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Chapter 1

Injective modules

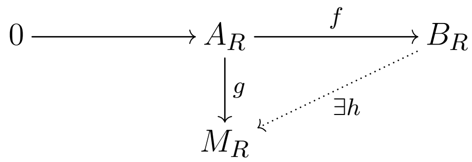
In this Chapter, we first recall notions concerning injective modules and injective envelopes. In Section 1.2, the construction of the maximal right ring of quotients of an arbitrary ring will be presented briefly. These rings of quotients will be used not only in this Chapter, but also in Chapter 5. In Section 1.3, as an example, we consider some properties of injective modules over a non-commutative free algebra. It is also proved that the maximal right ring of quotients of a free algebra R is the injective envelope of R_R as an R-module. In Section 4.2.3, we summarize some basic properties of the endomorphism ring of an injective module and an indecomposable injective module.

1.1 Some basic notions on injective modules, free modules and projective modules

In this Section, we briefly present some basic properties of injective modules, free modules and projective modules. These facts are elementary and can be found in any book of module theory such as [30], [14] or [4], so that we will omit the proofs of all these simple properties. In all cases, we indicate the citations (often [30] and [14]). Notice that the properties presented in this Section are just to help the readers review some needed elementary facts.

Injective modules and injective envelopes.

Let R be a ring. A right R-module M_R is called *injective* if, for any right R-modules A_R, B_R, any monomorphism $f\colon A_R \to B_R$ and any homomorphism $g\colon A_R \to M_R$, there exists a homomorphism $h\colon B_R \to M_R$ such that $g = hf$. In a diagram, this definition can be presented as follows:



Informally, in some books, this property is introduced by saying that any homomorphism $g\colon A_R \to M_R$ can be "extended" to $h\colon B_R \to M_R$.

Theorem 1.1.1. [30, Baer's Test] *A right R-module M_R is injective if and only if for any right ideal I of R, any homomorphism $g\colon I_R \to M_R$ can be extended to $h\colon R_R \to M_R$.*

Theorem 1.1.2. [30, Proposition 3.4] *A direct product of a family of injective modules is injective. In particular, a direct sum of finitely many injective modules is injective.*

Notice that a direct sum of an arbitrary family of injective modules is not necessarily injective. A direct sum of an arbitrary family of injective right R-modules is injective if and only if R is right Noetherian [4, Proposition 18.13].

Corollary 1.1.3. *All direct summands of an injective module are injective.*

Here, a submodule A_R of a right R-module M_R is said to be a *direct summand* of M_R if $A_R \oplus B_R = M_R$ for some submodule B_R of M_R.

Lemma 1.1.4. [30, Remarks 3.23] *Let M_R be a right R-module and A_R be a submodule of M_R. The following conditions are equivalent:*

1. *For any non-zero submodule B_R of M_R, $A_R \cap B_R \neq 0$.*
2. *For any non-zero element $x \in M_R$, there exists $r \in R$ such that $0 \neq xr \in A_R$.*

If two modules $A_R \leq M_R$ satisfy any of the properties in Lemma 1.1.4, then M_R is called an *essential extension* of A_R or A_R is said to be *essential* in M_R, denoted by $A_R \leq_e M_R$. For a right ideal I of the ring R, we say that I is an *essential right ideal* of R if $I_R \leq_e R_R$ as R-modules.

The dual notion of essential submodules is that of a superfluous submodules. A submodule A_R of a right module M_R is called *superfluous* (or *small*) in M_R if, for any submodule B_R of M_R with $A_R + B_R = M_R$, one has that $B_R = M_R$ (notation $A_R \leq_s M_R$).

Proposition 1.1.5. *Let M_R, N_R be right R-modules. Assume that $f : N_R \to M_R$ is a homomorphism, A_R, B_R are submodules of M_R and C_R is a submodule of N_R. Then*

1. *If A_R and C_R are essential in respectively M_R and N_R, then $A_R \oplus C_R$ is essential in $M_R \oplus N_R$.*
2. *If A_R and B_R are essential in M_R, then $A_R \cap B_R$ is essential in M_R.*
3. *If A_R is an essential submodule in B_R and B_R is essential in M_R, then A_R is essential in M_R.*
4. *If A_R is essential in M_R, then $f^{-1}(A_R)$ is essential in N_R.*

Moreover, if $M_R = N_R$, $A_R \leq_e M_R$ and f is an idempotent endomorphism of M_R, that is, $f^2 = f$, then $f(A_R) \leq_e f(M_R)$.

The proof is elementary.

Theorem 1.1.6. [30, Theorem 3.30 and Corollary 3.32] *Let $M_R \leq N_R$ be right R-modules. The following conditions are equivalent:*

1. *N_R is a maximal essential extension of M_R. That is, there is no proper extension module N'_R of N_R such that $M_R \leq_e N'_R$.*
2. *N_R is injective and is an essential extension of M_R.*
3. *N_R is minimal injective over M_R. That is, N_R is injective and there is no injective module M'_R such that $M_R \leq M'_R < N_R$.*

Moreover, if N'_R is another essential extension of M_R satisfying any of properties (1), (2) *and* (3)*, then there exists an isomorphism $f\colon N_R \to N'_R$ such that $f(m) = m$ for any $m \in M_R$.*

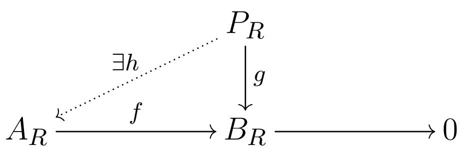
Corollary 1.1.7. *A module M_R is injective if and only if M_R is a direct summand of any module which contains M_R.*

It is proved that every module M_R is contained in an injective module which is an essential extension of M_R, denoted by $E(M_R)$. It is called an *injective envelope* or *injective hull* of M_R, denoted by $E(M_R)$. By Theorem 1.1.6, this injective envelope is unique up to isomorphisms. Moreover, if N_R is an injective module and M_R is a submodule of N_R, then there exists a direct summand of N_R such that it is an injective envelope of M_R.

Theorem 1.1.8. [30, Example 3.38] *For a family $\{ M_i \mid i = 1, 2, \cdots, n \}$ of finitely many right R-modules, $E(\bigoplus_{i=1}^{n} M_i) = \bigoplus_{i=1}^{n} E(M_i)$.*

Projective modules and hereditary rings.

Let R be a ring. A right R-module F_R is called *free* if it is isomorphic to a direct sum of copies of the module R_R. A right R-module P_R is called *projective* if, for any right R-modules A_R, B_R, any epimorphism $f\colon A_R \to B_R$ and any homomorphism $g\colon P_R \to B_R$, there exists a homomorphism $h\colon P_R \to A_R$ such that $g = fh$. In a diagram, this definition can be represented as follows:



Informally, this property say that any homomorphism $g\colon P_R \to B_R$ can be “lifted” to $h\colon P_R \to A_R$.

Proposition 1.1.9. [30, Corollary 2.6] *A right R-module P_R is projective if and only if it is isomorphic to a direct summand of some free right R-module.*

Corollary 1.1.10. *Every direct summand of a projective module is projective.*

A ring R is called *right hereditary* if every right ideal of R is projective as a right R-module. *Left hereditary* rings are defined similarly. A ring is said to be *hereditary* if it is left and right hereditary. Here are some examples of right (or left) hereditary rings.

Examples 1.1.11. 1. A direct product of two hereditary rings is a hereditary ring. It can be checked easily from the definition.

2. A non-zero module is called *simple* if its proper submodule it the zero submodule 0. A direct sum of a family of simple modules is said to be a *semisimple module*. We say that a ring R is a *semisimple artinian ring* if R_R is semisimple as a right R-module. It is possible to prove that R is semisimple artinian if and only if ${}_RR$ is semisimple as a left R-module, if and only if it is isomorphic to the direct product of finitely many matrix rings over division rings.

Every semisimple artinian ring is hereditary [14, Theorem 1.2]. In particular, the matrix ring $M_n(D)$, for some integer $n \geq 1$ and division ring D, is hereditary.

3. Let K be a field and $\{ x_i \mid \in I \}$ be a set of non-commutative indeterminates. Then every right (or left) ideal of the free algebra $R := K\langle \{ x_i \mid \in I \} \rangle$ is free as an R-module [11, Page 106]. Therefore, R is a hereditary ring.

Proposition 1.1.12. [40, Proposition I.9.5] *A ring R is right hereditary if and only if every factor module of an injective right R-module is injective.*

1.2 The maximal ring of quotients

There are two methods to define the maximal right ring of quotients of a ring R. The first one is to build it through the endomorphism ring $\mathrm{End}(E(R_R))$ (for examples, see [30, Chapter 5]) and the second one is to define as the direct limit over the set of all dense left ideals of R (for example, see [5, Chapter 2]). These are equivalent [30, Theorem 13.21]. In this Section, we use the second method.

Let R be a ring. Let M_R be a right R-module and A_R be a submodule of M_R. We say that A_R is *dense* in M_R if, for any $x_1, x_2 \in M_R$ with $x_1 \neq 0$, there exists $r \in R$ such that $x_1 r \neq 0$ and $x_2 r \in A_R$. A right ideal I of R is called *dense* if I_R is a dense submodule of R_R. The set of all dense right ideals of R is denoted by $\mathcal{D}_r(R)$.

By definition, every dense submodule of a module M_R is essential. However, the inverse is not true in general. It holds if the module M_R is non-singular. Recall a module M_R is called *non-singular* if, for any $m \in M_R$ with $\mathrm{ann}_R(m) = \{ r \in R \mid mr = 0 \} \leq_e R_R$, one has $m = 0$. A ring R is called *right non-singular* (resp. *left non-singular*) if the module R_R (resp. ${}_RR$) is non-singular. We say that R is *non-singular* if R is both right and left non-singular. For instance, every domain is non-singular. Here, a ring is said to be a *domain* if it has no left nor right zero-divisor.

Lemma 1.2.1. [30, Corollary 8.9] *Every essential submodule of a non-singular module is dense.*

Corollary 1.2.2. [30, Corollary 8.9] *In a right non-singular ring, the set of all dense right ideals coincides with the set of all essential right ideals.*

Proposition 1.2.3. *Let I and J be dense right ideals of R and $f\colon I_R \to R_R$ be a homomorphism. Then $I \cap J$ and $f^{-1}(J) = \{ x \in I \mid f(x) \in J \}$ are dense in R.*

The proof is elementary. See [5, Proposition 2.1.1].

From Proposition 1.2.3, $\mathcal{D}_r(R)$ is a directed set with the relation defined by $I \preceq J$ if $J \subseteq I$. For any $I \preceq J$ in $\mathcal{D}_r(R)$, put $\varphi_{I,J}\colon \operatorname{Hom}(I_R, R_R) \to \operatorname{Hom}(J_R, R_R), \varphi(f) = f|_J$, the restriction of f to J. One can check easily that $\varphi_{I,I}\colon \operatorname{Hom}(I_R, R_R) \to \operatorname{Hom}(I_R, R_R)$ is the identity of $\operatorname{Hom}(I_R, R_R)$ and $\varphi_{I_1,I_3} = \varphi_{I_1,I_2}.\varphi_{I_2,I_3}$ for any I, I_1, I_2, I_3 in $\mathcal{D}_r(R)$ with $I_1 \preceq I_2 \preceq I_3$. Thus, $(\operatorname{Hom}(I_R, R_R), \varphi_{I,J}), I, J \in \mathcal{D}_r(R)$, is a direct system over $\mathcal{D}_r(R)$. Hence, we can define the direct limit $Q_{mr}(R) := \varinjlim_{I \in \mathcal{D}_r(R)} \operatorname{Hom}(I_R, R_R)$.

Now, in $Q_{mr}(R)$, we consider the addition and multiplication defined as follows: for $(f, I), (g, J) \in Q_{rm(R)}$, where $I, J \in \mathcal{D}_r(R)$, $(f, I) + (g, J) := (f + g, I \cap J)$, $(f, I) \cdot (g, J) := (fg, g^{-1}(I))$. It is easy to check (or see [5, page 56]) that these operations are well-defined and that $Q_{mr}(R)$ is a ring with zero element $(0, R)$ and identity (Id_R, R). The ring $Q_{mr}(R)$ is called the *maximal right ring of quotients* of R.

Let S be a ring containing R as a subring. Assume that a is an element of S such that the right ideal $(R\colon a)_R = \{ r \in R \mid ar \in R \}$ is dense in R. Then we may consider an element $(\ell_a, (R\colon a)_R)$ of $Q_{mr}(R)$. Here, the homomorphism $\ell_a \in \operatorname{Hom}((R\colon a)_R, R_R)$ is the left multiplication determined by a, that is, $\ell_a(x) = ax$ for any $x \in (R\colon a)_R$.

Proposition 1.2.4. *The map $\iota\colon R \to Q_{mr}(R)$ defined by $\iota(a) = (\ell_a, R)$ is an injective homomorphism of rings. Thus, R may be considered as a subring of $Q_{mr}(R)$ via the map ι.*

The proof is elementary.

Proposition 1.2.5. *For any element $x \in Q_{mr}(R)$, $(R\colon x)_R$ is a dense right ideal of R. In particular, R_R is an essential submodule of $Q_{mr}(R)_R$.*

Proof. Let $x = (f, I) \in Q_{mr}(R)$, where $I \in \mathcal{D}_r(R)$ and $f \in \mathrm{Hom}(I_R, R_R)$. For any element $r \in I$, $xr = (f, I)(\ell_r, R) = (\ell_{f(r)}, R) \in R$. It follows that $I \subseteq (R\colon x)_R$. Hence, $(R\colon x)_R$ is dense in R. ■

Proposition 1.2.6. *For any element $x \in Q_{mr}(R)$, if $xJ = 0$ for some $J \in \mathcal{D}_r(R)$, then $x = 0$.*

Proof. Assume that $x = (f, I)$ for some $I \in \mathcal{D}_r(R)$ and $f \in \mathrm{Hom}(I_R, R_R)$. Then for any $r = (\ell_r, R) \in J$, $0 = xr = (\ell_{f(r)}, R)$ for all $r \in I \cap J$. Thus $f(I \cap J) = 0$. Therefore $x = (f, I) = (f|_{I\cap J}, I \cap J) = 0$. ■

Theorem 1.2.7. *Let Q be an extension ring of R. Assume that for any $a \in Q$, $(R\colon a)_R$ belongs to $\mathcal{D}_r(R)$ and if $aJ = 0$ for some $J \in \mathcal{D}_r(R)$, then $a = 0$. Then there exists a ring monomorphism $\phi\colon Q \to Q_{mr}(R)$ extending ι.*

Proof. Consider $\phi\colon Q \to Q_{mr}(R)$ defined by $\phi(a) = (\ell_a, (R\colon a)_R)$ for any $a \in Q$. Since $(R\colon a)_R$ belongs to $\mathcal{D}_r(R)$ for any $a \in Q$, ϕ is well-defined. It is elementary to check that ϕ is a ring homomorphism. We must show that ϕ extends ι and is injective. Indeed, for $a \in R$, $\phi(a) = (\ell_a, (R\colon a)_R) = (\ell_a, R) = a \in R$. Thus ϕ extends ι. Now if $0 = \phi(a) = (\ell_a, (R\colon a)_R)$, then $a(R\colon a)_R = 0$, which implies $a = 0$ by the hypothesis. ■

By Theorem 1.2.7, $Q_{mr}(R)$ is “maximal” among the rings having the properties in Propositions 1.2.4, 1.2.5 and 1.2.6.

The *maximal left ring of quotients* $Q_{ml}(R)$ of R is defined similarly. See [5, Section 2.2, Chapter 2].

1.3 An example

Let k be a field, n a positive integer. If $x_1, x_2, \ldots, x_n$ are n commuting indeterminats, then the polynomial ring $k[x_1, x_2, \cdots, x_n]$ is a noetherian ring. Injective modules over a noetherian ring were described in [33]. In this Section, we give some properties of injective modules over $k\langle x_1, x_2, \cdots, x_n\rangle$ in the case $x_1, x_2, \cdots, x_n$ are non-commuting indeterminates. Throughout this Section, *it is assumed that $x_1, x_2, \ldots, x_n$ are n non-commuting indeterminates, $n > 1$ and $R := k\langle x_1, x_2, \ldots, x_n\rangle$ denotes the free k-algebra.* Notice that R is a *free ideal ring*, that is, a ring in which every right (or left) ideal is a free R-module of unique rank (Example 1.1.11). In particular, R is a hereditary ring.

In the following, "countable" means either finite or of cardinality $\aleph_0$, the first infinite cardinal.

A set $\{ f_\lambda \mid \lambda \in \Lambda \}$ of elements of R is called *right linearly independent* over R if, for any $m > 0$, a subset $\{ f_{\lambda_1}, f_{\lambda_2}, \cdots, f_{\lambda_m} \}$ of m elements of $\{ f \mid \lambda \in \Lambda \}$ and $r_1, r_2, \ldots, r_n \in R$ with $f_{\lambda_1} r_1 + f_{\lambda_2} r_2 + \cdots + f_{\lambda_m} r_m = 0$, one has $r_1 = r_2 = \cdots = r_m = 0$. Since R is a domain, it is easy to check that $\{ f_\lambda \mid \lambda \in \Lambda \}$ is right linearly independent over R if and only if $\sum_{\lambda \in \Lambda} f_\lambda R = \bigoplus_{\lambda \in \Lambda} f_\lambda R$.

Lemma 1.3.1. *Every right ideal I of R is generated by a right linearly independent set of countably many elements of I.*

PROOF. Let I be a right ideal of R. Then I is a free right R-module, so that it has a free set of generators f_λ, $\lambda \in \Lambda$. Thus $I = \bigoplus_{\lambda \in \Lambda} f_\lambda R$, which implies that the f_λ's are right linearly independent. Since they are right linearly independent over R, they are a fortiori linearly independent over k. But R is countably dimensional over k, so that Λ is countable. ∎

Proposition 1.3.2. *A right R-module M_R is injective if and only if for every right linearly independent set $\{ f_\lambda \mid \lambda \in \Lambda \}$ over R, with Λ a countable set, and every family $\{ x_\lambda \mid \lambda \in \Lambda \}$ of elements of M_R, there exists $x \in M_R$ such that $x f_\lambda = x_\lambda$ for every $\lambda \in \Lambda$.*

PROOF. The module M_R is injective if and only if, for every right ideal I of R, any homomorphism $I_R \to M_R$ extends to R_R. By Lemma 1.3.1, I_R is generated by a right linearly independent set $\{ f_\lambda \mid \lambda \in \Lambda \}$, with Λ countable, that is, $I = \bigoplus_{\lambda \in \Lambda} f_\lambda R$ and $f_\lambda R \cong R_R$. Thus a homomorphism $I_R \to M_R$ is completely determined by the image of the f_λ's, which can be arbitrary elements x_λ of M_R. Thus a homomorphism $\varphi \colon I_R \to M_R$ with $\varphi(f_\lambda) = x_\lambda$ extends to R_R if and only if there exists $x \in M_R$ such that $x f_\lambda = x_\lambda$ for every $\lambda \in \Lambda$. ■

Proposition 1.3.3. *Every homomorphic image of an injective right R-module is injective.*

PROOF. This follows from Proposition 1.1.12 because R is a hereditary ring. ■

We recall the following Lemma, that appears in Bergman [6, Lemma 1].

Lemma 1.3.4. [6, Lemma 1] *Let S be a ring, κ an infinite regular cardinal such that every right ideal of S can be generated by less than κ elements, and $\{ M_i \mid i \in I \}$ a family of injective right S-modules. Then the submodule $\prod_{i \in I}^{\kappa} M_i$ of the direct product $\prod_{i \in I} M_i$ whose elements are the $x \in \prod_{i \in I} M_i$ with support of cardinality $< \kappa$ is an injective submodule of $\prod_{i \in I} M_i$.*

Here, an infinite cardinal κ is called a *regular cardinal* if for any set I with cardinal $|I| < \kappa$ and any family of sets $A_i, i \in I$, such that $|A_i| < \kappa$ for every $i \in I$, then the cardinal $|\cup_{i \in I} A_i| < \kappa$.

Proposition 1.3.5. *Let $\aleph_1$ be the second smallest infinite cardinal. For every family $\{ M_i \mid i \in I \}$ of injective right R-modules, the submodule $\prod_{i \in I}^{\aleph_1} M_i$ of the direct product $\prod_{i \in I} M_i$ whose elements are the $x \in \prod_{i \in I} M_i$ with countable support is injective.*

PROOF. The statement is implied directly from Lemma 1.3.4 since $\aleph_1$ is a regular cardinal. ■

Theorem 1.3.6. *The module $Q_{mr}(R)_R$ is an injective envelope of R_R.*

PROOF. By Proposition 1.2.5, R_R is essential in $Q_{mr}(R)_R$. Now R is non-singular, so that, from [5, Theorem 2.1.15], $Q_{mr}(R)_R$ is injective. Thus, $Q_{mr}(R)_R$ is an injective envelope of R_R. ■

1.4 The endomorphism ring of an injective module

In this Section, we recall some well known results on the endomorphism ring of an injective module.

Recall that the *Jacobson radical* $J(R)$ of an arbitrary ring R is the intersection of all maximal left ideals of R. It is easily proved that $J(R)$ is equal to the intersection of all maximal right ideals of R and, hence, $J(R)$ is a two-sided ideal of R.

Theorem 1.4.1. [30, Theorem 13.1] *Let M_R be an injective right R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. Then*

1. *The Jacobson radical $J(S)$ of S is $\{ f \in S \mid \ker f \leq_e M_R\}$.*
2. *The ring $S/J(S)$ is a von Neumann regular ring. That is, for every $\overline{f}$ in $S/J(S)$, there exists $\overline{g}$ in $S/J(S)$ such that $\overline{f} = \overline{fgf}$.*
3. *Idempotents of $S/J(S)$ can be lifted to idempotents of S. That is, for any $\overline{f}$ in $S/J(S)$, if $\overline{f}^2 = \overline{f}$, there exists g in S such that $g^2 = g$ and $\overline{g} = \overline{f}$.*
4. *The quotient ring $S/J(S)$ is a right self-injective ring. That is, $S/J(S)$ is an injective right $S/J(S)$-module.*

A non-zero right R-module M_R is called *indecomposable* if, for two submodules A, B of M_R with $A \oplus B = M_R$, one has $A = 0$ or $B = 0$. A module is said to be *indecomposable injective* if it is injective and indecomposable. The class of indecomposable injective modules and their endomorphism rings were studied in [33].

Theorem 1.4.2. [33, Theorem 2.6] *An injective module is indecomposable if and only if its endomorphism ring is a local ring.*

Here, a ring R is called *local* if R has a unique maximal left ideal, equivalently if R has a unique maximal right ideal, equivalently if the set of non-invertible elements of R is a two-sided ideal, equivalently if the Jacobson radical $J(R)$ of R is the set of all non-invertible elements of R.