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4 Introduction to Part II

4.1 Motivation

The model we study in this part of the thesis fits into the class of continuum percolation models, which have been studied by both mathematicians and physicists. Their first appearance can be traced back (at least) to [Gil61] under the name of random plane networks. This paper provides a model for infinite communication networks of stations with finite range: each pair of points of a Poisson point process on $\mathbb{R}^2$ is connected whenever their distance is less than a prescribed threshold $R > 0$. From a percolation point of view this may also be described by a Boolean percolation model (see Section 5.2). Another application mentioned in [Gil61] is the modeling of a contagious infection, where each individual gets infected when it has distance less than $R$ to an infected individual.

A subclass of continuum percolation models follows the following recipe: draw a random set of points (a Poisson point process, for instance) and attach to each of the points a geometric object, like a disk with a random radius (Boolean model) or a segment with a random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: we attach to each point of a Poisson point process on $\mathbb{R}^d$, $d \geq 1$, a Brownian path (a path of a Wiener sausage when $d \geq 4$).

Notations. For $d \geq 1$, we denote by $\text{Leb}_d$ the Lebesgue measure on $\mathbb{R}^d$. $\| \cdot \|$ and $\| \cdot \|_{\infty}$ stand for the Euclidean norm and sup-norm on $\mathbb{R}^d$, respectively. For any set $A$, the symbol $A^c$ refers to the complement $\mathbb{R}^d \setminus A$. The open ball with center $z$ and radius $r$ with respect to the Euclidean norm is denoted by $B(z, r)$, while $B_{\infty}(z, r)$ stands for the same ball with respect to the sup-norm. Furthermore, for $0 < r < r'$ we denote by $A(r, r') = B(0, r') \setminus B(0, r)$ and $A_{\infty}(r, r') = B_{\infty}(0, r') \setminus B_{\infty}(0, r)$ the annulus delimited by the balls of radii $r$ and $r'$ with respect to the Euclidean norm and sup-norm, respectively. Moreover, given a $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$, we denote its $i$-th component by $(B^i_t)_{t \geq 0}$, $i \in \{1, 2, \ldots, d\}$. Finally, for $I \subseteq \mathbb{R}^+$ we denote by $B_I$ the set $\{B_t, t \in I\}$. The symbol $\mathbb{P}^a$ denotes the law of a standard Brownian motion starting at $a$. In case of two or more independent copies we add a superscript, i.e., $\mathbb{P}^{a_1, a_2}$.

4.2 Overview

For $\lambda > 0$, let $(\Omega_p, A_p, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \text{Leb}_d$ is defined. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{ (B^x_t)_{t \geq 0}, x \in \mathcal{E} \}$ such that $B^x_0 = x$ for each $x \in \mathcal{E}$, and such that
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\((B_t^x - x)_{t \geq 0}\) is independent of \(E\). A more rigorous definition is provided in Section 5.1.3 below, where ergodic properties are obtained along the way. We study for \(t, r \geq 0\) the occupied set:

\[
O_{t,r} := \bigcup_{x \in E} \bigcup_{0 \leq s \leq t} B(B_s^x, r). \tag{4.1}
\]

We write \(O_t\) instead of \(O_{t,0}\).

Two points \(x\) and \(y\) of \(\mathbb{R}^d\) are said to be \textit{pairwise connected} in \(O_{t,r}\) if and only if there exists a continuous function \(\gamma : [0, 1] \rightarrow O_{t,r}\) such that \(\gamma(0) = x\) and \(\gamma(1) = y\). A subset of \(O_{t,r}\) is connected if and only if all of its points are pairwise connected. In the following a connected subset of \(O_{t,r}\) is called a component. A component \(C\) is bounded if and only if there exists an \(R > 0\) such that \(C \subseteq B(0, R)\). Otherwise, the component is said to be unbounded. A \textit{cluster} is a connected component that is maximal in the sense that it is not strictly contained in any other connected component.

We are interested in the percolative properties of the occupied set: Is there an unbounded cluster for large \(t\)? Is it unique? What happens for small \(t\)? Since an elementary monotonicity argument shows that \(t \mapsto O_{t,r}\) is non-decreasing, the first and the third question may be rephrased as follows: Is there a percolation transition in \(t\)?

\section*{4.3 Preliminaries on Boolean percolation}

The model of Boolean percolation is discussed in detail in [MR96], and we refer to this book for a discussion that goes beyond the description we are giving here.

Let \(\rho\) be a probability measure on \((0, \infty)\) and let \(\chi\) be the Poisson point process on \(\mathbb{R}^d \times [0, \infty)\) with intensity \((\lambda \times \text{Leb}_d) \otimes \rho\). We denote the corresponding probability measure by \(\mathbb{P}_{\lambda, \rho}\). A point \((x, r(x)) \in \chi\) is interpreted to be the open ball in \(\mathbb{R}^d\) with center \(x\) and radius \(r(x)\). Furthermore, we let \(E\) be the projection of \(\chi\) onto \(\mathbb{R}^d\). Boolean percolation deals with properties of the random set

\[
\Sigma = \bigcup_{x \in E} B(x, r(x)). \tag{4.2}
\]

Let \(C(y), y \in \mathbb{R}^d\), denote the cluster of \(\Sigma\) that contains \(y\). If \(y \notin \Sigma\), then \(C(y) = \emptyset\).

\textbf{Theorem 4.3.1} ([Gou08], Theorem 2.1). \textit{For all probability measures \(\rho\) on \((0, \infty)\) the following assertions are equivalent:}

(a) \(\int_0^\infty x^d \rho(dx) < \infty\). \tag{4.3}

(b) \textit{There exists a} \(\lambda_0 \in (0, \infty)\text{ such that for all } \lambda < \lambda_0, \}

\[\mathbb{P}_{\lambda, \rho}(\exists y \in \mathbb{R}^d : \text{Leb}_d(C(y)) = \infty) = 0. \tag{4.4}\]
Moreover, if (a) holds, then there exists a $C = C(d) > 0$ such that (4.4) is satisfied for all

$$
\lambda < C \left( \int_0^\infty x^d \rho(dx) \right)^{-1}.
$$

(4.5)

It is immediate from Theorem 4.3.1 that

$$
\lambda_c(\rho) := \inf \left\{ \lambda > 0 : \exists y \in \mathbb{R}^d : \text{Leb}(C(y)) = \infty \right\} > 0.
$$

(4.6)

Moreover, from the remark on page 52 of [MR96] it also follows that $\lambda_c(\rho) < \infty$ as soon as $\rho((0, \infty)) > 0$. A more geometric way to characterize (4.6) is via crossing probabilities. For that, fix $N_1, N_2, \ldots, N_d > 0$ and let $\text{CROSS}(N_1, N_2, \ldots, N_d)$ be the event that the set $[0, N_1] \times [0, N_2] \times \cdots \times [0, N_d]$ contains a component $C$ such that $C \cap \{0\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \emptyset$ and $C \cap \{N_1\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \emptyset$. The critical value $\lambda_{\text{CROSS}}$ with respect to this event is defined by

$$
\lambda_{\text{CROSS}}(\rho) = \inf \left\{ \lambda > 0 : \lim_{N \to \infty} \mathbb{P}_{\lambda, \rho}(\text{CROSS}(N, 3N, \ldots, 3N)) > 0 \right\}.
$$

(4.7)

It is proved in [MMS86] that

$$
\lambda_c(\rho) = \lambda_{\text{CROSS}}(\rho),
$$

(4.8)

provided $\rho$ has compact support.

### 4.4 Main results on Brownian percolation

Fix $\lambda > 0$.

**Theorem 4.4.1.** [No percolation for $d = 1$] Let $d = 1$. Then for all $t \geq 0$ the set $\mathcal{O}_t$ has almost surely no unbounded cluster.

**Theorem 4.4.2.** [Percolation phase transition and uniqueness for $d \in \{2, 3\}$] Let $d \in \{2, 3\}$. Then there exists a $t_c = t_c(\lambda, d) > 0$ such that, for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster, but, for $t > t_c$, $\mathcal{O}_t$ has a unique unbounded cluster.

Let $d \geq 4$, $r > 0$ and let $\delta_r$ be the Dirac measure at $r$. Denote by $\lambda_c(\delta_r)$ the critical value for $\mathcal{O}_{0,r}$, i.e., for all $\lambda < \lambda_c(\delta_r)$ the set $\mathcal{O}_{0,r}$ has almost surely no unbounded cluster, but for $\lambda > \lambda_c(\delta_r)$ $\mathcal{O}_{0,r}$ has a unique unbounded cluster (see also (4.6)). It follows from Theorem 4.3.1 that $\lambda_c(\delta_r) > 0$ and $\lim_{r \to 0} \lambda_c(\delta_r) = \infty$.

**Theorem 4.4.3.** [Percolation phase transition and uniqueness for $d \geq 4$] Let $d \geq 4$ and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then there exists a $t_c = t_c(\lambda, d, r) > 0$ such that, for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster, but, for $t > t_c$, $\mathcal{O}_t$ has a unique unbounded cluster.