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9

Electromagnetic Hopf knot in tetrad formalism

In this chapter we search for a tetrad adapted to the field of an electromagnetic Hopf knot, such that as many of the spin coefficients as possible are vanishing or otherwise do have a simple expression. It mainly consists of calculations. Doing these by hand, one soon acquires an intuition of how to change the tetrad in order to achieve a desired expression for one of the coefficients. The choice presented here is only one of the nicer possibilities. The results of this chapter are needed in the next.

9.1 introduction

Among the many exact solutions of Einstein's equations with a source, almost all sources are unphysical. Only very few known exact solutions correspond to a realistic source, such as the Schwarzschild or Kerr solution. In the rest of this thesis we will be concerned with finding an exact solution due to a source that might be of physical relevance.

Parallel light beams have been considered as a source to Einstein's equations, leading to plane gravitational waves [89, 90]. Plane waves in general relativity played an important role in the acceptance of the existence of gravitational radiation.
Experimental physicists are currently thinking about the production in the laboratory of electromagnetic knots. Although the extend of a Hopf knot is infinite, and in this respect an experimental realization can only approximate

the full Hopf knot, in contrast to a plane electromagnetic wave the energy of an electromagnetic Hopf knot is finite and very much concentrated [1].

In chapter 10 we will try to find an exact solution of the combined system of Einstein and Maxwell equations with as sole source an electromagnetic Hopf knot. Starting with a source, a fluid, an electromagnetic field or anything else, considered in flat space-time, thus neglecting gravity, that is fully understood and specified in terms of familiar Minkowski coordinates $\{t, x, y, z\}$ and using this as input for the energy-momentum tensor in Einstein's equations in order to calculate the metric leads to a problem that is peculiar to general relativity: in order to calculate the metric we need the source, but to describe the source correctly, we must know the metric. In the problem at hand this peculiarity leads to a comfortable freedom in the definition of an electromagnetic Hopf knot in the context of curved space-time. In chapter 10 it will become clear how we can trim our sails according to the current wind.
Due to the complicated t, x, y, z dependence of the Robinson congruence and the fact that this congruence is null, geodesic and shear free, the Newman-Penrose formalism seems most suited for solving our problem. We therefore first need to know what a Hopf knot in flat space-time looks like in this formalism.

9.2 Hopf knot in flat space-time in Newman-Penrose formalism

A possible tetrad adapted to the congruence of Robinson consists of the following basis.

$$\boxed{\begin{aligned} &\mathbf{l} = \tfrac{1}{A}(S, \mathbf{S}) \\ &\mathbf{n} = A(\tfrac{1}{2S}, -\tfrac{\mathbf{S}}{2S^2}) \\ &\mathbf{m} = \tfrac{e^{i\theta}}{\sqrt{2S}}(0, \mathbf{F}) \\ &\bar{\mathbf{m}} = \tfrac{e^{-i\theta}}{\sqrt{2S}}(0, \mathbf{F}^*) \end{aligned}} \tag{9.1}$$

in which A and θ are unspecified real functions, and

$S = |\mathbf{S}| = |\mathbf{E} \times \mathbf{B}| = E^2 = \epsilon$ (energy density, for $\mathbf{E} \cdot \mathbf{B} = 0, |\mathbf{E}| = |\mathbf{B}|$)

$E = |\mathbf{E}|$

$\mathbf{F} = \mathbf{E} + i\mathbf{B}$

A nice property of this choice of tetrad is the direct relation between any of the vector fields contained in it with the electric, magnetic or Poynting vector of the electromagnetic field under consideration. We take

$$\mathbf{F} = \mathbf{F}^{RS}(t, \mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix} \tag{9.2}$$

which defines an electromagnetic Hopf knot (equation (4.1)). The first vector field from the tetrad therefore defines a Robinson congruence: we know from chapter 4 ((4.9), (4.10)) that $\mathbf{l}$ can be written as $\mathbf{l} = 2e^{\psi}e^{\psi^c}k^0(k^\alpha)$, with $(k^\alpha) = (k^0, \mathbf{k})$, the Robinson congruence associated with (4.2), and $|k^0| = k^0 = |\mathbf{k}|$.

Since we use the Minkowski metric, the following relations hold. These can be seen almost without effort from what we know from electromagnetic null fields.

$m^\mu m_\mu = 0 = \bar{m}^\mu \bar{m}_\mu$
$m^\mu \bar{m}_\mu = -1$
$l^\mu m_\mu = 0 = l^\mu \bar{m}_\mu$
$l^\mu l_\mu = 0 = n^\mu n_\mu$
$l^\mu n_\mu = 1$
$m^\mu n_\mu = 0 = \bar{m}^\mu n_\mu$

The tetrad thus qualifies as a Newman-Penrose tetrad (section **7.4**).

9.2.1 the spin coefficients

We now calculate the spin coefficients (section **7.4**), including the optical scalars, and use the fact that we are working in Minkowski space, so the Christoffel symbols vanish. First we derive general expressions for κ, σ, ρ with respect to (9.1), the boxed equations below. They can be formulated straightforwardly in terms of properties of the electromagnetic field considered. For example, κ is proportional to the projection along the Poynting vector of the directional derivative in the direction of the Poynting vector of the Rieman-Silberstein vector. The fields A and θ could be adjusted in such a way as to make this proportionality factor equal to 1. But we will need a different choice, the one that makes $\epsilon = 0$. As soon as we start using (9.2) the spin coefficients will receive the label Rob.

$\kappa = \gamma_{(3)(1)(1)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu;\nu} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}$

and thus (for the tetrad (9.1) defined above)

$\kappa = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot(-\frac{\mathbf{S}}{A})_{,\nu}\,\frac{S^\nu}{A}$

It follows that (remember $D = l^\mu\partial_\mu$)

$$\boxed{\kappa = \frac{-e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot D\frac{\mathbf{S}}{A} = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot D\mathbf{F}} \text{ since } \mathbf{F}\cdot\mathbf{S} = 0$$

$\sigma = \gamma_{(3)(1)(3)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta\frac{\mathbf{S}}{A}$, or

$$\boxed{\sigma = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot\delta\mathbf{F}}$$

$\rho = \gamma_{(3)(1)(4)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}$

The difference with respect to σ is that now we have a directional derivative $\bar{m}^\mu\partial_\mu$, thus

$$\boxed{\rho = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot\delta^*\mathbf{F}}$$

$\epsilon = \frac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}) = \frac{1}{2}(\mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} + \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu}) =$

$\frac{1}{2}\{\frac{1}{2S}DS + \frac{A}{2}D\frac{1}{A} + \frac{-A\mathbf{S}}{2S^2}\cdot D\frac{-\mathbf{S}}{A} + \frac{-1}{2S}\mathbf{F}\cdot D\mathbf{F}^* - \frac{e^{i\theta}}{2E}\mathbf{F}\cdot\mathbf{F}^* D\frac{e^{-i\theta}}{E}\} =$
$\frac{1}{2}\{\frac{1}{2S}DS - \frac{1}{2A}DA + \frac{1}{2S}DS - \frac{1}{2A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^* + \frac{1}{E}DE + iD\theta\} =$
$\frac{1}{2}\{\frac{1}{S}DS - \frac{1}{A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^* + \frac{1}{2S}DS + iD\theta$

Thus $\epsilon = \frac{1}{2}\{\frac{3}{2S}DS - \frac{1}{A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^* + iD\theta\}$

Although we did not calculate κ yet, we do know that the first tetrad vector field is aligned with a geodesic Robinson congruence. Therefore we expect κ to be zero (see section **7.5** for the interpretation of some of the spin coefficients). Further, it is always possible to use an affine parameter along the geodesics (section **7.2.4**), and then $Re(\epsilon)$=0 also. Note that here we actually are considering an affine parameter *field*. We will try to find A and θ such that $\epsilon = 0$.

To compute the above expressions concretely, we first need the derivatives of the Riemann-Silberstein vector

$$\mathbf{F} = \frac{1}{((t-i)^2-r^2)^3}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix}.$$

$$\partial_t\mathbf{F} = \frac{-6(t-i)}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{-2}{((t-i)^2-r^2)^3}\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix}$$

$$\partial_x\mathbf{F} = \frac{6x}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}(x-iy)\\ i(x-iy)\\ -(t-i-z)\end{pmatrix}$$

$$\partial_y\mathbf{F} = \frac{6y}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}-i(x-iy)\\ (x-iy)\\ i(t-i-z)\end{pmatrix}$$

$$\partial_z\mathbf{F} = \frac{6z}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix}$$

The first term in these four derivatives is proportional to $\mathbf{F}$ and is therefore not needed in the calculation of the first three spin coefficients, κ, σ, ρ, because $\mathbf{S}\cdot\mathbf{F} = 0$. Therefore:

$$\mathbf{S}\cdot\partial_t\mathbf{F} = \frac{-2}{((t-i)^2-r^2)^3}\frac{(1+x^2+y^2+(t-z)^2)}{((r^2-(t^2+1))^2+4r^2)^3}\begin{pmatrix}2(x(t-z)+y)\\ 2(y(t-z)-x)\\ x^2+y^2-(t-z)^2-1\end{pmatrix}\cdot\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix} =$$
$$\frac{-2}{((t-i)^2-r^2)^3}\frac{(1+x^2+y^2+(t-z)^2)^2}{((r^2-(t^2+1))^2+4r^2)^3}(x-iy) = \frac{-2(x-iy)}{((t-i)^2-r^2)^3}S$$

Similarly:

$$\mathbf{S}\cdot\partial_x\mathbf{F} = \frac{2(t-z-i)}{((t-i)^2-r^2)^3}S$$

$$\mathbf{S}\cdot\partial_y\mathbf{F} = \frac{-2i(t-z-i)}{((t-i)^2-r^2)^3}S$$

$$\mathbf{S}\cdot\partial_z\mathbf{F} = \frac{2(x-iy)}{((t-i)^2-r^2)^3}S$$

We now calculate the spin coefficients explicitly.

$\kappa^{Rob.} \propto l^\mu \mathbf{S}\cdot\partial_\mu\mathbf{F} \propto (1+x^2+y^2+(t-z)^2)\mathbf{S}\cdot\partial_t\mathbf{F} + 2(x(t-z)+y)\mathbf{S}\cdot\partial_x\mathbf{F} + 2(y(t-z)-x)\mathbf{S}\cdot\partial_y\mathbf{F} + (x^2+y^2-(t-z)^2-1)\mathbf{S}\cdot\partial_z\mathbf{F} = 0$

This expresses the known fact that the integral curves of the vector field $\mathbf{l}$ are geodesics.

$\sigma^{Rob.} \propto m^\mu \mathbf{S}\cdot\partial_\mu\mathbf{F} \propto ((x-iy)^2-(t-i-z)^2)\mathbf{S}\cdot\partial_x\mathbf{F} + (i(x-iy)^2+i(t-i-z)^2)\mathbf{S}\cdot\partial_y\mathbf{F} - 2(x-iy)(t-i-z)\mathbf{S}\cdot\partial_z\mathbf{F} = 0$

As expected, the Robinson congruence is shear-free.

Returning now to the expression for $\epsilon^{Rob.}$, the real part vanishes for A such that $\frac{3}{2S}DS - \frac{1}{A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^*$ is imaginary. From $DS = \frac{1}{2}(\mathbf{F}\cdot D\mathbf{F}^* + \mathbf{F}^*\cdot D\mathbf{F})$ we infer that $A = S$ is the desired choice. If in addition we want $Im(\epsilon)=0$, we have to solve the following equation for θ:

$D\theta = \frac{i}{4S}(\mathbf{F}^*\cdot D\mathbf{F} - \mathbf{F}\cdot D\mathbf{F}^*) \quad\Leftrightarrow\quad (\partial_t + \frac{\mathbf{S}}{S}\cdot\nabla)\theta = \frac{2}{1+x^2+y^2+(t-z)^2}.$

The right hand side of this equation, combined with our knowledge of the electromagnetic Hopf knot justify the "ansatz" $\theta(t,x,y,z) = \theta(x^2+y^2, t-z)$. We write $\rho \equiv x^2+y^2, \xi \equiv t-z$ to obtain:

$(1+\xi^2)\partial_\xi\theta + 2\rho\xi\partial_\rho\theta = 1$

For $\theta \neq \theta(\rho)$ the equation is simple to solve:

$\partial_\xi\theta = \frac{1}{1+\xi^2}$, with solution $\tan\theta = \xi$ or $\theta = \tan^{-1}(t-z)$.

With the choices $\boxed{A = S}$ and $\boxed{\theta = \tan^{-1}(t-z)}$ we now also have $\epsilon^{Rob.} = 0$.

Note that this choice leads to $e^{i\theta} = \frac{1+i(t-z)}{\sqrt{1+(t-z)^2}}$.

Obtaining explicit expressions for the other coefficients requires lengthy but simple calculations, that will be presented in compressed form:

$\rho^{Rob.} = \frac{e^{i\theta}}{\sqrt{2S}S}\bar{m}^\mu\mathbf{S}\cdot\partial_\mu\mathbf{F} =$
$\frac{1}{2S^2}\frac{1}{((t+i)^2-r^2)^3}\{((x+iy)^2-(t+i-z)^2)\mathbf{S}\cdot\partial_x\mathbf{F} + (-i(x+iy)^2-i(t+i-z)^2)\mathbf{S}\cdot\partial_y\mathbf{F} + (-2(x+iy)(t+i-z))\mathbf{S}\cdot\partial_z\mathbf{F}\} =$

$\frac{1}{2S^2}\frac{1}{((t+i)^2-r^2)^3}\frac{2}{((t-i)^2-r^2)^3}S\{((x+iy)^2-(t+i-z)^2)(t-z-i) + (-i(x+iy)^2-i(t+i-z)^2)(-i)(t-z-i) + (-2(x+iy)(t+i-z))(x-iy)\} =$

$\frac{1}{S}\frac{1}{((r^2-(t^2+1))^2+4r^2)^3}\{((x+iy)^2-(t+i-z)^2)(t-z-i)+(-i(x+iy)^2-i(t+i-z)^2)(-i)(t-z-i)+(-2(x+iy)(t+i-z))(x-iy)\}=$

$\frac{1}{S}\frac{-2(1+x^2+y^2+(t-z)^2)}{((r^2-(t^2+1))^2+4r^2)^3}(t-z+i)=-2\frac{t-z+i}{1+x^2+y^2+(t-z)^2}$

So that

$$\boxed{\Theta^{Rob.}=\frac{2(t-z)}{1+x^2+y^2+(t-z)^2}}$$

and

$$\boxed{\omega^{Rob.}=\frac{-2}{1+x^2+y^2+(t-z)^2}}$$

From this and the fact that there is no freedom left for a suitable tetrad transformation to achieve $\rho=\rho^*$ (see section **7.4.4**), it follows that the Robinson congruence is not hyper-surface orthogonal (i.e. not proportional to the gradient of a scalar function).

For the propagation equations, expressing the change along the geodesics of the congruence, we find

$D\omega^{Rob.}=\frac{8(t-z)}{(1+x^2+y^2+(t-z)^2)^2}$ and $D\Theta^{Rob.}=\frac{4(1-(t-z)^2)}{(1+x^2+y^2+(t-z)^2)^2}$

From equations like this ($D(...)=f(x^\mu)$), of which one more will be given later in this chapter, we could take advantage in the next chapter, were, depending on some choices we make, we have to solve these kind of equations for (...).

We continue with the remaining spin coefficients.

$\pi^{Rob.}=\gamma_{(2)(4)(1)}=\mathbf{e}_{(2)}{}^{\mu}\mathbf{e}_{(1)}{}^{\nu}\mathbf{e}_{(4)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot D(\frac{-e^{-i\theta}}{\sqrt{2}S}\mathbf{F}^*)$

This expression contains a term proportional to $\mathbf{S}\cdot D\mathbf{F}^*=(\mathbf{S}\cdot D\mathbf{F})^*\propto(\kappa^{Rob.})^*=0$, and a term proportional to $\mathbf{S}\cdot\mathbf{F}^*=0$. Therefore $\pi^{Rob.}=0$.

$\lambda^{Rob.}=\gamma_{(2)(4)(4)}=\mathbf{e}_{(2)}{}^{\mu}\mathbf{e}_{(4)}{}^{\nu}\mathbf{e}_{(4)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot\delta^*(\frac{-e^{-i\theta}}{\sqrt{2}S}\mathbf{F}^*)$

This expression contains a term proportional to $\mathbf{S}\cdot\delta^*\mathbf{F}^*=(\mathbf{S}\cdot\delta\mathbf{F})^*\propto(\sigma^{Rob.})^*=0$, and a term proportional to $\mathbf{S}\cdot\mathbf{F}^* = 0$. Therefore $\lambda^{Rob.} = 0$.

$\mu^{Rob.} = \gamma_{(2)(4)(3)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} =$

$-\frac{\mathbf{S}}{2S}\cdot\delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = \frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\delta\mathbf{F}^* = \frac{1}{2}(\frac{e^{i\theta}}{S\sqrt{2S}}\mathbf{S}\cdot\delta^*\mathbf{F})^* = \frac{1}{2}(\rho^{Rob.})^* = \frac{-(t-z)+i}{1+x^2+y^2+(t-z)^2}$

$\nu^{Rob.} = \gamma_{(2)(4)(2)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\Delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = \frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\Delta\mathbf{F}^*$

Note that $\Delta = \partial_t - \frac{1}{2}D$ and $\mathbf{S}\cdot D\mathbf{F}^* \propto \kappa^* = 0$, thus

$\nu^{Rob.} = \frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\partial_t\mathbf{F}^* = \frac{-e^{-i\theta}(x+iy)}{\sqrt{2S}((t+i)^2-r^2)^3}$

$\tau^{Rob.} = \gamma_{(3)(1)(2)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\Delta(-\frac{\mathbf{S}}{S}) = \frac{e^{i\theta}}{\sqrt{2S}}\frac{1}{S}\mathbf{S}\cdot\Delta\mathbf{F} = 2(\nu^{Rob.})^*$

$\alpha^{Rob.} = \frac{1}{2}(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)})$ and

$\gamma_{(2)(1)(4)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\delta^*(-\frac{\mathbf{S}}{S}) = \frac{1}{4}\delta^*\frac{\mathbf{S}\cdot\mathbf{S}}{S^2} = 0,$

$\gamma_{(3)(4)(4)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta^*(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = -\frac{1}{2S}\mathbf{F}\cdot\delta^*\mathbf{F}^* - e^{i\theta}\sqrt{2S}\delta^*\frac{e^{-i\theta}}{\sqrt{2S}} =$

$-\frac{1}{2S}\mathbf{F}\cdot\delta^*\mathbf{F}^* + i\delta^*\theta + \frac{1}{2S}\delta^*S = \frac{1}{4S}(\mathbf{F}^*\cdot\delta^*\mathbf{F} - \mathbf{F}\cdot\delta^*\mathbf{F}^*) + i\delta^*\theta$

We have $\delta\theta = \frac{e^{i\theta}}{\sqrt{2S}}\frac{-2}{((t-i)^2-r^2)^3}(x-iy)(t-i-z)\partial_z\theta = -\tau^{Rob.}(t-i-z)\partial_{t-z}\theta = \frac{-\tau^{Rob.}(t-i-z)}{1+(t-z)^2}$,

therefore $i\delta^*\theta = \frac{-2i\nu^{Rob.}(t-z+i)}{1+(t-z)^2}$

The other terms turn out to be

$\frac{-1}{4S}\mathbf{F}\cdot\delta^*\mathbf{F}^* = \frac{e^{-i\theta}(x+iy)}{\sqrt{2S}((t+i)^2-r^2)^3} = -\nu^{Rob.}$ and $\frac{1}{4S}\mathbf{F}^*\cdot\delta^*\mathbf{F} = 3\frac{(t+i)^2-r^2}{(t-i)^2-r^2}\nu^{Rob.}$

These terms sum to $\alpha^{Rob.} = \frac{1}{2}[3\frac{(t+i)^2-r^2}{(t-i)^2-r^2} - \frac{(t-z)+i}{(t-z)-i}]\nu^{Rob.}$

$\beta^{Rob.} = \frac{1}{2}(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)})$ and

$\gamma_{(2)(1)(3)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\delta(-\frac{\mathbf{S}}{S}) = \frac{1}{4}\delta\frac{\mathbf{S}\cdot\mathbf{S}}{S^2} = 0,$

$\gamma_{(3)(4)(3)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = -\frac{1}{2S}\mathbf{F}\cdot\delta\mathbf{F}^* - e^{i\theta}\sqrt{2S}\delta\frac{e^{-i\theta}}{\sqrt{2S}} =$

$\frac{1}{4S}(\mathbf{F}^*\cdot\delta\mathbf{F} - \mathbf{F}\cdot\delta\mathbf{F}^*) + i\delta\theta = -2(\alpha^{Rob.})^*$ and this implies:

$\beta^{Rob.} = -(\alpha^{Rob.})^*$

$\gamma^{Rob.} = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)})$ and

$\gamma_{(2)(1)(2)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\Delta(-\frac{\mathbf{S}}{S}) = \frac{1}{4}\Delta\frac{\mathbf{S}\cdot\mathbf{S}}{S^2} = 0,$

$\gamma_{(3)(4)(2)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\Delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = -\frac{1}{2S}\mathbf{F}\cdot\Delta\mathbf{F}^* - e^{i\theta}\sqrt{2S}\Delta\frac{e^{-i\theta}}{\sqrt{2S}} =$

$\frac{1}{4S}(\mathbf{F}^*\cdot\Delta\mathbf{F} - \mathbf{F}\cdot\Delta\mathbf{F}^*) + i\Delta\theta$

Now use $\Delta = \partial_t - \frac{1}{2}D$ and $D\theta = \frac{i}{4S}(\mathbf{F}^*\cdot D\mathbf{F} - \mathbf{F}\cdot D\mathbf{F}^*)$ to get

$\gamma^{Rob.} = \frac{1}{8S}(\mathbf{F}^*\cdot\partial_t\mathbf{F} - \mathbf{F}\cdot\partial_t\mathbf{F}^*) + \frac{i}{2}\partial_t\theta = \frac{i}{1+x^2+y^2+(t-z)^2} - \frac{3i(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{1}{2}\frac{i}{1+(t-z)^2}$

We could try to express γ in terms of the other spin coefficients with the help of relations like

$\nu\tau = \frac{x^2+y^2}{(1+x^2+y^2+(t-z)^2)^2}, \quad \mu\mu^* = \frac{1+(t-z)^2}{(1+x^2+y^2+(t-z)^2)^2}$, or

$1+(t-z)^2 = \frac{\mu\mu^*}{(\nu\tau+\mu\mu^*)^2}, \quad x^2+y^2 = \frac{\nu\tau}{(\nu\tau+\mu\mu^*)^2},$

supplemented with formulas that contain α and β, but it is more important for the needs of the following chapter to note that $\gamma^{Rob.} \in \mathbb{I}$. Chapter 10 starts with a summary of the results obtained here.

9.2.2 the electromagnetic field

We now calculate the tetrad representation for the electromagnetic field tensor that corresponds to this particular electromagnetic Hopf knot. For this we need the following relations

$$(F_{\mu\nu}^{(asd)}) = \begin{pmatrix} 0 & F_x & F_y & F_z \\ -F_x & 0 & iF_z & -iF_y \\ -F_y & -iF_z & 0 & iF_x \\ -F_z & iF_y & -iF_x & 0 \end{pmatrix}$$

$$(F_{\mu\nu}^{(sd)}) = \begin{pmatrix} 0 & F_x^* & F_y^* & F_z^* \\ -F_x^* & 0 & -iF_z^* & iF_y^* \\ -F_y^* & iF_z^* & 0 & -iF_x^* \\ -F_z^* & -iF_y^* & iF_x^* & 0 \end{pmatrix}$$

$$(F_{\mu\nu}) = \tfrac{1}{2}(F_{\mu\nu}^{(asd)} + F_{\mu\nu}^{(sd)}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \text{ thus}$$

$$F_{ij} = \tfrac{i}{2}\epsilon_{ijk}(F^k - F^{*k}), \;\; (F^k) = (F_x, F_y, F_z) = (-F_k).$$

Using $\mathbf{F}\cdot\mathbf{F} = 0 = \mathbf{F}\cdot\mathbf{S}$, $\mathbf{F}\cdot\mathbf{F}^* = 2S$, we find for the Faraday tensor:

$\phi_0 = F_{(1)(3)} = F_{\mu\nu}l^\mu m^\nu =$

$F_{0j}\frac{e^{i\theta}}{\sqrt{2S}}F^j + F_{ij}\frac{S^i}{S}\frac{e^{i\theta}}{\sqrt{2S}}F^j = -\frac{1}{2}(F_j + F_j^*)F^j\frac{e^{i\theta}}{\sqrt{2S}} + \frac{i}{2}\epsilon_{ijk}(F^k - F^{*k})\frac{S^i}{S}\frac{e^{i\theta}}{\sqrt{2S}}F^j =$

$\frac{1}{2}(\mathbf{F}\cdot\mathbf{F} + \mathbf{F}^*\cdot\mathbf{F})\frac{e^{i\theta}}{\sqrt{2S}} + \frac{ie^{i\theta}}{2S\sqrt{2S}}\epsilon_{ijk}(F^k - F^{*k})S^iF^j = \frac{e^{i\theta}\sqrt{S}}{\sqrt{2}} + \frac{ie^{i\theta}}{2S\sqrt{2S}}(\mathbf{S}\cdot(\mathbf{F}\times\mathbf{F}) + \mathbf{S}\cdot(\mathbf{F}^*\times\mathbf{F})) =$

$\frac{e^{i\theta}\sqrt{S}}{\sqrt{2}} - \frac{e^{i\theta}}{S\sqrt{2S}}\mathbf{S}\cdot\mathbf{S} = 0.$

$\phi_1 = \frac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) =$

$-\frac{1}{4}F_{0j}\frac{S^j}{S} + \frac{1}{4}F_{i0}\frac{S^i}{S} - \frac{1}{2}F_{ij}\frac{S^i}{S}\frac{S^j}{S} + \frac{1}{2}F_{ij}\frac{1}{2S}F^{*i}F^j = -\frac{1}{2}F_{0j}\frac{S^j}{S} + \frac{i}{4}\frac{1}{2S}\epsilon_{ijk}(F^k - F^{*k})F^{*i}F^j = 0.$

$\phi_2 = F_{(4)(2)} = F_{\mu\nu}\bar{m}^\mu n^\nu =$

$\frac{e^{-i\theta}}{2\sqrt{2S}}F_{i0}F^{*i} - \frac{e^{-i\theta}}{2S\sqrt{2S}}F_{ij}F^{*i}S^j = -\frac{e^{-i\theta}}{4\sqrt{2S}}\mathbf{F}\cdot\mathbf{F}^* - \frac{ie^{-i\theta}}{4S\sqrt{2S}}\epsilon_{ijk}F^kF^{*i}S^j =$

$-\frac{e^{-i\theta}\sqrt{S}}{2\sqrt{2}} - \frac{e^{-i\theta}}{2S\sqrt{2S}}\mathbf{S}\cdot\mathbf{S} = -e^{-i\theta}\sqrt{\frac{S}{2}},$

so:

$$\boxed{\phi_2 = -\frac{e^{-i\theta}}{\sqrt{2}} \frac{1 + x^2 + y^2 + (t-z)^2}{((t^2 - r^2)^2 + 1 + 2(t^2 + r^2))^{\frac{3}{2}}}} \text{ and } \boxed{\phi_0 = 0 = \phi_1}.$$

This immediately leads to the energy-momentum tensor for the electromagnetic field in tetrad terms:

the only nonzero component is

$T_{(2)(2)} = -2\phi_2\phi_2^* = -S = -\frac{(1+x^2+y^2+(t-z)^2)^2}{((t^2-r^2)^2+1+2(t^2+r^2)^2)^3}.$

9.2.3 miscellanea

If this tensor acts as the only source for Einstein's equations we arrive at a space-time in which all Ricci coefficients except one vanish:

$R_{(2)(2)} = -2S$, or $\quad \Phi_{22} = S = \frac{(1+x^2+y^2+(t-z)^2)^2}{((t^2-r^2)^2+1+2(t^2+r^2)^2)^3}$

Looking back at the Maxwell equations in Newman-Penrose formalism (section **7.4.3**), we see that for the present electromagnetic field they read:

$\kappa\phi_2 = 0$, which is satisfied trivially because $\kappa = 0$
$D\phi_2 = \rho\phi_2$
$\sigma\phi_2 = 0$, which is satisfied because $\sigma = 0$
$\delta\phi_2 = (\tau - 2\beta)\phi_2$.

From the second equation we infer that $DS = -4S\frac{t-z}{1+x^2+y^2+(t-z)^2}$, as can be checked by direct calculation. This, like the explicit propagation equations for the expansion and rotation given above, could be of use when solving the combined system of Maxwell and Einstein equations with an electromagnetic Hopf knot as source. It is not difficult to add to this list more equations of the same form or with δ substituted for D, etc. However, since in the example given in the next chapter we do not use any of these, they will not be presented here. Other choices than those of chapter 10 have been considered in which this list is of importance.

9.2.4 another choice for the tetrad

We had good physical reasons to consider the tetrad in (9.1). This choice led to reasonable expressions for the spin coefficients, five of which turned out to vanish. However, for the sake of the problem posed in chapter 10, it can be advantageous to have an alternative at hand. Therefore, here we consider an alternative choice for the tetrad that will have an additional vanishing spin coefficient.

We perform a type 1 tetrad transformation (section **7.4.4**) to the tetrad (9.1) such that $\tau \to 0$. Since, with $\kappa = 0 = \sigma$, for type 1 transformations $\tau \to \tau + a\rho$, this will be achieved for

$$a = -\frac{\tau}{\rho} = -\frac{i}{\sqrt{2}} \frac{x-iy}{\sqrt{1+(t-z)^2}} \frac{(t-z-i)((t+i)^2-r^2)^{\frac{3}{2}}}{(t-z+i)((t-i)^2-r^2)^{\frac{3}{2}}},$$

thus $aa^* = \frac{1}{2}\frac{x^2+y^2}{1+(t-z)^2}$.

From equation (7.4) we find for the new tetrad:

$$\boxed{\begin{aligned} &\mathbf{l} = (1, \tfrac{\mathbf{S}}{S}) \\ &\mathbf{n} = \tfrac{1}{2}\tfrac{1+x^2+y^2+(t-z)^2}{1+(t-z)^2}(1,0,0,1) \\ &\mathbf{m} = a(1,0,0,1) + a\tfrac{t-z-i}{x-iy}(0,1,-i,0) \\ &\bar{\mathbf{m}} = a^*(1,0,0,1) + a^*\tfrac{t-z+i}{x+iy}(0,1,i,0) \end{aligned}} \tag{9.3}$$

Note that $\mathbf{n} = (\frac{1}{2} + aa^*)(1,0,0,1)$.

The spin coefficients can be calculated as before, or, much simpler, from the previous ones by the transformation formula in section **7.4.4**. For the latter strategy, use can be made of equations that are given in section **10.3**. This leads to the following expressions for the spin coefficients:

$$\sigma = \kappa = \tau = \epsilon = \pi = \lambda = 0,$$

$$\gamma = \frac{3i}{1+(t-z)^2} - \frac{6i(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{6i(x^2+y^2)}{1+(t-z)^2}\frac{t^2-r^2-1}{(t^2-r^2)^2+1+2(t^2+r^2)},$$

$$\rho = -2\frac{t-z+i}{1+x^2+y^2+(t-z)^2},$$

$$\alpha = \frac{6ia^*}{(t-i)^2-r^2},$$

$\beta = \frac{6ia}{(t+i)^2-r^2} - \frac{2a(t-z-i)}{1+x^2+y^2+(t-z)^2}$,

$\mu = \frac{t-z-i}{1+(t-z)^2}(((t-i)^2 - r^2)^3 - 1)$,

$\nu = \frac{2ia^*}{1+x^2+y^2+(t-z)^2} + \frac{ia^*}{1+(t-z)^2} - \frac{6ia^*(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{a^*(t-z-i)\{((t-i)^2-r^2)^3-1\}}{1+(t-z)^2}$

Part of the last expression comes from $\Delta a^* = \partial_t a^*$, which is valid since $Da^* = 0$. Note that Δ here refers to the original tetrad (9.1), and not (9.3). Again we have a simple relation between α and β: $\beta = -\alpha^* + a\rho^*$.

A type 1 tetrad transformation does not change the representation of an electromagnetic field for which $\phi_0 = 0 = \phi_1$, so with tetrad (9.3) we still have

$\phi_0 = 0 = \phi_1,\ \phi_2 = -\frac{1-i(t-z)}{\sqrt{2(1+(t-z)^2)}} \frac{1+x^2+y^2+(t-z)^2}{((t^2-r^2)^2+1+2(t^2+r^2))^{\frac{3}{2}}}$.