*-th Loewy invariants are finite, then the* β*-th Loewy invariant is finite for any* $\beta > \alpha + r$ *and* ${}_RM = \mathrm{soc}_{\omega+r}({}_RM)$*.*

4.2 Some examples

Recall that a left module over an arbitrary ring is said to be *quotient finite dimensional* (*q.f.d.*) if all its factor modules have finite Goldie dimension [23]. For instance, all uniserial modules are q.f.d., and all modules with Krull dimension are q.f.d. (Proposition 4.1.5). In the next result, we have collected some characterizations of artinian modules related to being Loewy modules.

Proposition 4.2.1. *The following conditions are equivalent for a left module* ${}_RM$ *over an arbitrary ring* R*:*

1. ${}_RM$ *is artinian.*
2. ${}_RM$ *is Loewy and has Krull dimension.*
3. ${}_RM$ *is Loewy and q.f.d.*
4. ${}_RM$ *is Loewy and its factor modules have finitely generated socles.*
5. *All factor modules of* ${}_RM$ *are essential extensions of their finitely generated socles.*

PROOF. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. (2) $\Rightarrow$ (3) follows from Propositions 4.1.5. (5) $\Rightarrow$ (1) is proved in [41, Proposition 2*]. ■

As Section 4.1, when the base ring R is commutative, the artinian R-modules are exactly the Loewy R-modules with finite Loewy invariants. In general, for a non-commutative ring, every artinian R-module is a Loewy R-module with finite Loewy invariants. In the next example, it is shown that *there exists a non-artinian module* ${}_RM$ *over a suitable non-commutative ring* R *that is a Loewy module of Loewy length* ω *with finite Loewy invariants.*

Example 4.2.2. Let k be a field, V_k a vector space over k of countable infinite dimension, and $\{ v_n \mid n \geq 0 \}$ be a basis of V_k. Let R be the set of all the endomorphisms $f \in \mathrm{End}(V_k)$ such that, for every $n \geq 0$, $f(v_{2n})$ belongs to the subspace $\langle v_0, v_2, v_4, \ldots, v_{2n} \rangle_k$ of V_k and $f(v_{2n+1})$ belongs to the subspace $\langle v_0, v_2, v_4, v_6, \ldots, v_{2n}, v_{2n+1} \rangle_k$. It is easily seen that R is a k-subalgebra of $\mathrm{End}(V_k)$, so that V inherits a left R-module structure. More precisely, ${}_RV$ is a left R-module with scalar multiplication defined by $fv = f(v)$ for every $f \in R$ and $v \in V$. Clearly, $\langle v_0, v_1, \ldots, v_n \rangle_k$ is a submodule of ${}_RV$ for every $n \geq 0$.

Lemma 4.2.3. *Let* $v = v_0\alpha_0 + v_1\alpha_1 + \cdots + v_n\alpha_n$ *be an element of* V *with* $\alpha_i \in k$ *for every* $i = 0, 1, 2, \ldots, n$ *and* $\alpha_n \neq 0$. *Then* $Rv \geq Rv_n$, *and* $Rv \geq Rv_{2m}$ *for every integer* m *with* $0 \leq 2m \leq n$.

PROOF. Let $f \in \mathrm{End}(V_k)$ be defined by $f(v_n) = v_n\alpha_n^{-1}$ and $f(v_i) = 0$ for every $i \neq n$. Then $f \in R$ and $f(v) = v_n$. This proves that $Rv \geq Rv_n$.

Similarly, for every m with $0 \leq 2m \leq n$, let $g \in \mathrm{End}(V_k)$ be defined by $g(v_n) = v_{2m}$ and $g(v_i) = 0$ for every $i \neq n$. Then $g \in R$ and $g(v_n) = v_{2m}$. Thus $gf(v) = v_{2m}$, and therefore $Rv \geq Rv_{2m}$. ■

We will now describe the n-th Loewy submodule of ${}_RV$ for any $n \geq 0$.

Lemma 4.2.4. *Let $n \geq 1$ be an integer. Then the n-th Loewy submodule $\mathrm{soc}_n({}_RV)$ of ${}_RV$ is $\langle v_0, v_1, v_2, \ldots, v_{2n-3}, v_{2n-2}\rangle_k$ and the Loewy invariants of ${}_RV$ are $d_0({}_RV) = 1$ and $d_n({}_RV) = 2$ for every $n \geq 1$.*

PROOF. We have $\mathrm{soc}_0({}_RV) = 0$ by definition. Let us first consider the case $n = 1$. We want to show that $\mathrm{soc}({}_RV) = Rv_0 = \langle v_0\rangle_k$. Now v_0 belongs to all non-zero submodules of ${}_RV$ by Lemma 4.2.3. That is, Rv_0 is the intersection of all non-zero submodules of ${}_RV$. It follows that Rv_0 is the unique simple submodule of ${}_RV$. Thus $\mathrm{soc}({}_RV) = Rv_0$.

We now proceed by induction on $n \geq 1$. Suppose that the n-th Loewy submodule $\mathrm{soc}_n({}_RV)$ of ${}_RV$ is ${}_R\langle v_0, v_1, v_2, \ldots, v_{2n-2}\rangle = \langle v_0, v_1, v_2, \ldots, v_{2n-2}\rangle_k$ for some $n \geq 1$. Consider the factor module $\overline{V} := {}_RV/\mathrm{soc}_n({}_RV)$. We will denote its elements as $\overline{v}$, where $v \in {}_RV$ and $\overline{v} = v + \mathrm{soc}_n({}_RV)$. From the definition of R, one sees that $R\overline{v_{2n-1}}$ is a one-dimensional vector space over k, hence a simple R-submodule of $\overline{V}$. Similarly for $R\overline{v_{2n}}$. Thus $R\overline{v_{2n-1}} \oplus R\overline{v_{2n}}$ is a semisimple submodule of $\overline{V}$ of composition length 2. By Lemma 4.2.3, this semisimple submodule is essential in $\overline{V}$. As the socle is the intersection of all essential submodules, it follows that $\mathrm{soc}(\overline{V}) = R\overline{v_{2n-1}} \oplus R\overline{v_{2n}}$. Thus $\mathrm{soc}_{n+1}({}_RV) = \langle v_0, v_1, \ldots, v_{2n}\rangle_k$. ■

By Lemma 4.2.4, the left R-module ${}_RV$ is a Loewy module of Loewy length ω with finite Loewy invariants. We will now show that it is not an artinian R-module.

For every $n \geq 0$, let

$$U_n := \langle v_0, v_2, \ldots, v_{2n}, v_{2n+1}, v_{2n+2}, \ldots\rangle_k$$

be the subspace of V_k generated by all the v_i with $i \geq 2n$ and all the v_i's with i even, $i < 2n$. It is easily seen that U_n is a submodule of ${}_RV$ and that $U_0 > U_1 > U_2 > \ldots$ is a strictly descending chain. Thus ${}_RV$ is not artinian.

It is proved that if ${}_RM$ is a Loewy module over a commutative ring R, α is an ordinal and r is a positive integer such that both $d_\alpha({}_RM)$ and $d_{\alpha+r}({}_RM)$ are finite, then $d_\beta({}_RM)$

is finite for every $\beta > \alpha + r$ and ${}_RM = \mathrm{soc}_{\omega+r}({}_RM)$ (Theorem 4.1.21). We will show that there exist modules ${}_RW$ over non-commutative rings R with completely arbitrary Loewy length and Loewy invariants. More precisely:

Example 4.2.5. Let k be a field, λ be an ordinal and f be a mapping of λ in the class $\mathrm{Card} \setminus \{0\}$ of all non-zero cardinals. Then there exists a local k-algebra R with a Loewy left module ${}_RW$ of Loewy length λ and Loewy invariants $d_\alpha({}_RW) = f(\alpha)$ for every ordinal $\alpha < \lambda$.

The construction is as follows. For every ordinal $\alpha < \lambda$, let W_α be a vector space over k of dimension $f(\alpha)$. Let $W_k := \bigoplus_{\alpha<\lambda} W_\alpha$ be the direct sum of all the vector k-spaces W_α, $\alpha < \lambda$. Let I be the set of all the endomorphisms $f \in \mathrm{End}(W_k)$ such that $f(W_\alpha) \subseteq \bigoplus_{\beta<\alpha} W_\beta$ for every $\alpha < \lambda$. Here, for $\alpha = 0$, we mean that the direct sum $\bigoplus_{\beta<\alpha} W_\beta$ is the direct sum of the empty family of vector subspaces, that is, the condition on f to belong to I is that $f(W_0) = 0$. Set $R := k \oplus I \subseteq \mathrm{End}(W_k)$. Then R turns out to be a k-subalgebra of $\mathrm{End}(W_k)$, the ring R is a local ring with maximal two-sided ideal I, residue division ring $R/I \cong k$, and ${}_RW_k$ has an R-k-bimodule structure.

We will now determine the cyclic non-zero submodules of the module ${}_RW$. Let w be a non-zero element of ${}_RW$, so that w can be written in a unique way as $w = w_1 + \cdots + w_n$ with $w_i \in W_{\alpha_i}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \lambda$ and $w_n \neq 0$. Let f be an element of I such that $f(w_n) = -w_1 - \cdots - w_{n-1}$ and $f(W_\alpha) = 0$ for every $\alpha < \lambda$, $\alpha \neq \alpha_n$. For such an f, we have that $f(w) = -w_1 - \cdots - w_{n-1}$ and $(1 + f)(w) = w_n$. It is now easy to see that the cyclic R-module Rw generated by w is $\langle w_n \rangle_k \oplus \left(\bigoplus_{\alpha<\alpha_n} W_\alpha\right)$. These are therefore all non-zero cyclic submodules of ${}_RW$. Since every module is the sum of its non-zero cyclic submodules, we get that the R-submodules of ${}_RW$ are exactly the improper submodule ${}_RW$ and the modules $U_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha\right)$, where β ranges over the set of all ordinals $< \lambda$ and U_β ranges over the set of all vector subspaces of W_β. Notice that: (1) the zero submodule of ${}_RW$ is obtained in this way for $\beta = 0$ and $U_\beta = 0$; (2) some submodules can be represented in this form in two different ways, because $W_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha\right) = 0 \oplus \left(\bigoplus_{\alpha<\beta+1} W_\alpha\right)$ for $\beta + 1 < \alpha$. If we want to represent the submodules of ${}_RW$ only once, we must take the improper submodule ${}_RW$ and all the representations of the form $U_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha\right)$

with β an ordinal $< \lambda$ and U_β a proper subspace of W_β. Now that we have a complete description of all submodules of ${}_RW$, it is easily seen that $\mathrm{soc}_\alpha({}_RW) = \bigoplus_{\beta<\alpha} W_\beta$ for every $\alpha \leq \lambda$. Moreover, $\mathrm{soc}_{\alpha+1}({}_RW)/\mathrm{soc}_\alpha({}_RW) \cong W_\alpha$ is an R-module of Goldie dimension $\dim_k(W_\alpha) = f(\alpha)$. This shows that ${}_RW$ is our required example of a Loewy module of Loewy length λ and Loewy invariants $d_\alpha({}_RW) = f(\alpha)$ for every $\alpha < \lambda$.

The module ${}_RW$ is artinian when its Loewy invariants are finite. More precisely, we will now prove that ${}_RW$ *is an artinian R-module if and only if $f(\alpha)$ is finite for every $\alpha < \lambda$.* Thus, for artinian modules over non-commutative rings, Loewy length and finite Loewy invariants can be completely arbitrary. This contrasts with the behavior of artinian modules over commutative rings R (Theorems 4.1.19, 4.1.20 and 4.1.21).

We already know that all artinian modules are Loewy modules with finite Loewy invariants. Conversely, assume that $f(\alpha)$ is a non-zero finite cardinal for every $\alpha < \lambda$, and take a descending chain $A_0 \geq A_1 \geq A_2 \geq \dots$ of submodules of ${}_RW$. If $A_n = {}_RW$ for every n, the chain is stationary. Similarly, the chain is stationary if $A_n = 0$ for some n. Thus we can suppose that there exists an index n such that $0 \neq A_i < {}_RW$ for every $i \geq n$. Then, for every $i \geq n$, $A_i = U_{\beta_i} \oplus \left(\bigoplus_{\alpha<\beta_i} W_\alpha\right)$ for a suitable ordinal $\beta_i < \lambda$ and a suitable proper vector subspace U_{β_i} of W_{β_i}. Thus we have a descending chain $\lambda > \beta_n \geq \beta_{n+1} \geq \beta_{n+2} \geq \dots$ of ordinals, necessarily stationary. Hence, there exists an index $m \geq n$ such that, if we set $W' := \left(\bigoplus_{\alpha<\beta_m} W_\alpha\right)$, then $A_i = U_{\beta_i} \oplus W'$ for every $i \geq m$, where U_{β_i} is a proper vector subspace of W_{β_m}. As W_{β_m} has finite dimension $f(\beta_m)$, the descending chain of subspaces $U_{\beta_m} \geq U_{\beta_{m+1}} \geq U_{\beta_{m+2}} \geq \dots$ is stationary, so that the chain A_i is stationary, and ${}_RW$ is artinian.

It is shown that if R is a commutative ring, ${}_RM$ is an artinian left module with simple socle, and $H := \mathrm{End}({}_RM)$, then M_H is the injective envelope of a simple right H-module (Theorem 4.1.18). The following example proves that if R is not commutative, then M_H is not necessarily the injective envelope of a simple H-module. In our example, M_H will turn out to be a semisimple H-module.

Example 4.2.6. Let R be the $\mathbb{R}$-algebra $R := \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$ and ${}_RM := \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$. Firstly, we show that ${}_RM$ is an artinian module with simple socle. The ring R is an $\mathbb{R}$-algebra of dimension 5 and ${}_RM$ is a vector space of dimension 4 over $\mathbb{R}$, so that ${}_RM$ is an artinian module. It is easy to check that $R\begin{pmatrix}0\\y\end{pmatrix}=\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$ for every $0\neq y\in\mathbb{C}$. Thus $\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$ is a simple submodule of ${}_RM$. Moreover, for every $a,b\in\mathbb{C}$ with $a\neq 0$, one has that $\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\a\end{pmatrix}\in\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$, which shows that $\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$ is essential in ${}_RM$. Since the socle is the intersection of all essential submodules, $\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$ turns out to be the socle of ${}_RM$.

Now we will see that the endomorphism ring $H:=\mathrm{End}({}_RM)$ of ${}_RM$ is isomorphic to $\mathbb{C}$. Let f be an element of H. Since $\begin{pmatrix}0\\\mathbb{C}\end{pmatrix}$ is the socle of ${}_RM$, $\begin{pmatrix}0\\1\end{pmatrix}f=\begin{pmatrix}0\\\alpha\end{pmatrix}$ for some $\alpha\in\mathbb{C}$. Hence, for every $b\in\mathbb{C}$, we have that

$$\begin{pmatrix}0\\b\end{pmatrix}f=\begin{pmatrix}0&0\\0&b\end{pmatrix}\left(\begin{pmatrix}0\\1\end{pmatrix}f\right)=\begin{pmatrix}0\\b\alpha\end{pmatrix}. \tag{4.2.1}$$

For any $a\in\mathbb{C}$, $\begin{pmatrix}a\\0\end{pmatrix}f=\begin{pmatrix}1&0\\0&0\end{pmatrix}\left(\begin{pmatrix}a\\0\end{pmatrix}f\right)=\begin{pmatrix}\beta_a\\0\end{pmatrix}$ for some $\beta_a\in\mathbb{C}$. From (4.2.1), we get that $\begin{pmatrix}0\\a\alpha\end{pmatrix}=\begin{pmatrix}0\\a\end{pmatrix}f=\begin{pmatrix}0&0\\1&0\end{pmatrix}\left(\begin{pmatrix}a\\0\end{pmatrix}f\right)=\begin{pmatrix}0\\\beta_a\end{pmatrix}$, which implies $\beta_a=\alpha a$. Therefore,

$$\begin{pmatrix}a\\b\end{pmatrix}f=\begin{pmatrix}a\\0\end{pmatrix}f+\begin{pmatrix}0\\b\end{pmatrix}f=\begin{pmatrix}a\alpha\\0\end{pmatrix}+\begin{pmatrix}0\\b\alpha\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\alpha.$$

In other words, for any element $f\in H$, there exists a unique $\alpha\in\mathbb{C}$ such that $(x)f=x\alpha$ for every $x\in{}_RM$. Thus $H\cong\mathbb{C}$ and M_H is a vector space of dimension 2 over $\mathbb{C}$.

4.3 Loewy modules whose Loewy invariants are finite

In this Section, we prove that some properties that are classically known to hold for artinian modules, also hold for Loewy modules with finite Loewy invariants.

Let ${}_RM$ be a left module over a ring R. It is clear that the socle $\mathrm{soc}({}_RM)$ of ${}_RM$ and, more generally, all its Loewy submodules $\mathrm{soc}_\alpha({}_RM)$, are fully invariant submodules of ${}_RM$, that is, $(\mathrm{soc}_\alpha({}_RM))f \leq \mathrm{soc}_\alpha({}_RM)$ for every ordinal α and every endomorphism f of ${}_RM$. Equivalently, $\mathrm{soc}_\alpha({}_RM)$ is a subbimodule of the bimodule ${}_RM_{\mathrm{End}({}_RM)}$ for every ordinal α. Thus we can consider the restrictions $f|_{\mathrm{soc}({}_RM)}$ and $f|_{\mathrm{soc}_\alpha({}_RM)}$, which are endomorphisms of $\mathrm{soc}({}_RM)$ and $\mathrm{soc}_\alpha({}_RM)$, respectively.

Proposition 4.3.1. *Let ${}_RM$ be a Loewy module whose Loewy invariants are all finite. The following conditions are equivalent for an endomorphism $f \in \mathrm{End}({}_RM)$:*

1. *f is an automorphism of ${}_RM$.*
2. *f is an injective endomorphism of ${}_RM$.*
3. *The restriction $f|_{\mathrm{soc}({}_RM)} \colon \mathrm{soc}({}_RM) \to \mathrm{soc}({}_RM)$ is injective.*
4. *The restriction $f|_{\mathrm{soc}({}_RM)} \colon \mathrm{soc}({}_RM) \to \mathrm{soc}({}_RM)$ is an automorphism of $\mathrm{soc}({}_RM)$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial, so that it suffices to show (4) $\Rightarrow$ (1). Let ${}_RM$ be a Loewy module with Loewy length λ and all Loewy invariants finite. Let f be an endomorphism of ${}_RM$ and assume that $f|_{\mathrm{soc}({}_RM)}$ is an automorphism of $\mathrm{soc}({}_RM)$. We must prove that f is an automorphism of ${}_RM$. Since the socle is essential in ${}_RM$, $f|_{\mathrm{soc}({}_RM)}$ injective implies f injective. We will show that f is surjective by induction on the Loewy length λ of ${}_RM$. The cases $\lambda = 0$ and $\lambda = 1$ are trivial. Suppose that the statement holds for every ordinal $\alpha < \lambda$. That is, every endomorphism of a Loewy module of Loewy length $\alpha < \lambda$ and finite Loewy invariants is surjective if the endomorphism restricted to its socle is an automorphism. If λ is a limit ordinal, then ${}_RM = \mathrm{soc}_\lambda({}_RM) = \cup_{\alpha<\lambda}\mathrm{soc}_\alpha({}_RM)$. For any $\alpha < \lambda$, the restriction of f to $\mathrm{soc}_\alpha({}_RM)$ is an automorphism of $L_\alpha({}_RM)$ by the inductive hypothesis. Hence f is surjective. If λ is a non-limit ordinal, then $\lambda = \alpha + 1$ for some ordinal α. Then f induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{soc}_\alpha({}_RM) & \longrightarrow & {}_RM & \longrightarrow & {}_RM/\mathrm{soc}_\alpha({}_RM) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f|\mathrm{soc}_\alpha({}_RM)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & \mathrm{soc}_\alpha({}_RM) & \longrightarrow & {}_RM & \longrightarrow & {}_RM/\mathrm{soc}_\alpha({}_RM) & \longrightarrow & 0.
\end{array}$$

Here f is injective and $f|_{\mathrm{soc}_\alpha({}_RM)}$ is an automorphism by the inductive hypothesis. From the Snake Lemma, we get an exact sequence

$$0 \longrightarrow \ker(\overline{f}) \longrightarrow 0 \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(\overline{f}) \longrightarrow 0.$$

In particular, $\overline{f}$ is a monomorphism. But ${}_RM/\mathrm{soc}_\alpha({}_RM)$ is a semisimple module of finite length, so that injective endomorphisms are automorphisms. Thus $\operatorname{coker}(\overline{f}) = 0$, so that $\operatorname{coker}(f) = 0$ and f is surjective. ■

Corollary 4.3.2. *Let ${}_RM$ be a Loewy module with finite Loewy invariants. Then the restriction homomorphism $\rho\colon \mathrm{End}({}_RM) \to \mathrm{End}(\mathrm{soc}({}_RM))$, $f \mapsto f|_{\mathrm{soc}({}_RM)}$, is local. In particular,* $\mathrm{End}({}_RM)$ *is a semilocal ring of dual Goldie dimension $\leq d_0({}_RM)$.*

Proof. The Corollary follows immediately from Proposition 4.3.1 and a result of Herbera and Shamsudding, see [14, Theorem 4.3(a)]. For another elementary proof, notice that the ring homomorphism ρ is local by Proposition 4.3.1. By hypothesis, $\mathrm{soc}({}_RM)$ is the direct sum of finitely many simple submodules of ${}_RM$, hence $\mathrm{End}(\mathrm{soc}({}_RM))$ is a semisimple artinian ring. As ρ is a local homomorphism of $\mathrm{End}({}_RM)$ into a semisimple artinian ring, the ring $\mathrm{End}({}_RM)$ must be semilocal by Theorem 4.1.7. ■

Thus Corollary 4.3.2 generalizes the famous result by Camps and Dicks that artinian modules have semilocal endomorphism rings [9, Corollary 6]. The next Corollary generalizes part of Theorem 4.1.18 to the non-commutative case.

Corollary 4.3.3. *Let ${}_RM$ be a Loewy module with finite Loewy invariants and simple socle. Then* $\mathrm{End}({}_RM)$ *is a local ring.*

Proof. This is the case $d_0({}_RM) = 1$ in Corollary 4.3.2, because semilocal rings of dual Goldie dimension 1 are local rings. ■

Thus if ${}_RM$ is a direct sum of modules satisfying the hypotheses of Corollary 4.3.3, that is, Loewy modules with finite Loewy invariants and simple socle, then there is uniqueness of direct-sum decompositions into indecomposables.

Proposition 4.3.1 and Corollary 4.3.2 have several implications, and we will now present some of them. Recall that two modules ${}_RM, {}_RN$ are said to be in the same *monogeny class* [13] if there exist a monomorphism ${}_RM \to {}_RN$ and a monomorphism ${}_RN \to {}_RM$. The notation for "${}_RM$ and ${}_RN$ are in the same monogeny class" is $[{}_RM]_m = [{}_RN]_m$. A class of modules $\mathcal{C}$ is said to be *mono-correct* [44] if, for every ${}_RM, {}_RN \in \mathcal{C}$, $[{}_RM]_m = [{}_RN]_m$ implies ${}_RM \cong {}_RN$. For instance, the class of all injective left R-modules is mono-correct [8].

Corollary 4.3.4. *The class of all Loewy left R-modules whose Loewy invariants are finite is mono-correct. More generally, any Loewy left R-module ${}_RM$ whose Loewy invariants are finite is* mono-correct*, in the sense that, for any left R-module ${}_RN$, $[{}_RM]_m = [{}_RN]_m$ implies ${}_RM \cong {}_RN$.*

Proof. Let ${}_RM, {}_RN$ be left R-modules and suppose that ${}_RM$ is a Loewy left R-module with finite Loewy invariants. If there exist a monomorphism $\varphi\colon {}_RM \to {}_RN$ and a monomorphism $\psi\colon {}_RN \to {}_RM$, then the injective endomorphism $\varphi\psi$ of ${}_RM$ is an automorphism by Proposition 4.3.1. Thus ψ is surjective, hence an isomorphism. ∎

Corollary 4.3.4 generalizes [44, 3.3.1] from artinian modules to Loewy modules with finite Loewy invariants. From [14, Proposition 4.9] and our Corollary 4.3.2, we get that:

Corollary 4.3.5. *Every Loewy left R-module ${}_RM$ whose Loewy invariants are finite has only finitely many direct summands up to isomorphism. More precisely, it has at most $2^{d_0({}_RM)}$ isomorphism classes of direct summands.*

Proof. By [14, Proposition 4.9], it suffices to show that the dual Goldie dimension $\operatorname{codim}(\operatorname{End}({}_RM)) \leq d_0({}_RM)$. Now the homomorphism $\rho\colon \operatorname{End}({}_RM) \to \operatorname{End}(\operatorname{soc}({}_RM))$ is a local homomorphism, so that $\operatorname{codim}(\operatorname{End}({}_RM)) \leq \operatorname{codim}(\operatorname{End}(\operatorname{soc}({}_RM)))$ [9, Corollary 2]. Moreover, $\operatorname{soc}({}_RM)$ is a semisimple module of composition length $d_0({}_RM)$. Thus $\operatorname{codim}(\operatorname{End}(\operatorname{soc}({}_RM)))$ is equal to the Goldie dimension $d_0({}_RM)$ of $\operatorname{soc}({}_RM)$. ∎

4.4 Max modules whose radical invariants are finite

We will now dualize the results in the previous Sections. If we try to dualize the previous situation, in which all Loewy factors $\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_\alpha({}_RM)$ are semisimple modules, the problem is that the radical factors $\mathrm{rad}_\alpha({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$ are only modules with zero radical, and not semisimple modules in general. The following two results consider the situation in which they are semisimple.

Lemma 4.4.1. *Let R be a ring and ${}_RM$ be a left R-module. Then $J(R)^n{}_RM \leq \mathrm{rad}_n({}_RM)$ for any positive integer n, and the equality holds when R is semilocal.*

PROOF. The first part of the statement follows from [14, Lemma 1.3] by induction on n. For the second, assume that R is semilocal. It suffices to show that $\mathrm{rad}({}_RM) = J(R)_RM$ for any module ${}_RM$. Now ${}_RM/J(R)_RM$ is a left $R/J(R)$-module, so that ${}_RM/J(R)_RM$ is semisimple. Since the radical of a semisimple module is 0 and $J(R)_RM \leq \mathrm{rad}({}_RM)$, one has that

$$\overline{0} = \mathrm{rad}({}_RM/J(R)_RM) = \mathrm{rad}({}_RM)/J(R)_RM.$$

Hence $\mathrm{rad}({}_RM) = J(R)_RM$. ∎

Proposition 4.4.2. *The following conditions are equivalent for a ring R:*

1. *The radical factor* $\mathrm{rad}_\alpha({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$ *is semisimple for every left R-module ${}_RM$ and every ordinal α.*
2. *The radical factor ${}_RM/\mathrm{rad}({}_RM)$ is semisimple for every left R-module ${}_RM$.*
3. *The ring R is semilocal.*

PROOF. For (1) ⇒ (2) take $\alpha = 0$. For (2) ⇒ (3) take ${}_RM = {}_RR$.

(3) ⇒ (1) By Lemma 4.4.1, $\mathrm{rad}_{\alpha+1}({}_RM) = J(R)\mathrm{rad}_\alpha({}_RM)$, so that

$$\mathrm{rad}_\alpha({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$$

is an $R/J(R)$-module. By (3), $R/J(R)$ is semisimple artinian. Therefore

$$\mathrm{rad}_\alpha({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$$

is a semisimple R-module. ■

Notice that the radical of a module is a fully invariant submodule. More generally, every module homomorphism $f\colon {}_RM \to {}_RN$ restricts to a homomorphism

$$f|_{\mathrm{rad}({}_RM)}\colon \mathrm{rad}({}_RM) \to \mathrm{rad}({}_RN)$$

and induces a homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RN/\mathrm{rad}({}_RN)$. By transfinite induction, we have that f restricts to a homomorphism $f|_{\mathrm{rad}_\alpha({}_RM)}\colon \mathrm{rad}_\alpha({}_RM) \to \mathrm{rad}_\alpha({}_RN)$ for every ordinal α and induces a homomorphism

$$\overline{f_\alpha}\colon {}_RM/\mathrm{rad}_\alpha({}_RM) \to {}_RN/\mathrm{rad}_\alpha({}_RN).$$

Proposition 4.4.3. *Let R be a semilocal ring, ${}_RM$ be a max left R-module with finite radical invariants and $f \in \mathrm{End}({}_RM)$ be an endomorphism of ${}_RM$. The following conditions are equivalent:*

1. *f is an automorphism of ${}_RM$.*
2. *The endomorphism f is surjective.*
3. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$.*
4. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is an automorphism of ${}_RM/\mathrm{rad}({}_RM)$.*

PROOF. The implications (1) ⇒ (2) ⇒ (3) are obvious. (3) ⇒ (4) follows from the fact that R is semilocal and ${}_RM$ has finite radical invariants, so that ${}_RM/\mathrm{rad}({}_RM)$ is a semisimple module of finite Goldie dimension (Proposition 4.4.2). Thus a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$ is an automorphism.

(4) ⇒ (1). Let ${}_RM$ be a max left module with radical length λ and all its radical invariants finite. Let f be a homomorphism of ${}_RM$ with $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ an automorphism of ${}_RM/\mathrm{rad}({}_RM)$. The radical of a max module is a superfluous submodule. Thus $\overline{f}$ surjective implies f surjective. It remains to show that f is injective. We will prove by transfinite induction that $\overline{f_\alpha}\colon {}_RM/\mathrm{rad}_\alpha({}_RM) \to {}_RM/\mathrm{rad}_\alpha({}_RM)$ is injective for every ordinal α, and the case $\alpha = \lambda$ will show that (1) holds.

The case $\alpha = 0$ is trivial, and the case $\alpha = 1$ is condition (4). Assume that our property holds for any ordinal $\beta < \alpha$. That is, $\overline{f_\beta}\colon {}_RM/\mathrm{rad}_\beta({}_RM) \to {}_RN/\mathrm{rad}_\beta({}_RN)$ is injective for every ordinal $\beta < \alpha$. If the ordinal α is limit, $x \in {}_RM$ and $(x)f \in \mathrm{rad}_\alpha({}_RM)$, then $(x)f \in \mathrm{rad}_\beta({}_RM)$ for every $\beta < \alpha$. By the inductive hypothesis, $x \in \mathrm{rad}_\beta({}_RM)$ for every $\beta < \alpha$, that is, $x \in \mathrm{rad}_\alpha({}_RM)$. This proves that $\overline{f_\alpha}$ is injective. If α is a non-limit ordinal, then $\alpha = \beta + 1$ for some ordinal β. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{rad}_\beta(M)/\mathrm{rad}_{\beta+1}(M) & \longrightarrow & M/\mathrm{rad}_{\beta+1}(M) & \longrightarrow & M/\mathrm{rad}_\beta(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \overline{f_{\beta+1}}} & & \downarrow{\scriptstyle \overline{f_\beta}} & & \\
0 & \longrightarrow & \mathrm{rad}_\beta(M)/\mathrm{rad}_{\beta+1}(M) & \longrightarrow & M/\mathrm{rad}_{\beta+1}(M) & \longrightarrow & M/\mathrm{rad}_\beta(M) & \longrightarrow & 0.
\end{array}$$

In this diagram, $\overline{f_{\beta+1}}$ is surjective and $\overline{f_\beta}$ is an automorphism by the inductive hypothesis. The Snake Lemma yields an exact sequence

$$0 \longrightarrow \ker(f') \longrightarrow \ker(\overline{f_{\beta+1}}) \longrightarrow 0 \longrightarrow \mathrm{coker}(f') \longrightarrow 0.$$

Thus f' is an epimorphism. Now $\mathrm{rad}_\beta({}_RM)/\mathrm{rad}_{\beta+1}({}_RM)$ is a semisimple module of finite composition length, so that its surjective endomorphisms are automorphisms. Hence $\ker(f') = 0$, so that $\ker(\overline{f_{\beta+1}}) = 0$, and $\overline{f_\alpha} = \overline{f_{\beta+1}}$ is injective. ■

In Proposition 4.4.3, if the commutativity is added to the ring R, that is, R is a commutative semilocal ring, then we find the same conclusion for any max module ${}_RM$ with only $r_0({}_RM)$ finite. That is:

Proposition 4.4.4. *Let R be a commutative semilocal ring, ${}_RM$ be a max R-module whose 0-th radical invariant is finite and $f \in \mathrm{End}({}_RM)$ be an endomorphism of ${}_RM$. The following conditions are equivalent:*

1. *f is an automorphism of ${}_RM$.*
2. *The endomorphism f is surjective.*
3. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$.*
4. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is an automorphism of ${}_RM/\mathrm{rad}({}_RM)$.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are like in the previous Proposition. As far as (4) $\Rightarrow$ (1) is concerned, let f be an endomorphism of ${}_RM$ with $\overline{f}$ an automorphism of ${}_RM/\mathrm{rad}({}_RM)$. The submodule $\mathrm{rad}({}_RM)$ is superfluous in ${}_RM$, so that f is a surjective endomorphism of ${}_RM$. Moreover, ${}_RM/\mathrm{rad}({}_RM)$ is finitely generated by hypothesis, so that $\mathrm{rad}({}_RM)$ superfluous in ${}_RM$ implies that ${}_RM$ is finitely generated. Now surjective endomorphisms of finitely generated modules over a commutative ring are isomorphisms, which implies that f is an automorphism of ${}_RM$. ■

Corollary 4.4.5. *Let ${}_RM$ be a max module with finite radical invariants over a semilocal ring R. Then the ring homomorphism*

$$\iota\colon \mathrm{End}({}_RM) \to \mathrm{End}({}_RM/\mathrm{rad}({}_RM)), \quad f \mapsto \overline{f},$$

is a local homomorphism. In particular, $\mathrm{End}({}_RM)$ *is a semilocal ring of dual Goldie dimension $\leq r_0({}_RM)$.*

PROOF. The ring homomorphism $\iota\colon \mathrm{End}({}_RM) \to \mathrm{End}({}_RM/\mathrm{rad}({}_RM))$ is local by Proposition 4.4.3. Now apply [9, Corollary 2]. ■

Compare Corollary 4.4.5 with [20, Theorem 3.3], where it is proved that finitely presented modules over a semilocal ring R have a semilocal endomorphism ring.

The result analogous to Corollary 4.3.3 is:

Corollary 4.4.6. *Let R be a semilocal ring and ${}_RM$ a max module with finite radical invariants and ${}_RM/\mathrm{rad}({}_RM)$ simple. Then* $\mathrm{End}({}_RM)$ *is a local ring.*

4.5 Modules over a perfect ring

By Theorems 4.1.11 and 4.1.15, every (right or left) module over a perfect ring is both Loewy and max.

Proposition 4.5.1. *Every left module over a perfect ring with finite* 0*-th radical invariant and finite* 0*-th Loewy invariant has a semilocal endomorphism ring. In particular,* *every finitely generated left module with finite Goldie dimension over a perfect ring has a semilocal endomorphism ring.*

PROOF. Let R be a perfect ring and ${}_RM$ be a left R-module whose socle $\mathrm{soc}({}_RM)$ and factor ${}_RM/\mathrm{rad}({}_RM)$ are the direct sums of finitely many simple submodules. Since ${}_RM$ is Loewy and max, we know that $\mathrm{soc}({}_RM)$ is essential and $\mathrm{rad}({}_RM)$ is superfluous in ${}_RM$. Moreover, the Goldie dimension $\dim({}_RM) = \dim(\mathrm{soc}({}_RM))$ and the dual Goldie dimension $\mathrm{codim}({}_RM) = \mathrm{codim}({}_RM/\mathrm{rad}({}_RM))$ are finite, so that the endomorphism ring of ${}_RM$ is semilocal [27, Theorem 3.3]. The second statement follows immediately from the first. ■

Theorem 4.5.2. *For a left module ${}_RM$ over a semiprimary ring R, the following conditions are equivalent:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is artinian.*
3. *All Loewy invariants of ${}_RM$ are finite.*
4. *All radical invariants of ${}_RM$ are finite.*

PROOF. The equivalence (1) $\Leftrightarrow$ (2) is well known (Theorem 4.1.6). We have that (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3), so it suffices to prove (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2). Let $J = J(R)$ be the Jacobson radical of R and k be the nilpotency index of J, that is, $J^k = 0$ and $J^\ell \neq 0$ for $\ell < k$.

To see that (4) $\Rightarrow$ (1), it suffices to show that there exists a positive integer m such that $\mathrm{rad}_m({}_RM) = 0$. This is trivial because $\mathrm{rad}_k({}_RM) = J^k{}_RM = 0$ by Lemma 4.4.1.

The proof of (3) $\Rightarrow$ (2) is by induction on k. Assume that ${}_RM$ has finite Loewy invariants. If $k = 1$, then R is semisimple, so that ${}_RM = \mathrm{soc}({}_RM)$, the direct sum of finitely many simple submodules, is artinian. Suppose that $k > 1$ and that every module with finite Loewy invariants over a semiprimary ring whose Jacobson radical has nilpotency index strictly less than k is artinian. We must show that ${}_RM$ is artinian. Since $\mathrm{soc}({}_RM)$ is the direct sum of finitely many simple modules, it is enough to show that ${}_RM/\mathrm{soc}({}_RM)$ is artinian. One has $J^k{}_RM = 0$, so $J^{k-1}M$ is annihilated by J, hence is an R/J-module, hence a semisimple R-module. Thus $J^{k-1}M \leq \mathrm{soc}({}_RM)$, and the factor module ${}_RM/\mathrm{soc}({}_RM)$ is a left R/J^{k-1}-module. Since $J(R/J^{k-1}) = J/J^{k-1}$ has nilpotence index $k-1$ and ${}_RM/\mathrm{soc}({}_RM)$ has finite Loewy invariants, ${}_RM/\mathrm{soc}({}_RM)$ is artinian as an R/J^{k-1}-module. Thus ${}_RM/\mathrm{soc}({}_RM)$ is also artinian as an R-module. ∎

Example 4.5.3. Let R be a semilocal ring and ${}_RM$ be a max left R-module. From Proposition 4.4.4, if R is commutative and ${}_RM$ has finite dual Goldie dimension, then the endomorphism ring $\mathrm{End}({}_RM)$ of ${}_RM$ is semilocal. In [20, Example 3.5] there is an example of a finitely generated module ${}_RM$ over a semiprimary ring R whose endomorphism ring is not semilocal. As R is semiprimary, every left R-module is max. Finitely generated modules over a semilocal ring have finite dual Goldie dimension. In particular, the 0-th radical invariant $r_0({}_RM)$ is finite. Since the endomorphism ring $\mathrm{End}({}_RM)$ is not semilocal, the module ${}_RM$ is not noetherian.

This example shows that in case ${}_RM$ has finite dual Goldie dimension but R is not commutative, the ring $\mathrm{End}({}_RM)$ is not necessarily semilocal. Moreover, it also proves that in Theorem 4.5.2, we cannot weaken (4) (i.e., that "all radical invariants of ${}_RM$ are finite") to "${}_RM$ has finite 0-th radical invariant". That is, the module ${}_RM$ in the example has finite 0-th radical invariant, but ${}_RM$ is not noetherian.

A module is called a *Shock module* if all its factor modules are max [22]. For example, every noetherian module over an arbitrary ring and all modules over a perfect ring are Shock. The dual of Proposition 4.2.1 is as follows.

Proposition 4.5.4. *The following conditions are equivalent for a left module ${}_RM$ over an arbitrary ring R:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is Shock and has Krull dimension.*
3. *${}_RM$ is Shock and q.f.d.*
4. *${}_RM$ is Shock and its factor modules have finitely generated socles.*

PROOF. (1) ⇒ (2) ⇒ (3) ⇒ (4) are elementary and (4) ⇒ (1) is proved in [38, Theorem 3.8]. ∎

Since semiprimary rings are perfect rings, the following Theorem generalizes the Hopkins-Levitzki Theorem 4.5.2. Its proof follows from Propositions 4.2.1 and 4.5.4.

Theorem 4.5.5. *Let R be a perfect ring. The following conditions are equivalent for a left R-module ${}_RM$:*

1. *$l({}_RM)$ is finite.*
2. *${}_RM$ is artinian.*
3. *${}_RM$ is noetherian.*
4. *${}_RM$ has Krull dimension.*
5. *${}_RM$ is q.f.d.*
6. *All factor modules of ${}_RM$ have finitely generated socles.*
7. *All factor modules of ${}_RM$ are essential extensions of finitely generated modules.*

From Theorem 4.5.5, we immediately get the following corollaries.

Corollary 4.5.6. *Every uniserial module over a perfect ring has a local endomorphism ring.*

PROOF. Every uniserial module is q.f.d., hence of finite composition length, and indecomposable. Indecomposable modules of finite composition length have a local endomorphism ring. ∎

Corollary 4.5.7. *Every module with Krull dimension over a perfect ring has Krull dimension* 0.

Corollary 4.5.8. *The following conditions are equivalent for a ring R:*

1. *R is left artinian.*
2. *R is perfect and left noetherian.*
3. *R is perfect and the left module ${}_RR$ has Krull dimension.*
4. *R is perfect and the left module ${}_RR$ is q.f.d.*
5. *R is perfect and, for any left ideal I of R, the left module ${}_RR/I$ has a finitely generated socle.*
6. *R is perfect and, for any left ideal I of R, the left module ${}_RR/I$ is an essential extension of a finitely generated module.*