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Chapter 3

Maximal ideals of the endomorphism ring of an injective module

It is well known that the endomorphism ring of an indecomposable injective module is local (Theorem 1.4.2), hence has a unique maximal right (left, two-sided) ideal. Similarly, the maximal right (left, two-sided) ideals of the endomorphism ring $\mathrm{End}(V_D)$ of a right vector space V_D over a division ring D are easy to describe [35]. In this case, the left ideals of $\mathrm{End}(V_D)$ correspond to the filters of the lattice $\mathcal{L}(V_D)$ of all subspaces of V_D, and the right ideals of $\mathrm{End}(V_D)$ correspond to the antifilters of $\mathcal{L}(V_D)$. In particular, maximal left ideals of $\mathrm{End}(V_D)$ correspond to ultrafilters, and maximal right ideals to ultraantifilters. The description of the maximal two-sided ideals of $\mathrm{End}(V_D)$ is also known [35, Theorem 4.3]. In the first Section we will epitomize these results.

The aim of the three Sections 3.2, 3.3 and 3.4 is to describe the maximal right ideals and maximal left ideals of the endomorphism ring S of an arbitrary injective module M_R over a ring R. In this case also, maximal left ideals of S correspond to ultrafilters of a suitable lattice $\mathcal{O}$, which is the set of orbits of the set of all direct summands of M_R under a suitable group action. In Section 3.2, we will present the construction of the lattice $\mathcal{O}$, which is based on a suitable equivalence between injective envelopes of a module, and then, in Section 3.3, we present the correspondence. In fact, this correspondence works not only for maximal left ideals, but also, more generally, for any left ideal containing the Jacobson radical $J(S)$ of S. Similarly, we have a one-to-one correspondence between the set of all right ideals of S containing $J(S)$ and the set of all antifilters of $\mathcal{O}$, which will be presented in Section 3.4.

Maximal two-sided ideals of S correspond to prime ideals of height one of the commutative monoid $V(M_R)$. Here $V(M_R)$ is the commutative monoid of all direct summands of the modules M_R^n, $n \geq 0$, up to isomorphism, with the operation induced by direct sum. The monoid $V(M_R)$ is canonically isomorphic to the monoid $V(S)$ of all finitely generated projective right S-modules up to isomorphism. If P denotes the prime ideal of height one of $V(S)$ corresponding to the maximal two-sided ideal I of S, then the monoid $V(S/I)$ turns out to be isomorphic to the reduced localization $(V(S)_P)_{\text{red}}$ of $V(S)$ at the prime ideal P. Moreover, there is a one-to-one correspondence between the set of all maximal two-sided ideals of S and the set of all ultracofilters of the lattice $\mathcal{L}_{ds}(M_R)$ of all direct summands of M_R. These results appear in Sections 3.5 and 3.6.

The last two Sections are devoted to considering the case in which M_R is an injective envelope of a direct sum of indecomposable injective modules, that is, an injective envelope of a direct sum $\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)}$, where the A_λ's are pair-wise non-isomorphic indecomposable injective modules and the κ_λ's are cardinals.

The main results in this Chapter have been published in [16].

3.1 Maximal ideals of the endomorphism ring of a vector space over a division ring

In this Section, we epitomize some main results on maximal left, maximal right and maximal two-sided ideals of $\text{End}(V_D)$, where V_D is a right vector space over a division ring D.

It is well known that if V_D is a finite dimensional vector space, then the endomorphism ring $\text{End}(V_D)$ is isomorphic to the matrix ring $M_n(D)$ with $n = \dim V_D$, the dimension of V_D. Hence, in this case, $\text{End}(V_D)$ has the unique maximal two-sided ideal 0.

In the case of infinite dimensional vector spaces, Orsatti and Rodino described maximal left, maximal right and maximal two-sided ideals in [35] as follows.

First we need to recall some notions concerning lattices. Let L be a lattice with order $\preceq$, meet $\wedge$ and join $\vee$. A subset F of L is called a *filter* if

(F1): $F \neq \emptyset$.

(F2): $V_1 \vee V_2 \in F$ for any $V_1, V_2 \in F$.

(F3): if $V \in F$, $U \in L$ and $V \preceq U$, then $U \in F$.

If the filter F of L is different from L, then we say that F is a *proper filter*. If F is proper and maximal among all proper filters of L (ordered by inclusion), then F is said to be an *ultrafilter*. The filter F is called *principal* if there exists an element $a \in F$ such that $F = \{ x \in L \mid a \preceq x \}$. Otherwise, F is called *free*.

The dual notion of a filter is that of an antifilter (or *ideal*). A subset A of L is called an *antifilter* if

(AF1): $A \neq \emptyset$.

(AF2): $V_1 \wedge V_2 \in A$ for any $V_1, V_2 \in A$.

(AF3): if $V \in L$, $U \in A$ and $V \preceq U$, then $V \in A$.

If the antifilter A of L is different from L, then we say A a *proper antifilter*. If the filter A is proper and maximal among all proper antifilters of L, then A is said to be an *ultraantifilter*. The antifilter A is called *principal* if there exists an element $a \in A$ such that $A = \{ x \in L \mid x \preceq a \}$. Otherwise, A is called *free*.

From now to the end of this Section, it is assumed that V_D is a vector space with $\dim(V_D) = d$, an infinite cardinal. Let $S := \mathrm{End}(V_D)$ be the endomorphism ring of V_D. Let $\mathcal{L}(V_D)$ be the set of all subspaces of V_D. Then $\mathcal{L}(V_D)$ is a lattice with the inclusion order, the join $V_1 \vee V_2 = V_1 + V_2$ and the meet $V_1 \wedge V_2 = V_1 \cap V_2$ for any $V_1, V_2 \in \mathcal{L}(V_D)$.

Theorem 3.1.1. [35, Proposition 2.3 and 2.4] *Let S and $\mathcal{L}(V_D)$ be as above.*

1. *For any left ideal I of S, the set $\mathcal{F}(I) := \{ \ker f \mid f \in I \}$ is a filter of $\mathcal{L}(V_D)$.*

2. *For any filter F of $\mathcal{L}(V_D)$, the set $\mathcal{I}(F) := \{ f \in S \mid \ker f \in F \}$ is a left ideal of S.*

Moreover, these associations are mutually inverse one-to-one correspondences between the set of all left ideals of S and the set of filters of $\mathcal{L}(V_D)$.

Corollary 3.1.2. [35, Corollary 2.7] *For any left ideal of I of S, the following conditions are equivalent:*

1. *I is generated by an idempotent element.*
2. *I is generated by an element.*
3. *$\mathcal{F}(I)$ is principal.*

Dually, we have results for the right side.

Theorem 3.1.3. [35, Proposition 3.3 and 3.4] *Let $S, \mathcal{L}(V_D)$ be as above.*

1. *For any right ideal I of S, the set $\mathcal{A}(I) := \{ f(V_D) \mid f \in I \}$ is an antifilter of $\mathcal{L}(V_D)$.*

2. *For any antifilter A of $\mathcal{L}(V_D)$, the set $\mathcal{I}(A) := \{ f \in S \mid f(V_D) \in A\}$ is a right ideal of S.*

Moreover, these associations are two mutually inverse one-to-one correspondences between the set of all right ideals of S and the set of antifilters of $\mathcal{L}(V_D)$.

Corollary 3.1.4. [35, Corollary 3.8] *For any right ideal I of S, the following conditions are equivalent:*

1. *I is generated by an idempotent element.*
2. *I is generated by an element.*
3. *$\mathcal{A}(I)$ is principal.*

Next are some results concerning two-sided ideals of S.

Theorem 3.1.5. [35, Theorem 4.3] *Let S, V_D, d be as above.*

1. *For any infinite cardinal $c \leq d$, the set $L_c := \{ f \in S \mid \dim f(V_D) < c \}$ is a proper two-sided ideal of S.*

2. *For any proper two-sided ideal I of S, there exists an infinite cardinal $c \leq d$ such that $I = L_c$.*

Moreover, these associations are two mutually inverse one-to-one correspondences between the set of all infinite cardinals $c \leq d$ and the set of proper two-sided ideals of S.

Corollary 3.1.6. $L_d = \{ f \in S \mid \dim f(V_D) < d \}$ *is the unique maximal two-sided ideal of S.*

3.2 The lattice $\mathcal{O}$

From now to the end of this Chapter, M_R will always be an injective right module over an arbitrary ring R and S will be the endomorphism ring of M_R.

Lemma 3.2.1. *Let B be a submodule of M_R and N_1, N_2 be two direct summands of M_R with $B \leq_e N_1$ and $B \leq_e N_2$. Then there exist an essential submodule A of M_R containing B and an automorphism f of M_R such that $N_2 = f(N_1)$ and $f(a) = a$ for every $a \in A$.*

Proof. Suppose $M_R = N_1 \oplus N_1'$ and $M_R = N_2 \oplus N_2'$. Set $B_1 := B \oplus N_1'$, $B_2 := B \oplus N_2'$ and $A := B_1 \cap B_2$. Then B_1 and B_2 are essential in M_R, hence so is A. Since both N_1 and N_2 are injective essential extensions of B, there is an isomorphism $g \colon N_1 \to N_2$ that is the identity on B. Extend g to an isomorphism $h \colon N_1 + A \to N_2 + A$ setting $h(x + a) = g(x) + a$ for every $x \in N_1$ and every $a \in A$. Now extend h to an isomorphism $f \colon M_R \to M_R$. Then $f(N_1) = N_2$ and $f(a) = a$ for every $a \in A$. ■

Let $\mathrm{Aut}(M_R)$ be the group of all automorphisms of M_R, that is, the group of all invertible elements of S. Let G be the subgroup of $\mathrm{Aut}(M_R)$ consisting of the automorphisms φ of M_R for which there exists an essential submodule A of M_R with $\varphi(a) = a$ for every $a \in A$, that is, the automorphisms of M_R that are the identity on some essential submodule of M_R. One has that G is a normal subgroup of $\mathrm{Aut}(M_R)$ [24].

The group G acts on the set $\mathcal{L}_{ds}(M_R)$ of all direct summands of M_R in a natural way. Let $\mathcal{O} := \mathcal{L}_{ds}(M_R)/G$ be the set of all orbits GN, $N \in \mathcal{L}_{ds}(M_R)$. Define a relation $\preceq$ on the set $\mathcal{O}$, setting, for every $M', M'' \in \mathcal{L}_{ds}(M_R)$, $GM' \preceq GM''$ if there exists an essential submodule A of M_R with $M' \cap A \leq M''$.

Lemma 3.2.2. *The relation $\preceq$ is a partial order on the set $\mathcal{O}$.*

Proof. First of all, one must show that the relation $\preceq$ is well defined. Suppose that $M', M'', N', N'' \in \mathcal{L}_{ds}(M_R)$, $GM' = GN'$, $GM'' = GN''$, and that there exists an essential submodule A of M_R with $M' \cap A \leq M''$. Then there are automorphisms φ and ψ of M_R such that $N' = \varphi(M')$, $N'' = \psi(M'')$, φ is the identity on an essential submodule A' of M_R and ψ is the identity on an essential submodule A''. Then $N' \cap A' \cap A'' \cap A =$

$\varphi(M') \cap A' \cap A'' \cap A = M' \cap A' \cap A'' \cap A$, and similarly $N'' \cap A'' \cap A' \cap A = M'' \cap A'' \cap A' \cap A$. Thus $N' \cap A' \cap A'' \cap A = M' \cap A' \cap A'' \cap A \leq M'' \cap A' \cap A'' \cap A = N'' \cap A' \cap A'' \cap A \leq N''$. This proves that the relation $\preceq$ is well defined.

The relation $\preceq$ is clearly reflexive and transitive. As far as symmetry is concerned, suppose $M', M'' \in \mathcal{L}_{ds}(M_R)$, $GM' \preceq GM''$ and $GM'' \preceq GM'$. Then there exist essential submodules A, A' of M_R with $M' \cap A \leq M''$ and $M'' \cap A' \leq M'$. Thus $M' \cap A \cap A' = M'' \cap A \cap A'$. Now $A \cap A'$ is an essential submodule of M_R, so that $M' \cap A \cap A'$ is essential in M' and $M'' \cap A \cap A'$ is essential in M''. By Lemma 3.2.1, one has that $GM' = GM''$. ■

Lemma 3.2.3. *The partially ordered set $\mathcal{O}$ is a lattice with*

$$GM_1 \wedge GM_2 := G(E(M_1 \cap M_2)) \qquad \textit{and} \qquad GM_1 \vee GM_2 := G(E(M_1 + M_2))$$

for any pair $GM_1, GM_2 \in \mathcal{O}$.

Proof. The most important thing in this Lemma is to prove that the meet $\wedge$ and the join $\vee$ defined above are well-defined. The proof of the rest is elementary. Let $GM_1, GM_2, GM_1', GM_2' \in \mathcal{O}$ such that $GM_1 = GM_1'$ and $GM_2 = GM_2'$. Then there exist automorphisms φ, ψ and essential submodules A, B of M_R such that $M_1' = \varphi(M_1), M_2' = \psi(M_2)$, $\varphi(a) = a$ for any $a \in A$ and $\psi(a) = a$ for every $a \in B$. Let $C = M_1 \cap M_2 \oplus N$ where N is a submodule of M_R such that $E(M_1 \cap M_2) \oplus N = M_R$. Then C is essential in M_R. Now, $M_1 \cap M_2 \cap A \cap B \cap C = \varphi(M_1) \cap \psi(M_2) \cap A \cap B \cap C = M_1' \cap M_2' \cap A \cap B \cap C \leq E(M_1' \cap M_2')$. Because $A \cap B \cap C$ is essential in M_R, $GE(M_1 \cap M_2) \preceq GE(M_1' \cap M_2')$. Since the symmetry of GM_1, GM_2 and GM_1', GM_2', one has $GE(M_1 \cap M_2) = GE(M_1' \cap M_2')$, which means that the meet $\wedge$ is well-defined. Similarily for the join $\vee$. ■

3.3 Maximal left ideals

Theorem 3.3.1. *Associate to any left ideal I of S containing $J(S)$ the subset $\mathcal{F}(I) := \{ G \ker e \mid e \in I, e^2 = e \}$ of $\mathcal{O}$. Conversely, associate to any filter F of $\mathcal{O}$ the subset*

$\mathcal{I}(F) := \{\, he + j \mid G\ker e \in F, h \in S, j \in J(S) \,\}$ *of* S*. Then this defines two mutually inverse one-to-one correspondences between the set of all left ideals of* S *containing* $J(S)$ *and the set of all filters of* $\mathcal{O}$*.*

PROOF. First of all, we will prove that $\mathcal{F}(I)$ is a filter of $\mathcal{O}$ and $\mathcal{I}(F)$ is a left ideal of S for every left ideal I of S containing $J(S)$ and every filter F in $\mathcal{O}$. Suppose I is a left ideal of S containing $J(S)$. Clearly $GM = G\ker 0 \in \mathcal{F}(I)$, which is therefore non-empty. Now, given $G\ker e \in \mathcal{F}(I)$ where e is an idempotent of I and $GN \in \mathcal{O}$ such that $G\ker e \preceq GN$, we must show that $GN \in \mathcal{F}(I)$. Let e' be an idempotent of S such that $\ker e' = N$. As $G\ker e \preceq GN = G\ker e'$, there exists an essential submodule A of M_R such that $\ker e \cap A \leq \ker e'$. By Lemma 2.3.7, there exist $g \in S, j \in J(S)$ such that $e' = ge + j$. Hence $e' \in I$. Therefore $GN \in \mathcal{F}(I)$, as we wanted to prove. In order to conclude the proof that $\mathcal{F}(I)$ is a filter, fix two elements $GN_1, GN_2 \in \mathcal{F}(I)$. Thus $GN_1 = G\ker e_1$ and $GN_2 = G\ker e_2$ for idempotents $e_1, e_2 \in I$. Since $S/J(S)$ is Von Neumann regular and idempotents of $S/J(S)$ can be lifted to idempotents of S, $Se_1 + Se_2 + J(S) = Se + J(S)$ for some idempotent $e \in S$ (Theorem 1.4.1). Hence, $e \in I$ and $e_1 = s_1e + j_1, e_2 = s_2e + j_2$ for some $s_1, s_2 \in S$ and $j_1, j_2 \in J(S)$. Thus, $A = \ker j_1 \cap \ker j_2$ is an essential submodule of M_R and $\ker e \cap A \leq \ker e_1 \cap \ker e_2$. This shows that $G\ker e \preceq GE(\ker e_1 \cap \ker e_2)$. Therefore, $GN_1 \wedge GN_2 = GE(\ker e_1 \cap \ker e_2) \in \mathcal{F}(I)$. This proves that $\mathcal{F}(I)$ is a filter in $\mathcal{O}$.

Similarly, we have, dually, $\mathcal{I}(F)$ is a left ideal of S containing $J(S)$ for every filter F in $\mathcal{O}$.

Now we will prove that $\mathcal{F}(\mathcal{I}(F)) = F$ and $\mathcal{I}(\mathcal{F}(I)) = I$. The inclusion $\mathcal{F}(\mathcal{I}(F)) \subseteq F$ is trivial. For the inclusion $F \subseteq \mathcal{F}(\mathcal{I}(F))$, notice that an arbitrary element of $F \subseteq \mathcal{O}$ can be written in the form $G\ker e$ for some idempotent $e \in S$. Then $e \in \mathcal{I}(F)$. It follows that $G\ker e \in \mathcal{F}(\mathcal{I}(F))$. Thus $\mathcal{F}(\mathcal{I}(F)) = F$. Similarly, the proof of the inclusion $I \subseteq \mathcal{I}(\mathcal{F}(I))$ is easy. For the opposite inclusion, suppose $e \in \mathcal{I}(\mathcal{F}(I))$. Then $G\ker e \in \mathcal{F}(I)$. Thus there exists an idempotent $e' \in I$ such that $G\ker e = G\ker e'$. In particular, $\ker e' \cap A \leq \ker e$. From Lemma 2.3.7, we have that $e = ge' + j$ for suitable $g \in S, j \in J(S)$. This proves that $e \in I$, and $I = \mathcal{I}(\mathcal{F}(I))$. ∎

Corollary 3.3.2. *Maximal left ideals of S correspond to ultrafilters of $\mathcal{O}$.*

3.4 Maximal right ideals

In this section, we will turn our attention to maximal right ideals.

Theorem 3.4.1. *Associate to any right ideal I of S containing $J(S)$ the subset $\mathcal{A}(I) := \{ Ge(M_R) \mid e \in I, e^2 = e \}$ of $\mathcal{O}$. Conversely, associate to any antifilter A of $\mathcal{O}$ the subset $\mathcal{I}(A) := \{ eh + j \mid e, h \in S,\ j \in J(S),\ e^2 = e,\ Ge(M_R) \in A \}$ of S. Then these associations are two mutually inverse one-to-one correspondences between the set of all right ideals of S containing $J(S)$ and the set of all antifilters of $\mathcal{O}$.*

Proof. We first show that $\mathcal{A}(I)$ is an antifilter of $\mathcal{O}$ for any right ideal I of S containing $J(S)$. From $0 \in I$, it follows that $G0 = G0(M_R) \in \mathcal{A}(I)$.

Now assume that $Ge'(M_R) \preceq Ge(M_R)$, where e, e' are idempotents in S and $Ge(M_R) \in \mathcal{A}(I)$. Then there exists an essential submodule A of M_R with $e'(M_R) \cap A \leq e(M_R)$. By Lemma 2.3.5 and $e \in I$, $e' \in eS + J(S) \subseteq I$. Hence, $Ge'(M_R) \in \mathcal{A}(I)$. Suppose that $Ge_1(M_R), Ge_2(M_R) \in \mathcal{A}(I)$, with e_1, e_2 idempotents belonging to I. Since Theorem 1.4.1, $S/J(S)$ is Von Neumann regular and idempotents of $S/J(S)$ can be lifted to idempotents of S, it follows that there exists an idempotent $e_3 \in S$ such that $(e_1S + e_2S + J(S))/J(S) = (e_3S + J(S))/J(S)$. In particular, $e_3 \in I$, $e_1 \in e_3S + J(S)$ and $e_2 \in e_3S + J(S)$. Hence, $Ge_3(M_R) \in \mathcal{A}(I)$, and, by Lemma 2.3.5, $e_1(M_R) \cap A_1 \leq e_3(M_R)$ and $e_2(M_R) \cap A_2 \leq e_3(M_R)$ for some essential submodules A_1, A_2 of M_R. Therefore, $Ge_1(M_R), Ge_2(M_R) \preceq Ge_3(M_R)$, which implies $Ge_1(M_R) \vee Ge_2(M_R) \preceq Ge_3(M_R)$. This shows that $Ge_1(M_R) \vee Ge_2(M_R) \in \mathcal{A}(I)$. One has $\mathcal{A}(I)$ is an antifilter of $\mathcal{O}$.

Next thing we will show is the fact that $\mathcal{I}(A)$ is a right ideal of S containing $J(S)$ for any antifilter A of $\mathcal{O}$. The only non-trivial thing to show is that $\mathcal{I}(A)$ is additively closed. For this, it suffices to prove that if $e_1, e_2 \in S$ are idempotents of S with $Ge_1(M_R), Ge_2(M_R) \in A$, then $e_1S + e_2S + J(S)$ is of the form $e_3S + J(S)$ for a suitable idempotent $e_3 \in S$ with $Ge_3(M_R) \in A$. Using the argument in the first paragraph, there exists idempotent $e_3 \in S$ such that $(e_1S + e_2S + J(S))/J(S) = (e_3S + J(S))/J(S)$

and $Ge_1(M_R) \vee Ge_2(M_R) \preceq Ge_3(M_R)$. It suffices to prove $Ge_3(M_R) \in A$. Indeed, we have $e_3 = e_1h_1 + e_2h_2 + j$ for some $h_1, h_2 \in S$ and $j \in J(S)$. Hence if put $B := \ker j$, then B is essential in M_R and $e_3(M_R) \cap B \leq e_1(M_R) + e_2(M_R)$, which implies $Ge_3(M_R) \preceq GE(e_1(M_R) + e_2(M_R)) = Ge_1(M_R) \vee Ge_2(M_R)$. Therefore, $Ge_3(M_R) = Ge_1(M_R) \vee Ge_2(M_R) \in A$.

Now we will show that $\mathcal{A}(\mathcal{I}(A)) = A$ and $\mathcal{I}(\mathcal{A}(I)) = I$. Let $Ge(M_R) \in A$ for some idempotent $e \in S$. Then $e \in \mathcal{I}(A)$, whence $Ge(M_R) \in \mathcal{A}(\mathcal{I}(A))$. Conversely, let $Ge(M_R) \in \mathcal{A}(\mathcal{I}(A))$ for some idempotent $e \in \mathcal{I}(A)$. Then there exist $e', g \in S$ and $j \in J(S)$ such that $e'^2 = e', Ge'(M_R) \in A$ and $e = e'g + j$. Using Lemma 2.3.5, one has that $e(M_R) \cap B \leq e'(M_R)$ for some essential submodule B of M_R. Hence, $Ge(M_R) \preceq Ge'(M_R)$, which implies $Ge(M_R) \in A$. Therefore $A = \mathcal{A}(\mathcal{I}(A))$.

Let $f \in I$. Then there exists an idempotent $e \in S$ such that $(fS + J(S))/J(S) = (eS + J(S))/J(S)$. Hence $e \in I$ and $f = eg + j$ for some $g \in S$ and $j \in J(S)$. We have that $Ge(M_R) \in \mathcal{A}(I)$, so $f \in \mathcal{I}(\mathcal{A}(I))$. Conversely, let $f \in \mathcal{I}(\mathcal{A}(I))$. Then there exists idempotent $e \in S$ such that $Ge(M_R) \in \mathcal{A}(I), f = es + i$ for some $s \in S, i \in J(S)$. Hence, $Ge(M_R) = Ge'(M_R)$ for some idempotent $e' \in I$. In particular, $e(M_R) \cap B \leq e'(M_R)$ for some essential submodule B of M_R. By Lemma 2.3.5, $e \in e'S + J(S) \subseteq I$. Therefore $f \in I$. This proves that $I = \mathcal{I}(\mathcal{A}(I))$. ■

Corollary 3.4.2. *Maximal right ideals of S correspond to ultraantifilters of $\mathcal{O}$.*

3.5 Maximal two-sided ideals

Recall that, for any module M_R, it is possible to define a commutative monoid $V(M_R)$ as follows. Given the set $\mathcal{S}(M_R)$ of all direct summands of the modules M_R^n, $n \geq 0$, fix a complete set $V(M_R)$ of representatives of the modules in the set $\mathcal{S}(M_R)$ up to isomorphism. Thus, for every $A \in \mathcal{S}(M_R)$, there is a unique $\langle A \rangle \in V(M_R)$ with $A \cong \langle A \rangle$. Direct sum induces an addition $+$ on the set $V(M_R)$, defined by setting, for every $\langle A \rangle, \langle B \rangle \in V(M_R)$, $\langle A \rangle + \langle B \rangle := \langle A \oplus B \rangle$. Then $V(M_R)$ with this operation becomes a commutative monoid, which naturally describes the direct sum decompositions in the set

$\mathcal{S}(M_R)$. The algebraic pre-order of $V(M_R)$ is defined as follows: If $\langle X\rangle, \langle Y\rangle \in V(M_R)$ then $\langle X\rangle \leq \langle Y\rangle$ if and only if X is isomorphic to a direct summand of Y

For any ring T, we will denote by $V(T) = V(T_T)$ the commutative monoid of a set of representatives of all finitely generated projective right T-modules up to isomorphism. If S denotes the endomorphism ring of a right R-module M_R, the monoid $V(M_R)$ turns out to be isomorphic to the monoid $V(S)$ [14, Theorem 4.7].

Recall that a commutative additive monoid C is *reduced* if $x, y \in C$ and $x+y = 0$ imply $x = y = 0$, that is, if $U(C) = 0$, where $U(C)$ denotes the group of invertible elements of the monoid C, that is, the set of the elements $a \in C$ with an "additive inverse" $-a$. The commutative monoid $V(M_R)$ is reduced for every module M_R. For every commutative monoid C, the monoid $C_{\text{red}} := C/U(C)$ is a reduced monoid.

A *prime ideal* of a commutative monoid C is a proper subset P of C such that, for any $x, y \in C$, one has $x + y \in P$ if and only if either $x \in P$ or $y \in P$. The complements of the prime ideals of C are exactly the *divisor-closed submonoids* of C, that is, the submonoids D of C such that, for all $x, y \in C$, $x + y \in D$ implies $x \in D$ and $y \in D$. The *spectrum* $\operatorname{Spec}(C)$ of a commutative monoid C is the set of all prime ideals of C.

The following Lemma is well known and easy to prove. A generalized version of it appears in [43, Corollary 4.1].

Lemma 3.5.1. *If X is a direct summand of $M_1 \oplus \cdots \oplus M_n$, where n is a positive integer and $M_1, \ldots, M_n$ are injective modules, then $X \cong X_1 \oplus \cdots \oplus X_n$, where X_i is a suitable direct summand of M_i for every $i = 1, \ldots, n$.*

For any ring S and any subset X of the commutative monoid $V(S)$, we will denote by $\operatorname{Tr}_S(X)$ the *trace* of X in S, that is, the sum of all images $f(A)$, where A ranges over X and f ranges over $\operatorname{Hom}(A, S_S)$. Observe that $\operatorname{Tr}_S(X)$ is a two-sided ideal of S. A two-sided ideal I of S is called a *trace ideal* if $I = \operatorname{Tr}_S(X)$ for some subset X of $V(S)$. A *maximal trace ideal* is a trace ideal that is maximal in the set of all proper trace ideals.

Lemma 3.5.2. *Let M_R be an injective module over a ring R. Denote by S the endomorphism ring of M_R. Assume that I is a two-sided ideal of S. Then I is a trace ideal if and only if I is generated by its idempotents.*

Proof. By Lemma 3.5.1, every direct summand of M_R^n is isomorphic to $X_1 \oplus \cdots \oplus X_n$, where each X_i is a direct summand of M_R. Applying the functor $\mathrm{Hom}(M_R, -)\colon$ Mod-$R \to$ Mod-S, we see that every finitely generated projective right S-module is isomorphic to a direct sum of finitely many cyclic projective right S-modules, that is, S-modules isomorphic to eS for suitable idempotents $e \in S$ [14, Theorem 4.7]. It follows that the trace of a projective S-module $e_1S \oplus \cdots \oplus e_nS$ is isomorphic to the two-sided ideal $\sum_{i=1}^n Se_iS$. Thus the trace ideals are generated by idempotent elements of S. Conversely, a two-sided ideal I generated by a set of idempotents $\{\, e_\lambda \mid \lambda \in \Lambda \,\}$ is the trace of the set $X = \{\, \langle e_\lambda S \rangle \mid \lambda \in \Lambda \,\}$. ∎

A prime ideal P of a commutative monoid C is a prime ideal *of height one* if it is minimal in the set of all prime non-empty ideals of C. From [19, Theorem 2.1(c)], we immediately get that:

Proposition 3.5.3. *Let M_R be an injective module over a ring R and S be the endomorphism ring of M_R. Then there is an order-reversing one-to-one correspondence between the set of all trace ideals of the ring S and the spectrum* $\mathrm{Spec}(V(M_R))$ *of the commutative monoid $V(M_R)$. In particular, maximal trace ideals correspond to prime ideals of height one of $V(M_R)$.*

Lemma 3.5.4. *Every maximal two-sided ideal of S is generated by its idempotents and $J(S)$.*

Proof. Let I be a maximal two-sided ideal of S. Then $I \supseteq J(S)$ and $S/J(S)$ is a Von Neumann regular ring (Theorem 1.4.1). Hence the right ideal $I/J(S)$ of $S/J(S)$, which is the sum of its finitely generated right ideals, is a sum of two-sided ideals of $S/J(S)$ generated by idempotents of $S/J(S)$. Every idempotent of $S/J(S)$ lifts to an idempotent of S (Theorem 1.4.1), so that I is generated as a right ideal by its idempotents and $J(S)$. Therefore I is generated a fortiori as a two-sided ideal by its idempotents and $J(S)$. ∎

Proposition 3.5.5. *If A is a maximal two-sided ideal of S, then the two-sided ideal generated by the idempotents of S is a maximal trace ideal of S. Conversely, if B is a maximal trace ideal of S, then $B + J(S)$ is a maximal two-sided ideal of S.*

Proof. Let A be a maximal two-sided ideal of S and let A' be the two-sided ideal of S generated by all the idempotents of A. Let $C \supseteq A'$ be any other proper trace ideal of S, that is, a two-sided ideal generated by its idempotents. By Lemma 3.5.4, $A = A' + J(S) \subseteq C + J(S)$. By the maximality of A, it follows that either $C + J(S) = A$ or $C + J(S) = S$. If $C + J(S) = A$, then all idempotents of C are in A, hence in A'. Thus $A' = C$ in this case. If $C + J(S) = S$, then $C = S$, because the right S-module $J(S)$ is a superfluous in the right S-module S by Nakayama's Lemma. This proves that A' is a maximal trace ideal of S.

Conversely, let B be a maximal trace ideal of S. We want to prove that $B + J(S)$ is a maximal two-sided ideal in S. Notice that $B + J(S)$ is proper, otherwise $B + J(S) = S$ implies $B = S$ because $J(S)$ is superfluous right ideal of S. Let B' be a maximal two-sided ideal of S containing $B + J(S)$. Now $B \subseteq B + J(S) \subseteq B'$, so that all idempotents of B are in B'. Let B'' be the two-sided ideal of S generated by all the idempotents of B'. Then $B \subseteq B''$, and these are proper two-sided ideals because B' is a proper two-sided ideal. By the maximality of B, we have that $B = B''$. By Lemma 3.5.4, $B' = B'' + J(S) = B + J(S)$. ■

Lemma 3.5.6. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of the ring S and the set of all maximal trace ideals of S.*

Proof. The correspondence associates to every maximal two-sided ideal A of S the two-sided ideal of S generated by the idempotents of A. This is a maximal trace ideal by Proposition 3.5.5. The inverse correspondence associates to any maximal trace ideal B of S, the maximal two-sided ideal $B + J(S)$ of S (Proposition 3.5.5). These two correspondences are mutually inverse by Lemma 3.5.4. ■

From Proposition 3.5.3 and Lemma 3.5.6, we can conclude that:

Theorem 3.5.7. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of the ring S and the set of all prime ideals of height one in the commutative monoid $V(M_R)$.*

3.6 Reduced localization

Let P be a prime ideal of a commutative monoid C. In $C \times C\backslash P$, we define a relation $\sim$ as follows: for $(x,s),(x',s') \in C \times C\backslash P$, $(x,s) \sim (x',s')$ if there exists $t \in C$ such that $x + s' + t = x' + s + t$. It is easy to check that $\sim$ is an equivalence relation in $C \times C\backslash P$. Put $C_P := (C \times C\backslash P)/\sim$ and every class $[(x,s)]_\sim$ of C_P is denoted by $x - s$ for any $x \in C, s \in C\backslash P$. One can check that C_P with the addition defined by $(x-s)+(x'-s') = (x+x')-(s+s')$ for $x-s, x'-s' \in C_P$. The monoid C_P is called the *localization* of C at P. Let $U(C_P)$ be the set of additive inverses of C_P. The monoid $(C_P)_{\text{red}} := (C_P)_{U(C_P)}$ is called the *reduced localization* of C at P. Hence, the canonical homomorphism $C \to (C_P)_{\text{red}}$, defined by $x \mapsto x - 0 + U(C_P)$, is surjective.

Theorem 3.6.1. *Let S be the endomorphism ring of an injective module, I a maximal two-sided ideal of S and P the corresponding prime ideal of height one of $V(S)$. Then the reduced localization $(V(S)_P)_{\text{red}}$ and the monoid $V(S/I)$ are canonically isomorphic. The isomorphism maps an element $\langle A_S\rangle - \langle 0_S\rangle + U(V(S))$ of $(V(S)_P)_{\text{red}}$ to the element $\langle A_S/A_SI\rangle$ of $V(S/I)$.*

Proof. We must prove that the monoid homomorphism

$$V(\pi)\colon V(S) \to V(S/I),$$

induced by the canonical projection $\pi\colon S \to S/I$ and defined by $V(\pi)(\langle A_S\rangle) = \langle A_S/A_SI\rangle$, is surjective, and that $V(\pi)(\langle A_S\rangle) = V(\pi)(\langle B_S\rangle)$ if and only if there exist $C_S, D_S \in V(S)\setminus P$ with $A_S \oplus C_S \cong B_S \oplus D_S$.

In order to prove that the monoid homomorphism $V(\pi)\colon V(S) \to V(S/I)$ is surjective, notice that $I \supseteq J(S)$ and $S/J(S)$ is Von Neumann regular, so that S/I is Von Neumann regular. Thus every finitely generated projective S/I-module is a finite direct sum of projective cyclic S/I-modules, that is, principal right ideals $\overline{e}(S/I)$ of S/I, where $\overline{e}$ is an idempotent of S/I [25, Proposition 2.6]. Therefore it suffices to prove that idempotents of S/I lift to idempotents of S. This is asserted by [21, Theorem 19.27 (c)].

It remains to show that for every A_S, B_S finitely generated projective S-modules, $V(\pi)(A_S) = V(\pi)(B_S)$ if and only if there exist $C_S, D_S \in V(S)\setminus P$ with $A_S \oplus C_S \cong B_S \oplus$

D_S. Let I' be the maximal trace ideal generated by all the idempotent elements of I. The canonical projection $\pi\colon S \to S/I$ is the composite mapping of the canonical projections $\pi'\colon S \to S/I'$ and $\pi''\colon S/I' \to S/I$. Thus $V(\pi) = V(\pi'') \circ V(\pi')$. By [37, Theorem 1.5] (also see [19, Theorem 2.1(d)]), for every pair finitely generated projective S-modules A_S, B_S, one has $V(\pi')(A_S) = V(\pi')(B_S)$ if and only if there exist $C_S, D_S \in V(S) \setminus P$ with $A_S \oplus C_S \cong B_S \oplus D_S$. Therefore in order to conclude the proof, it is enough to show that, for every pair A_S, B_S of projective S-modules, $A_S/A_SI \cong B_S/B_SI$ implies $A_S/A_SI' \cong B_S/B_SI'$. We claim that $J(S/I') = I/I'$. To prove the claim, notice that $J(S/I')$ is the intersection of all M/I' where M ranges over the set of all maximal right ideals of S containing I'. But every maximal right ideal of S contains $J(S)$, hence contains I' if and only if it contains $I = I' + J(S)$. Now S/I is a simple ring, so that $J(S/I) = 0$. Thus the intersection of the maximal right ideals M of S containing I is I. This concludes the proof of our claim.

Now assume that A_S, B_S are finitely generated projective S-modules and that A_S/A_SI is isomorphic to B_S/B_SI. Then $A_S/A_SI', B_S/B_SI'$ are finitely generated projective S/I'-modules that are isomorphic modulo $J(S/I')$, because

$$\begin{aligned}(A_S/A_SI')/(A_S/A_SI')J(S/I') &= (A_S/A_SI')/(A_S/A_SI')(I/I') = \\ &= (A_S/A_SI')/(A_SI/A_SI') \cong A_S/A_SI,\end{aligned}$$

and similarly for B_R. Thus $A_S/A_SI \cong B_S/B_SI$ implies $A_S/A_SI' \cong B_S/B_SI'$, as desired. ■

Lemma 3.6.2. *If M_R is an injective module and M_1, M_2 are two direct summands of M_R with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M_R.*

Proof. The submodules M_1, M_2 are injective, so that $M_1 + M_2 = M_1 \oplus M_2$ is injective. Hence $M_1 \oplus M_2$ is a direct summand of M_R. ■

Let $L := \mathcal{L}_{ds}(M_R)$ be the set of all direct summands of M_R. Then L with set inclusion is a partially ordered set. We will write $X \perp Y$ if $X, Y \in L$ and $X \cap Y = 0$. In this case, that is, if $X \perp Y$, then the upper bound $X \vee Y$ exists and is $X \oplus Y$. Thus we have a partially defined addition $+$ on the set L, defined by $X + Y := X \oplus Y$ for every $X, Y \in L$ with $X \perp Y$. Notice that there is a canonical mapping can: $L \to V(M_R)$ defined by

$\mathrm{can}(X) = \langle X \rangle$ for every $X \in L$. Its image is the closed interval of $V(M_R)$ consisting of all $\langle X \rangle \in V(M_R)$ with X a direct summand of M_R, that is, the interval of all $\langle X \rangle \in V(M_R)$ with $\langle 0 \rangle \leq \langle X \rangle \leq \langle M_R \rangle$.

We will say that a subset C of L is a *cofilter* in L if:

(1) $C \neq \emptyset$.

(2) $X \in L$, $Y \in C$ and $X \cong Y$ imply $X \in C$.

(3) $X \in L$, $Y \in C$ and $X \leq Y$ imply $X \in C$.

(4) $X, Y \in C$ and $X \perp Y$ imply $X + Y \in C$.

Proposition 3.6.3. *There is an order-preserving one-to-one correspondence between the set $\mathcal{C}$ of all cofilters in L and the set $\mathcal{D}$ of all divisor-closed submonoids of $V(M_R)$. It associates to any cofilter C in L the submonoid of $V(M_R)$ generated by* $\mathrm{can}(C)$. *The inverse correspondence associates to any divisor-closed submonoid D of $V(M_R)$ the cofilter* $\mathrm{can}^{-1}(D)$ *of L.*

Proof. We first prove that the submonoid $[\mathrm{can}(C)]$ of $V(M_R)$ generated by $\mathrm{can}(C)$ for some cofilter C of L is a divisor-closed submonoid of $V(M_R)$. If $X \in V(M_R)$, $Y \in [\mathrm{can}(C)]$ and $X \leq Y$, then $Y = \langle Y_1 \rangle + \cdots + \langle Y_n \rangle$ for suitable $Y_i \in C$. From Lemma 3.5.1, we have that $X \cong X_1 \oplus \cdots \oplus X_n$ with $X_i \leq Y_i$ for every $i = 1, \ldots, n$. But C is a cofilter, so that $X_i \in C$ for every i. Thus $X = \langle X_1 \rangle + \cdots + \langle X_n \rangle \in [\mathrm{can}(C)]$.

Conversely, the proof that $\mathrm{can}^{-1}(D)$ is a cofilter for every divisor-closed submonoid D of $V(M_R)$ is trivial.

In order to prove that the two correspondences are inverse to the other, we must prove that $\mathrm{can}^{-1}([\mathrm{can}(C)]) = C$ for every $C \in \mathcal{C}$ and $[\mathrm{can}(\mathrm{can}^{-1}(D))] = D$ for every $D \in \mathcal{D}$. The inclusion $\mathrm{can}^{-1}([\mathrm{can}(C)]) \supseteq C$ follows from (2). Conversely, assume $X \in \mathrm{can}^{-1}([\mathrm{can}(C)])$. Then $X \in L$ and $\langle X \rangle \in [\mathrm{can}(C)]$, so that $\langle X \rangle = \langle X_1 \rangle + \cdots + \langle X_n \rangle$ for some $X_i \in C$. Thus $X_1 \oplus \cdots \oplus X_n$ is isomorphic to a direct summand of M_R, i.e., $M_R = M_1 \oplus \cdots \oplus M_n \oplus M'$ with $M_i \cong X_i$. By (2), each $M_i \in C$, and $M_1 \oplus \cdots \oplus M_n \in C$ by (4). Thus $X \in C$ by (2).

Finally, fix a divisor-closed submonoid $D \in \mathcal{D}$. Then $\mathrm{can}(\mathrm{can}^{-1}(D)) = D \cap \mathrm{can}(L)$, and we must show that $[D \cap \mathrm{can}(L)] = D$. The inclusion $[D \cap \mathrm{can}(L)] \subseteq D$ is obvious. Conversely, if $\langle X \rangle \in D$, then $\langle X \rangle \in V(M_R)$, and we can suppose that X is a direct summand of M_R^n. By Lemma 3.5.1, we can suppose $X = X_1 \oplus \cdots \oplus X_n$ with every $X_i \in L$. As D is divisor-closed, each $\langle X_i \rangle$ is in D. This shows that each $\langle X_i \rangle$ is in $D \cap \mathrm{can}(L)$, so that $\langle X \rangle \in [D \cap \mathrm{can}(L)]$, as we wanted to prove. ∎

Every cofilter contains 0. The cofilter L is the improper cofilter. A cofilter maximal among all proper cofilters is an *ultracofilter*. Hence:

Theorem 3.6.4. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of S and the set of all ultracofilters of $L = \mathcal{L}_{ds}(M_R)$.*

A standard characterizations of ultrafilters U on a set X is that a filter U on X is an ultrafilter if and only if for every subset A of X, either $A \in U$ or $X \setminus A \in U$. Here is the analogue of this characterization for our ultracofilters.

Proposition 3.6.5. *A cofilter C of L is an ultracofilter if and only if $M_R \notin C$ and, for every $X \in L \setminus C$, there exist an integer $n \geq 0$, a module $X' \in L$ isomorphic to a submodule of X^n and an element Y of C such that $M_R = X' \oplus Y$.*

Proof. Let C be an ultracofilter. Since C is a proper subset of L, we have that $M_R \notin C$. Assume that $X \in L \setminus C$. Then $M_R = X \oplus Z$ for some $Z \in L$. By Proposition 3.6.3, $[\mathrm{can}(C)]$ is a maximal divisor-closed submonoid of $V(M_R)$, and $\mathrm{can}^{-1}([\mathrm{can}(C)]) = C$. Thus $X \notin \mathrm{can}^{-1}([\mathrm{can}(C)])$, that is, $\langle X \rangle \notin [\mathrm{can}(C)]$. By Lemma 3.5.1, the sum of two divisor-closed submonoids of $V(M_R)$ is a divisor-closed submonoid of $V(M_R)$. Hence if $[\![\langle X \rangle]\!]$ denotes the divisor-closed submonoid of $V(M_R)$ generated by $\langle X \rangle$, then $[\![\langle X \rangle]\!] + [\mathrm{can}(C)]$ is a divisor-closed submonoid of $V(M_R)$ properly containing $[\mathrm{can}(C)]$. From the maximality of $[\mathrm{can}(C)]$, it follows that $[\![\langle X \rangle]\!] + [\mathrm{can}(C)] = V(M_R)$. Thus $M_R = X' \oplus M_1 \oplus \cdots \oplus M_t$, where X' is isomorphic to a direct summand of X^n for some $n \geq 0$, and every M_i is in C. By property (4), we get that $Y := M_1 \oplus \cdots \oplus M_t \in C$.

Conversely, let C be a proper cofilter of L with the property that for every $X \in L \setminus C$ there exist $X' \in L$ isomorphic to a submodule of X^n for some n and $Y \in C$ with $M_R = X' \oplus Y$. Let C' be a cofilter of L properly containing L. Let X be in $C' \setminus C$. By hypothesis, there exist $X' \in L$ isomorphic to a submodule of X^n for some n and $Y \in C$ with $M_R = X' \oplus Y$. Then $X \in C'$ implies $\langle X^n \rangle \in [\mathrm{can}(C')]$, which is a divisor-closed submonoid, so that $\langle X' \rangle \in [\mathrm{can}(C')]$. Thus $X' \in C'$ and $Y \in C \subseteq C'$. It follows that $M_R = X' \oplus Y \in C'$, so $C' = L$. ■

3.7 Ultrafilters in the case $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$.

The following important result was proved in [43].

Proposition 3.7.1. [43, Corollary 4.2] *Let $M_R = E(\bigoplus_{i \in I} M_i)$ be an injective module that is an injective envelope of a direct sum of indecomposable injective submodules M_i, $i \in I$. Then any two such decompositions of M_R are isomorphic. Furthermore, if N is an injective submodule of M_R, there is a subset $J \subseteq I$ such that $M_R = N \oplus E(\bigoplus_{i \in J} M_i)$ and $N \cong E(\bigoplus_{i \in I \setminus J} M_i)$.*

In the following, $\{ A_\lambda \mid \lambda \in \Lambda \}$ will be a set of representatives of all indecomposable injective right R-modules up to isomorphism, and $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ will be an injective envelope of a direct sum of a family of indecomposable injective modules. Let N be a direct summand of M_R. By Proposition 3.7.1, one has that $N \cong E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\dim_\lambda(N))})$ for uniquely determined cardinals $\dim_\lambda(N)$. It is easy to check that if N_1, N_2 are two direct summands of M_R and $\lambda \in \Lambda$, then:

1. $\dim_\lambda(M_R) = \kappa_\lambda$;
2. If $N_1 \cong N_2$, then $\dim_\lambda(N_1) = \dim_\lambda(N_2)$;
3. If $N_1 \leq N_2$, then $\dim_\lambda(N_1) \leq \dim_\lambda(N_2)$;
4. $\dim_\lambda(N_1 \oplus N_2) = \dim_\lambda(N_1) + \dim_\lambda(N_2)$.

In the rest of this Section, we will describe the ultrafilters of the lattice $\mathcal{O} = \mathcal{L}_{ds}(M_R)/G$ in the case $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$.

Proposition 3.7.2. *Let $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. If B is an indecomposable injective submodule of M_R, then $\mathcal{F}_B := \{ GN \in \mathcal{O} \mid GB \preceq GN \}$ is an ultrafilter of $\mathcal{O}$.*

Proof. It is easily seen that $\mathcal{F}_B$ is a filter of the lattice $\mathcal{O}$. Let $\mathcal{G}$ be a filter strictly containing $\mathcal{F}_B$. We must show that $\mathcal{G} = \mathcal{O}$. In fact, fix $GN \in \mathcal{G}$ with $GN \notin \mathcal{F}_B$. Consider

$GB \wedge GN = G(E(B \cap N)) \in \mathcal{G}$. If $E(B \cap N) \neq 0$, then $B = E(B \cap N)$, because B is indecomposable injective. Thus $GB = G(E(B \cap N)) \preceq GN$. It follows that $GN \in \mathcal{F}_B$, a contradiction. Therefore $E(B \cap N) = 0$, so that $G0 \in \mathcal{G}$, and we can conclude that $\mathcal{G} = \mathcal{O}$. ∎

The following result describes all principal ultrafilters in $\mathcal{O}$.

Proposition 3.7.3. *Let $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. If $\mathcal{F}$ is a principal ultrafilter of $\mathcal{O}$, then there exists an indecomposable injective submodule B of M_R such that $\mathcal{F} = \mathcal{F}_B$.*

Proof. Let GN be the least element of $\mathcal{F}$. By Proposition 3.7.1, there exists an indecomposable injective submodule B of N. Then $GB \preceq GN$, so that the principal filter $\mathcal{F}_B$ generated by GB is a proper filter of $\mathcal{O}$ that contains $\mathcal{F}$. By the maximality of $\mathcal{F}$, one has that $\mathcal{F} = \mathcal{F}_B$. ∎

Corollary 3.7.4. *If $M_R = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where M_i is indecomposable injective, then all ultrafilters of $\mathcal{O}$ are principal, that is, of the form $\mathcal{F}_B$ for some indecomposable injective submodule B of M_R.*

Proof. We know that an ultrafilter $\mathcal{F}$ of $\mathcal{O}$ is free if and only if for every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $y \preceq x$ and $y \neq x$. Since every properly descending chain in $\mathcal{L}_{ds}(M_R)$ is finite, it follows that there is no free ultrafilter in $\mathcal{O}$. Now apply Proposition 3.7.3. ∎

Recall that a set $\{ N_i \mid i \in I \}$ of non-zero submodules of a module N_R is said to be *independent* if $N_{i_0} \cap \sum_{i \in I \setminus \{i_0\}} N_i = 0$ for every $i_0 \in I$. That is, $\sum_{i \in I} N_i = \bigoplus_{i \in I} N_i$.

Remark 3.7.5. If M_R is an injective R-module and $\{N_j |\ j \in J\}$ is an independent set of submodules of M_R, then $GE(\bigoplus_{j \in J} N_j) = GE(\bigoplus_{j \in J} E(N_j))$, because $\bigoplus_{j \in J} N_j$ is essential in $E(\bigoplus_{j \in J} E(N_j))$.

Lemma 3.7.6. *Let I be a subset of Λ, so that $M_R := E(\bigoplus_{\lambda \in I} A_\lambda)$ is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules. Then, for every $GN \in \mathcal{O}$, there exists a unique subset K_N of I such that*

$GN = GE(\bigoplus_{\lambda\in K_N} A_\lambda)$. *In particular, the lattices* $\mathcal{O}$ *and* $\mathcal{P}(I)$ *of all subsets of* I *are isomorphic.*

Proof. By Proposition 3.7.1, there exists a subset K_N of I such that N is isomorphic to $E(\bigoplus_{\lambda\in K_N} A_\lambda)$. Let T be a set of representatives of all indecomposable injective submodules of N up to isomorphism. It is easy to check that T is an independent set of submodules of N and $N = E(\bigoplus_{B\in T} B)$. By Proposition 3.7.1, for every $B \in T$, there exists a unique $\lambda_B \in I$ such that $B \cong A_{\lambda_B}$. If $B \cap A_{\lambda_B} = 0$, then B is a direct summand of $E(\bigoplus_{\lambda\in I\setminus\{\lambda_B\}} A_\lambda)$. This contradicts Proposition 3.7.1. Thus $B \cap A_{\lambda_B} \neq 0$. By Proposition 6.1, $K_N = \{\lambda_B \mid B \in T\}$. By Remark 3.7.5, one finds that $GN = GE(\bigoplus_{\lambda\in K_N} A_\lambda)$.
■

From the lattice isomorphism $\mathcal{O} \cong \mathcal{P}(I)$ of Lemma 3.7.6, we immediately get the following three results.

Theorem 3.7.7. *Let* I *be a subset of* Λ, *so that* $M_R = E(\bigoplus_{\lambda\in I} A_\lambda)$ *is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules. Then there is a one-to-one correspondence between the set of all filters of* $\mathcal{O}$ *and the set of all filters of the lattice* $\mathcal{P}(I)$ *of all subsets of* I. *It associates to any filter* $\mathcal{F}$ *of* $\mathcal{O}$ *the filter* $P(\mathcal{F}) = \{K_N \mid GN \in \mathcal{F}\}$ *of* $\mathcal{P}(I)$, *where* K_N *is defined as in Lemma 3.7.6. The inverse correspondence associates to any set* P *of* $\mathcal{P}(I)$ *the filter* $\mathcal{F}(P) = \{GE(\bigoplus_{\lambda\in K} A_\lambda) \mid K \in P\}$ *of* $\mathcal{O}$. *In particular, there is a one-to-one correspondence between the set of all ultrafilters of* $\mathcal{O}$ *and the set of all ultrafilters of the power set* $\mathcal{P}(I)$ *of* I.

Proposition 3.7.8. *Let* $M_R = E(\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)})$ *be an injective envelope of a direct sum of a family of indecomposable injective modules. If* $\mathcal{A}$ *is a principal ultraantifilter of* $\mathcal{O}$, *then there exists a direct-sum decomposition* $M_R = B \oplus B'$ *of* M_R *with* B' *indecomposable and* $\mathcal{A} = \{GN \in \mathcal{O} \mid GN \preceq GB\}$.

Theorem 3.7.9. *Let* I *be a subset of* Λ, *so that* $M_R = E(\bigoplus_{\lambda\in I} A_\lambda)$ *is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective* *modules. Then there is a one-to-one correspondence between the set of all ultraantifilters of $\mathcal{O}$ and the set of all ultraantifilters of the power set $\mathcal{P}(I)$ of I.*

The one-to-one correspondence in Theorem 3.7.9 is the same as that in Theorem 3.7.7.

3.8 Ultracofilters of $\mathcal{L}_{ds}(E(\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)}))$

In this Section, we will describe the ultracofilters of $\mathcal{L}_{ds}(E(\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)}))$, where the A_λ's are pair-wise non-isomorphic indecomposable injective modules. As the following Proposition shows, the behavior of the module $M_R = E(A^{(\kappa)})$, where A is an indecomposable injective module, is similar to that of vector spaces over a division ring (Corollary 3.1.6).

Proposition 3.8.1. *Let $A = A_\lambda$ be an indecomposable injective module, κ be a non-zero cardinal and $M_R = E(A^{(\kappa)})$. Then there is a unique ultracofilter $\mathcal{U}$ of $\mathcal{L}_{ds}(M_R)$. Moreover,*

1. *If κ is finite, then $\mathcal{U} = \{0\}$.*
2. *If κ is infinite, then $\mathcal{U} = \{\, N \in \mathcal{L}_{ds}(M_R) \mid \dim_\lambda(N) < \kappa \,\}$.*

Proof. Statement (1) is trivial, because when κ is finite, there are only two cofilters in $\mathcal{L}_{ds}(M_R)$, namely $\{0\}$ and $\mathcal{L}_{ds}(M_R)$.

(2) Suppose that κ is infinite. It is suffices to show that every proper cofilter is contained in $\mathcal{U} := \{\, N \in \mathcal{L}_{ds}(M_R) \mid \dim_\lambda(N) < \kappa \,\}$ and that $\mathcal{U}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Let $\mathcal{C}$ be a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Let U be an arbitrary element of $\mathcal{C}$. If $\dim_\lambda(U) = \kappa$, then $M_R \cong U$, so that $M_R \in \mathcal{C}$. This implies that $\mathcal{C} = \mathcal{L}_{ds}(M_R)$, a contradiction. Therefore $\dim_\lambda(U) < \kappa$. This shows that $\mathcal{C} \subseteq \mathcal{U}$. To conclude the proof, we must see that $\mathcal{U}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Obviously, $M_R \notin \mathcal{U}$ and $0 \in \mathcal{U}$, so that $\mathcal{U}$ is a non-empty proper subset of $\mathcal{L}_{ds}(M_R)$. Given N_1, N_2 in $\mathcal{L}_{ds}(M_R)$ with $N_1 \cong N_2$ and $N_2 \in \mathcal{U}$, then $\dim_\lambda(N_1) = \dim_\lambda(N_2) < \kappa$ implies that $N_1 \in \mathcal{U}$. If $N_1 \leq N_2$ and $N_2 \in \mathcal{U}$, then $\dim_\lambda(N_1) \leq \dim_\lambda(N_2) < \kappa$ implies that $N_1 \in \mathcal{U}$. Finally, if $N_1, N_2 \in \mathcal{U}$ and $N_1 \perp N_2$, then $\dim_\lambda(N_1) < \kappa$ and $\dim_\lambda(N_2) < \kappa$, so that

$$\dim_\lambda(N_1 + N_2) = \dim_\lambda(N_1 \oplus N_2) = \dim_\lambda(N_1) + \dim_\lambda(N_2) < \kappa + \kappa = \kappa.$$

Therefore $N_1 + N_2 \in \mathcal{U}$. ■

As a consequence, the endomorphism ring of $M_R = E(A_\lambda^{(\kappa)})$ has a unique maximal two-sided ideal. More precisely:

When κ is finite, the unique ultracofilter $\mathcal{U}$ of $L = \mathcal{L}_{ds}(M_R)$ is $\{0\}$, the corresponding divisor-closed submonoid of $V(M_R) \cong \mathbb{N}$ is the zero submonoid, the corresponding prime ideal of height one of $V(M_R)$ consists of all non-zero elements of $V(M_R)$, the corresponding trace ideal of $S = \mathrm{End}(M_R)$ is the zero ideal, and the unique maximal two-sided ideal of the endomorphism ring S is $J(S)$, consisting of all endomorphisms of M_R with essential kernel (Theorem 1.4.1).

When κ is infinite, the monoid $V(M_R)$ is isomorphic to the interval $[0, \kappa]$, the additive monoid of all cardinals $\leq \kappa$. The divisor-closed submonoid of $[0, \kappa]$ corresponing to $\mathcal{U}$ is the interval $[0, \kappa[$, the additive monoid of all cardinals $< \kappa$. The corresponding prime ideal of height one of $V(M_R)$ consists only of κ. The corresponding trace ideal of $S = \mathrm{End}(M_R)$ is the two-sided ideal I generated by all idempotent endomorphisms of M_R whose image is isomorphic to $E(A^{(\alpha)})$ for some $\alpha < \kappa$. Equivalently, I is the two-sided ideal of S generated by all endomorphisms of M_R whose image is contained in a submodule of M_R isomorphic to $E(A^{(\alpha)})$ for some $\alpha < \kappa$. The unique maximal two-sided ideal of the endomorphism ring S is $I + J(S)$.

Let $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. In view of Proposition 3.8.1, we will now introduce a kind of support of the submodules of M_R, which will be called the *maximal support* because we will use it to describe the maximal two-sided ideals of the endomorphism ring S of M_R. From now on, I will be the subset of Λ consisting of all indices $\lambda \in \Lambda$ with $\kappa_\lambda > 0$, so that $M_R = E(\bigoplus_{\lambda \in I} A_\lambda^{(\kappa_\lambda)})$ with $\kappa_\lambda > 0$ for every $\lambda \in I$. We can assume that $I \neq \emptyset$. For each direct summand N of M_R, let $\mathrm{supp}(N)$ (the *maximal support of* N) be the set of all $\lambda \in I$ satisfying the following property: if κ_λ is infinite, then $\dim_\lambda(N) = \kappa_\lambda$; if κ_λ is finite, then $\dim_\lambda(N) > 0$. Let N_1, N_2 be two direct summands of M_R. It is easily seen that:

1. If $N_1 \cong N_2$, then $\mathrm{supp}(N_1) = \mathrm{supp}(N_2)$.

2. If $N_1 \leq N_2$, then $\operatorname{supp}(N_1) \subseteq \operatorname{supp}(N_2)$.
3. If $N_1 \cap N_2 = 0$, then $\operatorname{supp}(N_1 \oplus N_2) = \operatorname{supp}(N_1) \cup \operatorname{supp}(N_2)$.

Lemma 3.8.2. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and let N be in $\mathcal{C}$. If J is a subset of I and δ_λ is an ordinal $\leq \dim_\lambda(N)$ for every $\lambda \in J$, then $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ belongs to $\mathcal{C}$.*

Proof. The module $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ is a direct summand of $E(\bigoplus_{\lambda\in I} A_\lambda^{(\dim_\lambda(N))})$, and $E(\bigoplus_{\lambda\in I} A_\lambda^{(\dim_\lambda(N))})$ is isomorphic to N (Proposition 3.7.1). It follows that $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ belongs to $\mathcal{C}$. ∎

In order to visualize the maximal support and the meaning of Lemma 3.8.2, notice that the quotient set of $\mathcal{L}_{ds}(M_R)$ modulo isomorphism is in a canonical one-to-one correspondence with the direct product $\prod_{\lambda\in I}[0, \dim_\lambda(M_R)]$ of the intervals of cardinal $[0, \dim_\lambda(M_R)]$ (Proposition 3.7.1).

Lemma 3.8.3. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and J be a subset of I. Set $\mathcal{C}^* := \{ V_1 \oplus V_2 \mid V_1 \in \mathcal{C},\ V_2 \in \mathcal{L}_{ds}(M_R),\ V_1 \perp V_2,$ and there exists $V_2' \leq E(\bigoplus_{j\in J} A_j^{(\kappa_j)})$ with $V_2' \cong V_2 \}$. Then $\mathcal{C}^*$ is a cofilter of $\mathcal{L}_{ds}(M_R)$ containing $\mathcal{C}$.*

Proof. Clearly $\mathcal{C}^*$ contains $\mathcal{C}$, so that condition (1) in the Definition of cofilter holds trivially. In order to prove (2), assume $V_1 \oplus V_2 \in \mathcal{C}^*$ and $X \in \mathcal{L}_{ds}(M_R)$ with $X \cong V_1 \oplus V_2$. Then $X = X_1 \oplus X_2$ with $X_1 \cong V_1$ and $X_2 \cong V_2$. Thus $X \in \mathcal{C}^*$. For (3), assume $X \in L_{ds}(M_R)$, $V_1 \oplus V_2 \in \mathcal{C}^*$ and $X \leq V_1 \oplus V_2$. By Lemma 3.5.1, $X \cong X_1 \oplus X_2$ with $X_1 \leq V_1$ and $X_2 \leq V_2$. It easily follows that $X_1 \oplus X_2 \in \mathcal{C}^*$. By (2), $X \in \mathcal{C}^*$ also. As far as (4) is concerned, assume $V_1 \oplus V_2 \perp U_1 \oplus U_2$. Then the sum $V_1 + V_2 + U_1 + U_2$ is direct, and $V_1 + V_2 + U_1 + U_2 = (V_1 \oplus U_1) \oplus (V_2 \oplus U_2) \in \mathcal{C}^*$. ∎

Proposition 3.8.4. *Let $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. For each cofilter $\mathcal{C}$ of $\mathcal{L}_{ds}(M_R)$, set*

$$P(\mathcal{C}) := \{ \operatorname{supp}(N) \mid N \in \mathcal{C} \}.$$

Then $P(\mathcal{C})$ is an antifilter of the power set $\mathcal{P}(I)$ of I. Moreover, if $\mathcal{U}$ is an ultracofilter and $P(\mathcal{U})$ is a proper subset of $\mathcal{P}(I)$, then $P(\mathcal{U})$ is an ultraantifilter of $\mathcal{P}(I)$.

Proof. Obviously $\emptyset \in P(\mathcal{C})$, so $P(\mathcal{C}) \neq \emptyset$. Suppose that $X \subseteq \mathrm{supp}(N)$ for some $N \in \mathcal{C}$. By Lemma 3.8.2, $N' := E(\bigoplus_{\lambda\in X} A_\lambda^{(\dim_\lambda(N))}) \in \mathcal{C}$. Therefore $X = \mathrm{supp}(N') \in P(\mathcal{C})$. Assume $X_1 = \mathrm{supp}(N_1), X_2 = \mathrm{supp}(N_2)$ with $N_1, N_2 \in \mathcal{C}$. Consider

$$U := E(\bigoplus_{\lambda\in X_1} A_\lambda^{(\dim_\lambda(N_1))}) \oplus E(\bigoplus_{\lambda\in X_2\setminus X_1} A_\lambda^{(\dim_\lambda(N_2))}).$$

By Lemma 3.8.2 again, $E(\bigoplus_{\lambda\in X_1} A_\lambda^{(\dim_\lambda(N_1))})$ and $E(\bigoplus_{\lambda\in X_2\setminus X_1} A_\lambda^{(\dim_\lambda(N_2))})$ belong to $\mathcal{C}$, so that $U \in \mathcal{C}$. Thus, $X_1 \cup X_2 = X_1 \cup (X_2\setminus X_1) = \mathrm{supp}(U) \in P(\mathcal{C})$. This proves that $P(\mathcal{C})$ is an antifilter of $\mathcal{P}(I)$.

Finally, we must show that $P(\mathcal{U})$ is maximal in the set of all proper antifilters of $\mathcal{P}(I)$ whenever $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$ and $P(\mathcal{U}) \neq \mathcal{P}(I)$. It suffices to prove that if J is a subset of I and $J \notin P(\mathcal{U})$, then $I \setminus J \in P(\mathcal{U})$. Now $J \notin P(\mathcal{U})$ implies that $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)}) \notin \mathcal{U}$. By Lemma 3.8.3, $\mathcal{U}^* := \{\, V_1 \oplus V_2 \mid V_1 \in \mathcal{U}, V_2 \in \mathcal{L}_{ds}(M_R),\ V_1 \perp V_2$, and there exists $V_2' \leq E(\bigoplus_{j\in J} A_j^{(\kappa_j)})$ with $V_2' \cong V_2 \,\}$ is a cofilter of $\mathcal{L}_{ds}(M_R)$ containing $\mathcal{U}$ and $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)})$. Because of the maximality of $\mathcal{U}$, one has $\mathcal{U}^* = \mathcal{L}_{ds}(M_R)$. Therefore there exists a direct-sum decomposition $M_R = V_1 \oplus V_2$ with $V_1 \in \mathcal{U}$, a direct summand V_2' of $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)})$ and $V_2 \cong V_2'$. Hence $\dim_\lambda(V_1) = \kappa_\lambda$ for every $\lambda \in I \setminus J$. In particular, $\mathrm{supp}(V_1) \supseteq I \setminus J$. But $\mathrm{supp}(V_1) \in P(\mathcal{U})$, so that $I \setminus J \in P(\mathcal{U})$, as we wanted to prove. ■

The proof of the following Proposition is straightforward and elementary.

Proposition 3.8.5. *Let $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. For any proper antifilter P of the power set $\mathcal{P}(I)$ of I, the set $\mathcal{C}(P) := \{\, N \mid \mathrm{supp}(N) \in P \,\}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$.*

Remark 3.8.6. We will now prove the following two facts.

1. The antifilter $P(\mathcal{C})$ in Proposition 3.8.4 is not necessarily a proper subset of $\mathcal{P}(I)$ whenever $\mathcal{C}$ is properly contained in $\mathcal{L}_{ds}(M_R)$.

2. The cofilter $\mathcal{C}(P)$ in $\mathcal{L}_{ds}(M_R)$ in Proposition 3.8.5 is proper whenever P is proper in $\mathcal{P}(I)$, but it is not necessarily an ultracofilter of $\mathcal{L}_{ds}(M_R)$ when P is an ultraantifilter in $\mathcal{P}(I)$.

To see this, let A_i, $i \in \mathbb{N}$, be pair-wise non-isomorphic indecomposable injective modules indexed in the set $\mathbb{N}$ of positive integers and $M_R = E(\bigoplus_{i\in\mathbb{N}} A_i^i)$. In order to prove (2), let P be any ultraantifilter of $\mathcal{P}(\mathbb{N})$ containing the antifilter of all finite subsets of $\mathbb{N}$. We will show that $\mathcal{C}(P)$ is not an ultraantifilter of $\mathcal{L}_{ds}(M_R)$ applying Proposition 3.6.5, that is, showing that there exists an element $X \in \mathcal{L}_{ds}(M_R)\backslash\mathcal{C}(P)$ such that $M_R \neq X'\oplus Y$ for every integer $n \geq 0$, every direct summand X' of M_R isomorphic to a submodule of X^n and every element Y of $\mathcal{C}(P)$.

Consider an element $X \cong E(\bigoplus_{i\in\mathbb{N}} A_i)$ of $\mathcal{L}_{ds}(M_R)$. As $\operatorname{supp}(X) = \mathbb{N}$, it follows that $X \notin \mathcal{C}(P)$. For every element Y of $\mathcal{C}(P)$, one has that $\operatorname{supp}(Y) \in P$, so $\mathbb{N} \setminus \operatorname{supp}(Y)$ is an infinite subset of $\mathbb{N}$. Therefore $\{\dim_i M \mid i \in \mathbb{N} \setminus \operatorname{supp}(Y)\}$ is not bounded. On the other hand, for any integer $n \geq 0$, any direct summand X' of X^n satisfying $X' \perp Y$ and any $i \in \mathbb{N}\setminus \operatorname{supp}(Y)$, one has that $\dim_i(X'\oplus Y) = \dim_i(X') + \dim_i(Y) = \dim_i(X') + 0 = \dim_i(X') \leq \dim_i(X^n) = n$. Thus $\{\dim_i(X' \oplus Y) \mid i \in \mathbb{N} \setminus \operatorname{supp}(Y)\}$ is bounded. Hence $X' \oplus Y \neq M_R$. By Proposition 3.6.5, $\mathcal{C}(P)$ is not an ultraantifilter of $\mathcal{L}_{ds}(M_R)$.

In order to prove (1), set

$$\mathcal{C} := \{ N \in \mathcal{L}_{ds}(M_R) \mid \text{there exists } m \in \mathbb{N} \text{ with } \dim_i N < m \text{ for every } i \in \mathbb{N}\}.$$

It is easy to check that $\mathcal{C}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$ and $P(\mathcal{C}) = \mathcal{P}(\mathbb{N})$.

Let us go back to our notation of $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{(\kappa_\lambda)})$ with the A_λ's pair-wise non-isomorphic indecomposable injective modules and $\kappa_\lambda > 0$ for every $\lambda \in I$. We will show in Theorems 3.8.8 and 3.8.10 that the difficulties stated in Remark 3.8.6 do not arise when either $\{\lambda \in I \mid \kappa_\lambda < \infty\}$ is a finite subset of I or there exists a positive integer m such that $\kappa_\lambda < m$ for every $\lambda \in I$. We will show that, in these cases, $\mathcal{C}(P)$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$ when P is an ultraantifilter of $\mathcal{P}(I)$ and $P(\mathcal{U})$ is an ultraantifilter of $\mathcal{P}(I)$ when $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$.

Lemma 3.8.7. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and let N be a module in $\mathcal{C}$. If there exists a positive integer m such that $\kappa_\lambda < m$ for every $\lambda \in I$, then $E(\bigoplus_{\lambda\in\operatorname{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}$.*

Proof. For every $t = 1, 2, \ldots, m-1$, set $J_t := \{\, \lambda \in \operatorname{supp}(N) \mid \kappa_\lambda \geq t \,\}$, so that $E(\bigoplus_{\lambda\in\operatorname{supp}(N)} A_\lambda^{\kappa_\lambda}) \cong \bigoplus_{t=1}^{m-1} E(\bigoplus_{\lambda\in J_t} A_\lambda)$. By Lemma 3.8.2, if J is a subset of $\operatorname{supp}(N)$ and δ_λ is an ordinal $\leq \dim_\lambda(N)$ for every $\lambda \in J$, then $E(\bigoplus_{\lambda\in J} A_\lambda^{\delta_\lambda})$ belongs to $\mathcal{C}$. Apply this Lemma 3.8.2 to the subset J_t of $\operatorname{supp}(N)$ with $\delta_\lambda = 1$ for every $\lambda \in J_t$, so that $E(\bigoplus_{\lambda\in J_t} A_\lambda) \in \mathcal{C}$. Then $E(\bigoplus_{\lambda\in\operatorname{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}$ by properties (2) and (4) in the Definition of cofilter. ■

Theorem 3.8.8. *Let m be a positive integer such that $\kappa_\lambda < m$ for every $\lambda \in I$.*

1. *If $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$, then $P(\mathcal{U})$ is an ultraantifilter of the lattice $\mathcal{P}(I)$.*

2. *Conversely, if P is an ultraantifilter of the lattice $\mathcal{P}(I)$, then $\mathcal{C}(P)$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$.*

Proof. 1. By Proposition 3.8.4, it suffices to show that $P(\mathcal{U})$ is a proper subset of $\mathcal{P}(I)$. Now, if $I \in P(\mathcal{U})$, then there exists $N \in \mathcal{U}$ such that $I = \operatorname{supp}(N)$. By Lemma 3.8.7, $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{\kappa_\lambda}) = E(\bigoplus_{\lambda\in\operatorname{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{U}$, so that $\mathcal{U} = \mathcal{L}_{ds}(M_R)$ by property (3) in the definition of cofilter. This is a contradiction.

2. We already know that $\mathcal{C}(P)$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$ by Proposition 3.8.5. Suppose that $\mathcal{C}'$ is a cofilter strictly containing $\mathcal{C}(P)$. Let U be an element of $\mathcal{C}'$ not in $\mathcal{C}(P)$. Since P is an ultraantifilter of $\mathcal{P}(I)$, either $\operatorname{supp}(U)$ belongs to P or $I\setminus\operatorname{supp}(U)$ belongs to P. But $U \notin \mathcal{C}(P)$, so that $I\setminus\operatorname{supp}(U) \in P$. In particular, $E(\bigoplus_{\lambda\in I\setminus\operatorname{supp}(U)} A_\lambda^{\kappa_\lambda}) \in \mathcal{C}(P)$ by the definition of $\mathcal{C}(P)$. Applying Lemma 3.8.7 to the module $U \in \mathcal{C}'$, we find that $E(\bigoplus_{\lambda\in\operatorname{supp}(U)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}'$. Thus both $E(\bigoplus_{\lambda\in I\setminus\operatorname{supp}(U)} A_\lambda^{\kappa_\lambda})$ and $E(\bigoplus_{\lambda\in\operatorname{supp}(U)} A_\lambda^{\kappa_\lambda})$ belong to $\mathcal{C}'$, so that their direct sum M_R belongs to $\mathcal{C}'$. It follows that $\mathcal{C}' = \mathcal{L}_{ds}(M_R)$. ■

Let us go back to the general case of the κ_λ's possibly infinite or unbounded.

Lemma 3.8.9. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and N be a module in $\mathcal{C}$. If J is a finite subset of* $\operatorname{supp}(N)$*, then $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$.*

Proof. Set $J_1 := \{ \lambda \in J \mid \kappa_\lambda < \infty \}$ and $J_2 := \{ \lambda \in J \mid \kappa_\lambda = \infty \}$. Then J_1 and J_2 are finite sets and $J_1 \cup J_2 = J$. By Lemma 3.8.2 applied to the set J_1 with $\delta_\lambda = 1$ for all λ, we see that $E(\bigoplus_{\lambda \in J_1} A_\lambda)$ is in $\mathcal{C}$. By Lemma 3.8.2 applied to the set J_2 with $\delta_\lambda = \kappa_\lambda$ for all λ, we see that $E(\bigoplus_{\lambda \in J_2} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$. By Lemma 3.8.2 applied to the set $\{\lambda\}$ with $\delta_\lambda = 1$, we see that A_λ belongs to $\mathcal{C}$ for every $\lambda \in J$. Since J_1 is finite and $\kappa_\lambda < \infty$ for every $\lambda \in J_1$, $E(\bigoplus_{\lambda \in J_1} A_\lambda^{(\kappa_\lambda)}) = \bigoplus_{\lambda \in J_1} A_\lambda^{\kappa_\lambda}$ belongs to $\mathcal{C}$ by property (4) of the definition of cofilter. Thus $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)}) \cong E(\bigoplus_{\lambda \in J_1} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus_{\lambda \in J_2} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$ by property (4) again. ∎

Theorem 3.8.10. *Suppose that $\{ \lambda \in I \mid \kappa_\lambda < \infty \}$ is finite.*

1. *If $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$, then $P(\mathcal{U})$ is an ultraantifilter of the lattice $\mathcal{P}(I)$.*

2. *Conversely, if P is an ultraantifilter of the lattice $\mathcal{P}(I)$, then $\mathcal{C}(P)$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$.*

Proof. 1. By Proposition 3.8.4, it suffices to show that $P(\mathcal{U})$ is proper in $\mathcal{P}(I)$. If $I \in P(\mathcal{U})$, then there exists $N \in \mathcal{U}$ such that $I = \operatorname{supp}(N)$. We have N is isomorphic to $E(\bigoplus_{\lambda \in \operatorname{supp}(N)} A_\lambda^{(\dim_\lambda(N))}) = E(\bigoplus_{\lambda \in K} A_\lambda^{(\dim_\lambda(N))}) \oplus E(\bigoplus_{\lambda \in \operatorname{supp}(N) \setminus K} A_\lambda^{(\dim_\lambda(N))})$. Here K is the set of all $\lambda \in \operatorname{supp}(N)$ satisfying the dimension $\dim_\lambda(N)$ is finite. One has $E(\bigoplus_{\lambda \in I \setminus K} A_\lambda^{(\dim_\lambda(N))}) = E(\bigoplus_{\lambda \in \operatorname{supp}(N) \setminus K} A_\lambda^{(\dim_\lambda(N))}) \in \mathcal{U}$ because it is a direct summand of N (property (3) in the definition of cofilter). Moreover, by Lemma 3.8.9 applied to the cofilter $\mathcal{U}$ and the finite subset K of $\operatorname{supp}(N)$, we see that $E(\bigoplus_{\lambda \in K} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{U}$. Hence $M_R = E(\bigoplus_{\lambda \in I} A_\lambda^{(\kappa_\lambda)}) = E(\bigoplus_{\lambda \in K} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus_{\lambda \in I \setminus K} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{U}$. Therefore, $\mathcal{U}$ is not proper, which is a contradiction.

2. Let P be an ultraantifilter of $\mathcal{P}(I)$ and $\mathcal{C}'$ be a cofilter strictly containing $\mathcal{C}(P)$. Then there exists an element U in $\mathcal{C}' \setminus \mathcal{C}(P)$. Since P is an ultraantifilter of $\mathcal{P}(I)$, either $\operatorname{supp}(U)$ belongs to P or $I \setminus \operatorname{supp}(U)$ belongs to P. But $U \notin \mathcal{C}(P)$, so that $I \setminus \operatorname{supp}(U) \in P$.

Hence $E(\bigoplus_{\lambda \in I \setminus \mathrm{supp}(U)} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}(P)$. Now $J := \{ \lambda \in \mathrm{supp}(U) | \dim_\lambda(U) < \infty \}$ is a finite subset of I. By Lemma 3.8.9, we get that $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. By Lemma 3.8.2 applied to the element U of $\mathcal{C}'$ and the subset $\mathrm{supp}(U) \setminus J$ of $\mathrm{supp}(N)$ with $\delta_\lambda = \kappa_\lambda$ for all λ, we obtain that $E(\bigoplus_{\lambda \in \mathrm{supp}(U) \setminus J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. Hence $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus_{\lambda \in \mathrm{supp}(U) \setminus J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. Thus $M_R \in \mathcal{C}'$ and $\mathcal{C}' = \mathcal{L}_{ds}(M_R)$. ■

Theorem 3.8.11. *If either the subset $\{ \lambda \in I \mid \kappa_\lambda < \infty \}$ is finite or there exists a positive integer m such that $\kappa_\lambda < m$ for every $\lambda \in I$, then there is a one-to-one correspondence between the set of all ultraantifilters of $\mathcal{P}(I)$ and the set of all ultracofilters of $\mathcal{L}_{ds}(M_R)$.*

Proof. The correspondence is that defined in Propositions 3.8.4 and 3.8.5. By Theorem 3.8.8 and 3.8.10, it suffices to prove that $\mathcal{C}(P(\mathcal{U})) = \mathcal{U}$ and $P(\mathcal{C}(P)) = P$ for any ultraantifilter P of $\mathcal{P}(I)$ and any ultracofilter $\mathcal{U}$ of $\mathcal{L}_{ds}(M_R)$. It is clear that $\mathcal{U} \subseteq \mathcal{C}(P(\mathcal{U}))$. As $\mathcal{C}(P(\mathcal{U})) \neq \mathcal{L}_{ds}(M_R)$ and $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$, it follows that $\mathcal{U} = \mathcal{C}(P(\mathcal{U}))$. Similarly for the equality $P(\mathcal{C}(P)) = P$. ■

The next two corollaries follow directly from Theorem 3.8.11.

Corollary 3.8.12. *Let $A_1, A_2, \cdots, A_n$ be n indecomposable injective modules which are pair-wise non-isomorphic, and $\kappa_1, \kappa_2, \cdots, \kappa_n$ be n non-zero cardinals. Then the endomorphism ring of $M_R = E(\bigoplus_{i=1}^{n} A_i^{(\kappa_i)})$ has exactly n maximal two-sided ideals.*

Corollary 3.8.13. *Let $M_R = E(\bigoplus_{i \in I} A_i)$ be an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules A_i. Then there is a one-to-one correspondence between the set of all maximal two-sided ideals of the endomorphism ring $S = \mathrm{End}(M_R)$ of M_R and the set of all ultraantifilters of the power set $\mathcal{P}(I)$ of I.*