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Chapter 2

Quasi-duo rings and endomorphism rings of injective modules

Let M_R be an injective module over R. It is well known that M_R is indecomposable if and only if the endomorphism ring $S := \mathrm{End}(M_R)$ of M_R is a local ring (Theorem 1.4.2). In this trivial case, every maximal right ideal and maximal left ideal of S is two-sided. Rings in which all maximal right ideals and maximal left ideals are two-sided are called *quasi-duo*. Hence, the endomorphism ring of an indecomposable injective module is quasi-duo. The aim of this Chapter is to consider injective modules whose endomorphism rings are quasi-duo. Such injective modules will be proved to be *square-free*, that is, they contain no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R. Conversely, we also show that if M_R is square-free injective, then S is quasi-duo.

In Sections 2.1 and 2.2, we give some examples of square-free modules and quasi-duo rings. In Section 2.3 we present the main results of this Chapter, which appear in [7].

2.1 Square-free injective modules

In this Section, we recall some notions and properties of square-free injective modules.

A right module M_R over a ring R is called *square-free* if it contains no non-zero submodule isomorphic to $A \oplus A$ for some submodule A of M_R. An injective module which is square-free is said to be *square-free injective*.

Proposition 2.1.1. *An injective right R-module M_R is square-free if and only if it contains no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R.*

PROOF. If M_R is square-free, then obviously M_R contains no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R.

Conversely, assume that M_R is not square-free. Then there exists a non-zero submodule B of M_R such that N is a submodule of M_R and $N \cong B \oplus B$. Let $E(N)$ and $A := E(B)$ be injective envelopes of N and B respectively in M_R. By Theorem 1.1.8, $E(N) \cong A \oplus A$. Notice that $E(N)$ and A are non-zero direct summands of M_R, so that M_R contains the direct summand $E(N)$ which is isomorphic to $A \oplus A$. ■

Obviously, indecomposable injective modules are square-free injective. Since the injective envelope of a simple module is indecomposable, one has.

Proposition 2.1.2. *The injective envelope of a simple module is square-free injective.*

The following Lemma is trivial (see [8]).

Lemma 2.1.3. *Let A_R, B_R be simple right R-modules and $E(A_R)$, $E(B_R)$ be injective envelopes respectively of A_R and B_R. Then $E(A_R)$ is isomorphic to $E(B_R)$ if and only if A_R is isomorphic to B_R.*

The following Proposition is trivial.

Proposition 2.1.4. *Over a semisimple artinian ring, there are only finitely many square-free (injective) modules up to isomorphisms.*

Proposition 2.1.5. *Let $\{ M_\lambda \mid \lambda \in \Lambda \}$ be a family of simple R-modules and $M_R = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a semisimple module. The injective envelope $E(M_R)$ is square-free if and only if $\{ M_\lambda \mid \lambda \in \Lambda \}$ is a family of pair-wise non-isomorphic modules.*

PROOF. We have $E(M_R) = E(\bigoplus_{\lambda\in\Lambda} M_\lambda) = E(\bigoplus_{\lambda\in\Lambda} E(M_\lambda))$ where $E(M_\lambda)$ is the injective envelope of M_λ. The "if" part is from Lemma 2.1.3 and the other part is from [43, Corollary 4.2] and Lemma 2.1.3. ■

2.2 Quasi-duo rings

A ring R is called *left duo* (resp. *left quasi-duo*) if every left ideal (resp. maximal left ideal) of R is a two-sided ideal. *Right duo* (resp. *right quasi-duo*) rings are defined similarly. If R is both left and right duo (resp. quasi-duo), then R is said to be *duo* (resp. *quasi-duo*). Of course, every duo ring is quasi-duo, commutative rings and division rings are duo.

Proposition 2.2.1. *A ring R is quasi-duo if and only if $R/J(R)$ is quasi-duo.*

PROOF. The statement is trivial since all maximal left ideals and maximal right ideals of R contain the Jacobson radical $J(R)$. ■

Proposition 2.2.2. *The matrix ring $R = M_n(D)$ over a division ring D is quasi-duo if and only if $n = 1$.*

PROOF. If $n = 1$, then $R = D$ is quasi-duo. If $n > 1$, then the set of all matrices whose first column is 0 is a maximal left ideal of R. On the other hand, R has only two two-sided ideals 0 and R. Hence, R is not quasi-duo. ■

Proposition 2.2.3. *The ring $R = R_1 \times R_2 \times \cdots \times R_n$ is quasi-duo if and only if each ring R_i is quasi-duo.*

PROOF. The result is elementary because I_i is a left (resp. right or two-sided) ideal of R_i if and only if $R_1 \times \cdots \times I_i \times \cdots \times R_n$ is a left (resp. right or two-sided) ideal of R. ■

For any positive integer n, a ring R is called *type n* if the quotient ring $R/J(R)$ is isomorphic to $D_1 \times D_2 \times \cdots \times D_n$ where D_i is a division ring. From Proposition 2.2.1, 2.2.2 and 2.2.3, the following Corollary follows immediately.

Corollary 2.2.4. *Every ring of type n is quasi-duo.*

Now let us give some examples of quasi-duo rings.

Example 2.2.5. The following rings are quasi-duo:

1. Local rings. Hence, so are the endomorphism rings of indecomposable injective modules (Theorem 1.4.2).

2. The endomorphism rings of couniform projective modules ([2, Theorem 8.7] and Corollary 2.2.4). Recall that a non-zero right R-module M_R is said to be *couniform* (or *hollow*) if, for any submodules A, B of M_R, $M_R = A + B$ implies $A = M_R$ or $B = M_R$. By the definition of superfluous submodules, M_R is couniform if and only if every proper submodule of M_R is superfluous. A *couniform projective* module is a module which is projective and couniform.

3. The endomorphism rings of non-zero uniserial modules ([14, Theorem 9.1] and Corollary 2.2.4). Here, recall that an *uniserial module* is a module whose set of submodules is totally ordered by inclusion. That is, for any two submodules N_1 and N_2, either $N_1 \leq N_2$ or $N_2 \leq N_1$.

4. The endomorphism rings of cyclically presented modules over a local ring ([2, Theorem 2.1] and Corollary 2.2.4). Notice that a right module over a ring R is *cyclically presented* if it is isomorphic to R/aR for some element $a \in R$.

5. The endomorphism ring of the kernel of a non-zero homomorphism between indecomposable injective modules ([18] and Corollary 2.2.4).

Notice that examples of right quasi-duo rings which are not left quasi-duo are unknown [31, Question 7.7]. Of course, if there exists such a ring R, then the opposite ring R^{op} of R is left quasi-duo and not right quasi-duo.

2.3 The endomorphism ring of a square-free injective module

Let M_R be a right module over a ring R and N be a direct summand of M_R, that is, $M_R = N \oplus N'$ for some submodule N' of M_R. Throughout this section, ι_N denotes the embedding of N into M_R, and π_N denotes the projection of M_R onto N.

Proposition 2.3.1. *Let M_R be an injective R-module. If the endomorphism ring $S = \operatorname{End}(M_R)$ of M_R is either left quasi-duo or right quasi-duo, then M_R is square-free.*

PROOF. Assume that S is left quasi-duo and there exist direct summands A, B, C of M_R such that $A \oplus B \oplus C = M_R$ and A is isomorphic to B. Call I a maximal left ideal of S containing $s = \iota_{B\oplus C}\pi_{B\oplus C}$. Then I is a two-sided ideal of S by hypothesis. Let α be an isomorphism from A to B. Define $f := \iota_B\alpha\pi_A$ and $g := \iota_A\alpha^{-1}\pi_B + \iota_C\pi_C$. Then $sf + gs \in I$ and

$$\begin{aligned}\ker(sf+gs) &= \{\, a+b+c \in A \oplus B \oplus C \mid (sf+gs)(a+b+c) = 0 \,\} \\ &= \{\, a+b+c \in A \oplus B \oplus C \mid \alpha(a) + \alpha^{-1}(b) + c = 0 \,\} = 0.\end{aligned}$$

Hence, $sf + gs$ is injective. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A \oplus B \oplus C = M_R & \xrightarrow{sf+gs} & M_R \\ & & \big\downarrow \mathrm{Id}_{M_R} & \swarrow h & \\ & & M_R & & \end{array}$$

Since M_R is injective, there exists a homomorphism h of S such that $h(sf+gs) = 1$, which implies $I = S$. Contradiction. Therefore, if S is left quasi-duo, then M_R is square-free.

Similarly to the right side. ■

Lemma 2.3.2. *Let M_R be a square-free injective R-module. If N_1, N_2 are two isomorphic direct summands of M_R, then N_1 and N_2 are two injective envelopes of $N_1 \cap N_2$.*

PROOF. Let A_1, A_2 be injective envelopes of $N_1 \cap N_2$ respectively in N_1, N_2 and B_1, B_2 be respectively direct summands of N_1, N_2 such that

$$N_1 = A_1 \oplus B_1, N_2 = A_2 \oplus B_2.$$

Since $N_1 \cong N_2$ and $B_1 \cap A_2 = 0$, B_1 is isomorphic to a direct summand C of N_2 and since $B_1 \cap N_2 = 0$, $B_1 + N_2 = B_1 \oplus N_2$ is injective, which implies that M_R contains $B_1 \oplus C$, with $B_1 \cong C$, as a direct summand. Therefore $B_1 = 0$. Similarly, $B_2 = 0$. Thus, N_1, N_2 are two injective envelopes of $N_1 \cap N_2$. ■

Lemma 2.3.3. *Let M_R be a square-free injective R-module with $M_R = M_1 \oplus M_2$. If N is a direct summand of M_R, then N is an injective envelope of $(N \cap M_1) \oplus (N \cap M_2)$.*

Proof. Let A be an injective envelope of $N \cap M_1$ in N and B be a direct summand of N such that $N = A \oplus B$. Consider $\pi := \pi_{M_2}$, the projection of $M_1 \oplus M_2$ onto M_2, and the restriction $\pi|_B : B \to M_2$. Because $B \cap M_1 = 0$, $\pi|_B$ is injective. Hence, B is isomorphic to a direct summand C of M_2. By Lemma 2.3.2, B is an injective envelope of $B \cap C$. Therefore, B is an injective envelope of $N \cap M_2 = B \cap M_2 \geq B \cap C$. This implies N is an injective envelope of $(N \cap M_1) \oplus (N \cap M_2)$. ■

Lemma 2.3.4. *Let M_R be an injective R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. For any element $f \in S$, there exist $e_1, e_2, g_1, g_2, h_1, h_2 \in S$ and $i_1, i_2, j_1, j_2 \in J(S)$ such that e_1, e_2 are idempotents and*

$$e_1 = fg_1 + i_1, f = e_1h_1 + j_1,$$

$$e_2 = g_2f + i_2, f = h_2e_2 + j_2.$$

Proof. Notice that any finitely generated left ideal of a von Neumann regular ring is generated by an idempotent. Now the ring $S/J(S)$ is a Von Neumann regular ring and any idempotent of $S/J(S)$ can be lifted to an idempotent of S (see Theorem 1.4.1). Hence, for any $f \in S$, there exists an idempotent $e_1 \in S$ such that

$$(fS + J(S))/J(S) = (e_1S + J(S))/J(S).$$

Therefore, $e_1 = fg_1 + i_1, f = e_1h_1 + j_1$ for some $g_1, h_1 \in S$ and $i_1, j_1 \in J(S)$. Similarly to $e_2 = g_2f + i_2, f = h_2e_2 + j_2$. ■

Lemma 2.3.5. *Let M_R be an injective R-module, $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R and e_1, e_2 be idempotents of S. Then $e_1 = e_2s + j$ for some $s \in S, j \in J(S)$ if and only if $e_1(M_R) \cap A \leq e_2(M_R)$ for some essential submodule A of M_R.*

PROOF. Suppose that $e_1 = e_2s + j$ for some $s \in S$ and $j \in J(S)$. Then $A := \ker j$ is essential in M_R and for any $e_1(x) \in A$, $e_1(x) = (e_1 - j)(e_1(x)) = e_2s(x) \in e_2(M_R)$. Hence, $e_1(M_R) \cap A \leq e_2(M_R)$. Conversely, assume that $e_1(M_R) \cap A \leq e_2(M_R)$ for some essential submodule A of M_R. If we put $A_1 := e_1(M_R) \cap A$ and $A_2 := (1 - e_1)(M_R) \cap A$, then $A_1 \oplus A_2$ is essential in A. Hence, $A_1 \oplus A_2$ is essential in M_R. Put $\pi := \pi_{e_1(M_R)}$ and $\iota := \iota_{e_1(M_R)}$. By $A_1 \leq e_1(M_R) \cap e_2(M_R)$, for any $a + b \in A_1 \oplus A_2$, $(e_1 - e_2\iota\pi)(a + b) = 0$. Therefore, $e_1 - e_2\iota\pi \in J(S)$. Hence, $e_1 = e_2s + j$ with $s = \iota\pi \in S, j = e_1 - e\iota\pi \in J(S)$. ∎

Proposition 2.3.6. *Let M_R be an injective R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. If M_R is square-free, then every right ideal of S containing $J(S)$ of S is a two-sided ideal. In particular, if M_R is square-free, then S is right quasi-duo.*

PROOF. Let I be a right ideal of S containing $J(S)$. Let $f \in I$ and $\phi \in S$. We must show that $\phi f \in I$. By Lemma 2.3.4, there exist $e, g, h \in S$ and $i, j \in J(S)$ such that $e = fg+i$ is an idempotent and $f = eh+j$. The element e belongs to I and $\phi f = \phi eh+\phi j$, so that it suffices to show that $\phi e \in I$. Indeed, let N' be a direct summand of M_R such that $M_R = N \oplus N'$ with $N = e(M_R)$, and M_1 be an injective envelope of $\ker \phi$ in M_R. Then there exists a direct summand M_2 of M_R such that $M_R = M_1 \oplus M_2$. Let N_1, N_2 be respectively injective envelopes of $N \cap M_1, N \cap M_2$ in N. By Lemma 2.3.3, we may assume that

$$M_R = N \oplus N' = N_1 \oplus N_2 \oplus N'.$$

Consider the restriction $\phi|_{N_2}\colon N_2 \to N_1 \oplus N_2 \oplus N'$ of ϕ to N_2. It is easy to check that $\phi|_{N_2}$ is injective. Hence, $N_2 \cong \phi|_{N_2}(N_2)$ and by Lemma 2.3.2, $A = N_2 \cap \phi|_{N_2}(N_2)$ is an essential submodule of N_2. Put $B := \phi|_A^{-1}(A)$ with $\phi|_A : A \to N_2$ the restriction of ϕ to A. Then B is also an essential submodule of N_2. Write $\psi\colon N_2 \to N_2$ for a homomorphism extending $\phi|_A$ and let $\psi' = \iota_{N_2}\psi\pi_{N_2}$. One has $\psi'(M_R) \leq N_2 \leq N = e(M_R)$, which implies from Lemma 2.3.5 that $\psi' \in eS \subseteq I$. Moreover, since for any $a+b+c \in (\ker \phi \cap N_1) \oplus B \oplus N'$, $(\phi e - \psi')(a + b + c) = 0$ and $(\ker \phi \cap N_1) \oplus B \oplus N'$ is essential in M_R, $\phi e - \psi'$ belongs to $J(S)$. This shows that $\phi e \in I$. ∎

Lemma 2.3.7. *Let M_R be an injective R-module, $S = \operatorname{End}(M_R)$ be the endomorphism ring of M_R. For any two elements f, g of S, $g = hf + j$ for some $h \in S, j \in J(S)$ if and only if* $\ker f \cap A \leq \ker g$ *for some essential submodule A of M_R.*

Proof. Assume that $g = hf + j$ for some $h \in S$ and $j \in J(S)$. Then $\ker g \geq \ker(hf) \cap \ker j \geq \ker f \cap A$ with $A = \ker j \leq_e M_R$.

Conversely, assume that $\ker f \cap A \leq \ker g$ for a submodule $A \leq_e M_R$. Let M_1 be an injective envelope of $\ker g$ in M_R, and N_1 be an injective envelope of $\ker f \cap A$ in M_1. Assume that $M_R = N_1 \oplus N_2 \oplus M_2$ for some direct summands N_2, M_2 respectively of M_1 and M_R. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N_2 \oplus M_2 & \xrightarrow{f|_{N_2 \oplus M_2}} & M_R \\ & & \Big\downarrow g|_{N_2 \oplus M_2} & \swarrow h & \\ & & M_R & & \end{array}$$

Here, $f|_{N_2 \oplus M_2}$ and $g|_{N_2 \oplus M_2}$ are the restrictions of f and g to $N_1 \oplus M_2$. Since M_R is injective, there exists $h : M_R \to M_R$ such that

$$g|_{N_2 \oplus M_2} = h\ f|_{N_2 \oplus M_2}.$$

Because $(g - hf)(a + b + c) = 0$ for any $a + b + c \in (\ker f \cap A) \oplus N_2 \oplus M_2$ which is essential in M_R, $j = g - hf \in J(S)$. ■

Proposition 2.3.8. *Let M_R be an injective R-module and $S = \operatorname{End}(M_R)$ be the endomorphism ring of M_R. If M_R is square-free, then every left ideal of S containing the Jacobson radical $J(S)$ of S is a two-sided ideal. In particular, if M_R is square-free, then S is left quasi-duo.*

Proof. Let I be a left ideal of S containing $J(S)$. Let f, ϕ be elements of S with $f \in I$. We must show that $f\phi \in I$. By Lemma 2.3.4, there exist $e, g, h \in S$ and $i, j \in J(S)$ such that $e = gf + i$ is an idempotent and $f = he + j$. Hence, $e \in I$ and $f\phi = he\phi + j\phi$. It suffices to show that $e\phi \in I$. Indeed, put $N_1 := \ker e$ and let N_2 be a direct summand of M_R such that $M_R = N_1 \oplus N_2$. Consider $\psi := \pi_{N_2} \phi \iota_{N_1} : N_1 \to N_2$. Let N_1' be an injective envelope of $A := \ker\phi$ in N_1. Then exists a direct summand N_1'' of M_R such that $N_1 = N_1' \oplus N_1''$. Since $\ker\psi|_{N_1''} = N_1'' \cap A = 0$, $\psi|_{N_1''}$ is injective. This implies N_1'' is isomorphic to a direct summand C of N_2. Hence, M_R contains a direct summand isomorphic to $C \oplus C$. By hypothesis, $C = 0$. In other words, A is essential in N_1. Now, one has that ϕ can be written as the matrix $\phi = \begin{pmatrix} \pi_{N_1}\phi\iota_{N_1} & \pi_{N_1}\phi\iota_{N_2} \\ \pi_{N_2}\phi\iota_1 & \pi_{N_2}\phi\iota_{N_2} \end{pmatrix}$. Then $\ker(e\phi) = \phi^{-1}(\ker e) = \phi^{-1}(N_1) \geq \ker(\pi_{N_2}\phi\iota_1) \oplus \ker(\pi_{N_2}\phi\iota_{N_2}) \geq A = (A \oplus N_2) \cap N_1 = (A \oplus N_2) \cap \ker e$. Since $A \oplus N_2$ is essential in M_R, by Lemma 2.3.7, there exist $h \in S, j \in J(S)$ such that $e\phi = he + j$. Thus $e\phi \in I$. ■

Now the following is the main result of this Chapter.

Theorem 2.3.9. *Let M_R be an injective right R-module and $S = End(M_R)$. Then the following conditions are equivalent*

1. *M_R is square-free.*
2. *S is left quasi-duo.*
3. *S is right quasi-duo.*
4. *Every left ideal of S containing $J(S)$ is a two-sided ideal of S.*
5. *Every right ideal of S containing $J(S)$ is a two-sided ideal of S.*

Proof. (1) ⇒ (4) is from Proposition 2.3.8.

(1) ⇒ (5) is Proposition 2.3.6.

(4) ⇒ (2) and (5) ⇒ (3) are obvious.

Finally, (2) ⇒ (1) and (3) ⇒ (1) is Proposition 2.3.1,

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