Author: Dalhuisen, Jan Willem
Title: The Robinson congruence in electrodynamics and general relativity
Issue Date: 2014-03-25

The Robinson Congruence in Electrodynamics and General Relativity

Jan Willem Dalhuisen

ISBN: 978-90-8593-000-6

The Robinson Congruence in Electrodynamics and General Relativity

PROEFSCHRIFT

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus prof. mr. C.J.J.M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 25 maart 2014
klokke 10:00 uur

door

Jan Willem Dalhuisen

geboren te Elst (Gld)
in 1966

Promotiecommissie:

Promotor:	Prof. dr. D. Bouwmeester	Universiteit Leiden en University California Santa Barbara
Leden:	Prof. dr. N. Van den Bergh	Universiteit Gent
	Prof. dr. A. Burinskii	Russian Academy of Sciences, Moscow
	Prof. dr. E.R. Eliel	Universiteit Leiden
	Prof. dr. G. Nienhuis	Universiteit Leiden
	Dr. A. Boyarsky	Universiteit Leiden
	Dr. P. J. M. Bongaarts	Universiteit Leiden

The work presented in this thesis has been made possible by financial support from the Marie-Curie Program EU-CT-2006-042580.

Casimir PhD series, Delft-Leiden 2014-2

The limits of my language are the limits of my world.

Whereof one cannot speak, thereof one must be silent.

Ludwig Wittgenstein

Contents

Introduction

This thesis is an account of the research that started with a simple observation. In 2008 the investigations of W.T.M. Irvine and D. Bouwmeester with respect to curious solutions to Maxwell's equations culminated in an article in Nature Physics [1]. These solutions, which had been described earlier but less complete by A. Rañada [2], were dubbed linked and knotted beams of light. Although they are more general than the term 'light' suggests, 'linked and knotted' describes these solutions very well: a picture of all the electric field lines at a particular instant of time would reveal that they consist of circles, any two of which are linked in the manner of two neighboring Olympic rings. The complete collection of circles fill all of space in a tangled manner. 'Knotted' refers to this tangle, but more aptly, also to the mathematical notion of knot of which a circle is a simple example.
One of the investigators, D.B., had been a post-doc in the group of Roger Penrose and knew therefore very well the similarity between the new solutions and a picture representing a certain kind of twistor, drawn by its inventor in the standard reference for twistor theory [3]. This intriguing configuration is referred to in the twistor theory literature as the Robinson congruence.
Both pictures, of the electric field lines and the twistor, exhibit a geometry that is far from commonplace and a naturally justified question was therefore born: is there any deeper connection between these? It was my initial task, as a theorist in a predominant experimental quantum optics group, to address this question.
Confronted with this task it is also natural to search for the differences. There are many. A twistor is a mathematical object, whereas Maxwell equations describe accurately an import part of nature. Apart from an overall translational movement the twistor picture does not change in time. The circles of the electric field deform in a complicated way that is very hard to describe. There are other twistors, unrelated to the linked structure. Besides the electric field, the magnetic field exhibits the same structure, only rotated with respect to the electric one. Why then should there be a deeper correspondence?

Happily, as part of present research, it was found that a neat description of the linked and knotted solutions was possible in terms of a complex combination of the electric and magnetic field. From this it could be established that there is another physical quantity of the knotted solutions, the Poynting vector, also exhibiting the same structure as the electric or magnetic field, again rotated, now with respect to both. The surprise was the observation that the time development of the Poynting vector was simple and equal to the time development in the twistor picture. This fact was an important contribution to [4,5] and led to the conclusion that further investigation was warranted.

It was realized that the investigated structures related to Robinson congruences could be of interest in other fields of physics too. Plasma physics, hydrodynamics and general relativity were possible theories among others for which the knotted solutions could be 'generalized'. The connection to general relativity very much appealed to me although it implied that I would venture deeper into terra incognita for the research group that I was embedded in. My research supervisor encouraged this branching out of the research and arranged support from additional researchers. In the second half of my PhD research I have therefore worked closely together with a bachelor, a Masters, and another PhD student. Furthermore we had fruitful discussion with Prof. Iwo Białinicky-Birula (Warsaw), Prof. Roger Penrose (Oxford, and Lorentz professor 2011) and Prof. Alexander Burinskii (Moscow, and visiting professor summer 2012).
Whereas the additional students, with my support, focused on studies of twistors and Robinson congruences in plasma physics and in the linearized version of general relativity, I set myself the ambitious goal of searching for a solution of the full non-linear Einstein equations with a source term based on the electromagnetic knot. The approach that I took is based on the Newman-Penrose tetrad formalism and by the time of writing of this thesis several insights have been obtained that reduces the large number of coupled equations to a much smaller and more manageable set of equations. However a conclusive result has not yet been obtained. It might well be that the next step would require a significant numerical effort. As this thesis will show, my interest is mainly in analytic studies and future contributions from a researcher with better computer skills might be needed to complete this ambitious project. Apart from presenting my findings in a clear and precise way I have taken it therefore as my task to present my understanding of knotted structures of light, of special topics in twistor theory, and of the Newman-Penrose formalism in a form that will bring new researchers quickly up to date in this specialized field of research.

In the following I will describe and comment on the content of my thesis chapter by chapter.

Chapter 1 introduces the basic ingredients for this thesis: tetrads, spinors and twistors. With the possible exception of twistors, every physicist is familiar with these concepts. However, the way in which spinors are introduced here, as general mathematical objects without the for physicists self-evident association

with spin-$\frac{1}{2}$ particles, is not standard. Vectors and tensors can be built from spinors, and therefore it is possible to 'rewrite' many physical theories in 'spinor-language', and this will be done in later chapters.
The introduction of a null tetrad leads in a natural way to Infeld Van Der Waerden symbols, hybrid quantities that figure as a dictionary between 'tensor-language' and 'spinor-language'. In later chapters the null tetrad will turn up again, but now as a field of basis vectors in the Newman-Penrose formalism.
Twistors are introduced in the most basic and succinct manner as a combination of two spinors. Twistor theory is covering a broad and expanding range of topics but for this thesis only one instance of the relationships between subspaces of twistor space and subspaces of Minkowski space will be relevant: a null twistor as a null geodesic.
The important Kerr theorem is given, including a sketchy proof, and will play an essential role in later chapters.

Chapter 2 is about the geometry of the key structure in this thesis, the Hopf fibration, which is intimately linked to the Robinson congruence. Modern literature deals with the Hopf fibration with the help of mathematical tools from algebraic topology and discusses the geometry in question in just a few lines. Given the importance of a detailed understanding of all aspects of the Hopf fibration, we prefer a more basic treatment, in which all features are explicitly written out.

Chapter 3 recapitulates the basic equations of Maxwell theory in different forms. The familiar vector notation is used to define the units that will be used and to list the properties that will be of importance throughout this work. The Riemann-Silberstein vector and the Faraday tensor are introduced to rewrite the equations in 'RS-form' and in manifest covariant (relativistic) form. Finally, the equations are rewritten in spinor-language. Since this will be the least familiar, some basic examples are given.
All the presented forms will be used in later chapters.

Chapter 4 together with chapter 5 are two long chapters containing most of the new results that are presented in this thesis. It starts with a summary of what can be considered the germ of present research, Rañada's attempt at a topological theory of electromagnetism. This summary contains elucidations, partly in the form of an appendix, not present in the original work of Rañada as well as explicit expressions for the fields considered to make the connection with chapter 2. Although these expressions have appeared in the literature before [6], we show that in order to have a manageable expression for all times, they should be combined in the form of the Riemann-Silberstein vector (4.1). In this form these fields were found to be given in a different context and without further ado in [7] as a specific example of Trautman-Robinson fields. The Poynting vector, which has not been considered before, exhibits the structure of a Hopf fibration for all times (4.2). Alternative forms of these fields are also presented.

Next, with the help of a theorem by Robinson we show that the 'optical analogy' for null twistors can be extended to non-null twistors. This establishes a mathematical correspondence between a non-null twistor and an electromagnetic Hopf knot. Included is a discussion of *local* duality transformations in electrodynamics, for which literature is very scarce.
The method used for the optical analogy for a non-null twistor is also applied to a null twistor, leading to what we call a degenerate Robinson congruence. We note that this congruence can also be obtained by a sequence of operations on a plane wave solution, including conformal inversion and shifts of parameters into the complex plane. This sequence was brought to our attention by work of, and discussions with, I. Białinicky-Birula [7,8].
The main results of this chapter were published in [4].
We surmise that in the later part of chapter 4 there is a strong link with the work of E. Newman [9] but acknowledge that more research has to be done in this direction.

Chapter 5 was intended to study the said sequence of operations in more detail, but grew into a more general study of conformal transformations and in particular conformal inversion, after it was observed that not much was written about this elusive symmetry of vacuum Maxwell equations.
In the literature one encounters confusion about the interpretation of conformal transformations. Sometimes, without explicitly stating this, it is assumed that the transformation is just a change of coordinates, and that tensors transform accordingly, even when only Minkowski space is considered. This is incorrect, although results derived in this manner have been found to be valid. To avoid confusion we present a careful elementary exposition of the symmetry at hand, resulting in a prescription for obtaining new solutions from old ones. To the best of our knowledge this has not been done previously in this manner.
As a new result, a nice transformation formula for the Riemann-Silberstein vector under conformal inversion is derived and applied to some familiar fields. Among these examples is an electromagnetic Hopf knot, which, surprisingly, is transformed into another Hopf knot.

Chapter 6 discusses the Penrose transform, a contour integral formula that needs a twistor function as input and the evaluation of which gives a solution to a particular zero rest mass free field equation. It is shown what twistor function is needed in order to get an electromagnetic Hopf knot as a result of the contour integral. This twistor function, which was found by my fellow students J. Swearngin and A. Thompson, adapted slightly in order to meet the requirements, is then used in the Penrose transform to arrive at a solution to linearized Einstein equations. This solution, exhibiting the Hopf structure in different ways, is discussed.
The results are also discussed in [10,11].

Chapter 7 is a review of mathematical notions from differential geometry that are needed in the general theory of relativity, with an emphasis on the tetrad formalism. In the remaining chapters we lean strongly on this chapter.
The method of presentation is based on [12].

Chapter 8 presents two exact solutions to Einstein's equations. One contains a Robinson congruence, the other a degenerate Robinson congruence. Again, but different as in chapter 4, there is a relationship between the two via a complex shift. The solutions are arrived at via the method presented by Debney, Kerr and Schild [13], the relevant content of which is summarized and supplemented.
Although the solutions are not new, we do not known whether they have been published in this form. It is pointed out that there seems to be no physical interpretation attached to the curves of the congruence. Comparison with chapter 6, where the Robinson congruence appears as a linearized solution *with* interpretation suggests itself, but has not been carried out to completion.

Chapter 9 treats the electromagnetic Hopf knot in the Newman-Penrose formalism, neglecting the curvature of space-time due to the energy density of the knot. It leads to a suitable choice of tetrad fields adapted to the structure of the knot and an expression for the spin coefficients in terms of Minkowski coordinates. The knowledge of this description of the Hopf knot is needed in the next chapter, were we try to include gravity effects.
The chapter is based on many calculations that are presented in a much condensed form.

Chapter 10 presents an ambitious attempt to redo the calculation of chapter 9, but now taking into account the effect of the energy distribution on the curvature of space-time. This leads to a problem for which there is no obvious solution: in order to describe the source correctly we need the metric, but in order to calculate the metric we need to know the source. Ultimately this is related to the problem of interpretation of coordinates in general relativity. It is therefore allowed to take some freedom in the definition of the source, and here we show one attempt to do this in such a way that the relevant equations can be solved. This is "work in progress" and the assumptions that were chosen in simplifying the problem have not yet resulted in a solution although important insights have been obtained in how to address this challenging problem.