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The Robinson congruence in general relativity

8.1 introduction

In previous chapters the Robinson congruence or its projection on a time-slice, the Hopf fibration, was seen to correspond to a non-null twistor, an exact solution to source-free Maxwell equations and a solution to linearized Einstein equations. The purpose of this chapter is to show that it is also related to an exact solution in general relativity. To this end we summarize and, to suit our purposes, supplement an article by Debney, Kerr and Schild [13] in which a formalism was established that can be considered as a solution generating technique "avant-la-lettre".

Consider a metric that can be cast in so called Kerr-Schild form:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h\mathbf{e}_\mu \mathbf{e}_\nu,$$

in which η is the metric of Minkowski space (M^4 with usual coordinates {t,x,y,z} that will also be used (with different interpretation of course) as coordinates in the curved manifold), $h = h(x^\mu)$ and $\mathbf{e}$ is a null vector:

$$g^{\mu\nu}\mathbf{e}_\nu \mathbf{e}_\mu = \mathbf{e}^\mu \mathbf{e}_\mu = 0.$$

The contravariant form of the metric tensor can easily be seen to be

$g^{\mu\nu} = \eta^{\mu\nu} - 2h\mathbf{e}^{\mu}\mathbf{e}^{\nu}.$

It follows that $\eta^{\mu\nu}\mathbf{e}_{\mu}\mathbf{e}_{\nu} = 0$: $\mathbf{e}$ is also a null vector with respect to "auxiliary" Minkowski space. Raising and lowering of indices of vectors that are orthogonal to $\mathbf{e}$, including $\mathbf{e}$ itself, can be done with η.

In terms of the null coordinates $\{u, v, w, \bar{w}\}$:

$u \equiv \frac{1}{\sqrt{2}}(t+z)$

$v \equiv \frac{1}{\sqrt{2}}(t-z)$

$w \equiv \frac{1}{\sqrt{2}}(x+iy)$

$\bar{w} = \frac{1}{\sqrt{2}}(x-iy)$

we can write a general (covariant) vector with unit coefficient in front of du as

$\mathbf{e} = du + Adv + \bar{Y}dw + Yd\bar{w}$

(the case of vanishing coefficient in front of du can be treated by a suitable limiting process, but will be of no concern here)
and the Minkowski metric tensor as

$$(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The condition for $\mathbf{e}$ to be null with respect to η now gives $A = Y\bar{Y}$:

$\mathbf{e} = du + Y\bar{Y}dv + \bar{Y}dw + Yd\bar{w}, \quad \text{or}$

$$(\mathbf{e}_{\mu}) = \begin{pmatrix} 1 \\ Y\bar{Y} \\ \bar{Y} \\ Y \end{pmatrix} \quad \text{and} \quad (\mathbf{e}^{\mu}) = (\eta^{\mu\nu}\mathbf{e}_{\nu}) = \begin{pmatrix} Y\bar{Y} \\ 1 \\ -Y \\ -\bar{Y} \end{pmatrix}.$$

The line element in null coordinates is $ds^2 = 2dudv - 2dwd\bar{w} + 2h(du + Y\bar{Y}dv + \bar{Y}dw + Yd\bar{w})^2$, and therefore

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + 2h \begin{pmatrix} 1 & Y\bar{Y} & \bar{Y} & Y \\ Y\bar{Y} & (Y\bar{Y})^2 & Y\bar{Y}^2 & Y^2\bar{Y} \\ \bar{Y} & Y\bar{Y}^2 & \bar{Y}^2 & Y\bar{Y} \\ Y & Y^2\bar{Y} & Y\bar{Y} & Y^2 \end{pmatrix}.$$

The inverse can be calculated most directly from $g^{\mu\nu} = \eta^{\mu\nu} - 2h\mathbf{e}^\mu\mathbf{e}^\nu$:

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} - 2h \begin{pmatrix} (Y\bar{Y})^2 & Y\bar{Y} & -Y^2\bar{Y} & -Y\bar{Y}^2 \\ Y\bar{Y} & 1 & -Y & -\bar{Y} \\ -Y^2\bar{Y} & -Y & Y^2 & Y\bar{Y} \\ -Y\bar{Y}^2 & -\bar{Y} & Y\bar{Y} & \bar{Y}^2 \end{pmatrix}.$$

Using the null vector $\mathbf{e}$, we can form a Newman-Penrose tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, $\mathbf{l} = (l^\mu) \equiv (\mathbf{e}^\mu)$, $\mathbf{n} = (n^\mu)$, $\mathbf{m} = (m^\mu)$, $\bar{\mathbf{m}} = (\bar{m}^\mu)$, with, in null coordinates

$$(l^\mu) = \begin{pmatrix} Y\bar{Y} \\ 1 \\ -Y \\ -\bar{Y} \end{pmatrix}, \quad (n^\mu) \equiv \begin{pmatrix} 1 - hY\bar{Y} \\ -h \\ hY \\ h\bar{Y} \end{pmatrix}, \quad (m^\mu) \equiv \begin{pmatrix} -Y \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\bar{m}^\mu) \equiv \begin{pmatrix} -\bar{Y} \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Note that in these coordinates $\bar{m}^\mu$ is not simply the complex conjugate of m^μ, as would be the case in real coordinates. In terms of the more familiar $\{t, x, y, z\}$ these definitions would be more complicated. We will need only $\mathbf{e}$ in these coordinates:

$$(\mathbf{e}^\mu) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + Y\bar{Y} \\ -(Y + \bar{Y}) \\ i(Y - \bar{Y}) \\ -1 + Y\bar{Y} \end{pmatrix} \tag{8.1}$$

When the spinor field $(\Pi^A) = (\Pi^1, \Pi^2)$ corresponds to the direction of $\mathbf{e}$, considered as a vector in auxiliary Minkowski space, M^4, we find from $\mathbf{e}^\mu \propto \sigma^\mu{}_{AX'}\Pi^A\bar{\Pi}^{X'}$ and equations (8.1), (1.4) that

$$Y = -\frac{\Pi^1}{\Pi^2} = \frac{\Pi_2}{\Pi_1}. \tag{8.2}$$

8.2 Debney, Kerr and Schild

The result of the article by Debney, Kerr and Schild that is of interest here can be summarized as follows.

For arbitrary (complex) analytic functions Φ and Ψ, constant $q \in \mathbb{C}$ and constants $p, c, m \in \mathbb{R}$ we generate a solution of Einstein-Maxwell equations in terms of null coordinates $\{u, v, w, \bar{w}\}$:

$F \equiv \Phi(Y) + (qY + c)(w + Yv) - (pY + \bar{q})(u + Y\bar{w})$

generates, via $F \equiv 0$:

<u>e.m. field:</u>

$\phi_0 = 0, \quad \phi_1 = \frac{1}{2}\Psi(Y)P^{-2}\rho^2, \quad \phi_2 = -\frac{1}{2}(\partial_Y \Psi)(\partial_Y F)^{-2} + \frac{1}{2}\Psi(Y)(\partial_Y F)^{-3}\partial_Y \partial_Y F$

<u>metric:</u>

$g_{\mu\nu} = \eta_{\mu\nu} + 2h\mathbf{e}_\mu \mathbf{e}_\nu$

where

$P \equiv pY\bar{Y} + qY + \bar{q}\bar{Y} + c$

$\rho \equiv -P(\partial_Y F)^{-1}$

$h \equiv \frac{1}{2}mP^{-3}(\rho + \bar{\rho}) - \frac{1}{2}\Psi\bar{\Psi}P^{-4}\rho\bar{\rho}$

$\mathbf{e} \equiv du + Y\bar{Y}dv + Yd\bar{w} + \bar{Y}dw$ is null geodesic and shear-free

ϕ_0, ϕ_1, ϕ_2 and ρ are defined as in the Newman-Penrose formalism with $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ as in section **8.1**.

Comparison of this function F with equations (8.2) and (1.10), which is also valid in curved spaces of Kerr-Schild form, explains one part of the working of this prescription to generate solutions to the combined system of Maxwell and Einstein equations, the fact that $\mathbf{e}$ is geodesic and shear-free.

The defining equations of ϕ_0, ϕ_1 and ϕ_2 in terms of the tetrad components of the Faraday tensor in section **7.4.3** can be used to obtain the inverse relations, expressing the tetrad components in terms of the ϕ's: $F_{(1)(3)} = -F_{(3)(1)} = F^*_{(1)(4)} = \phi_o$, $F_{(1)(2)} = -F_{(2)(1)} = \phi_1 + \phi_1^*$, etc. Equation (7.2) can now be used to obtain the $F^{(k)(l)}$'s. From these, the calculation of $F^{(k)(l)}F_{(k)(l)}$ leads to $4(\phi_0\phi_2 + \phi_0^*\phi_2^*) - 4(\phi_1^2 + \phi_1^{*2})$. This shows that the formalism of Debney, Kerr and Schild does not lead to null electromagnetic fields, since, from section **3.3**, $2(\mathbf{E}^2 - \mathbf{B}^2) = F^{\mu\nu}F_{\mu\nu} = F^{(k)(l)}F_{(k)(l)} = 0$ only for $\Psi(Y) = 0$: when an electromagnetic field is present, at least one of the requirements for this field to be null is not satisfied.

8.3 examples

In [13] the choice $\Phi = iaY,\ q = 0$ and $p = \frac{1}{\sqrt{2}} = c$ was shown to lead to the now famous Kerr metric for $\Psi = 0$ and to its electrically charged version, the Kerr-Newman metric, for $\Psi = e$. Since the angular momentum of these solutions is proportional to a it follows that we arrive at the Schwarzschild and the charged Schwarzschild or Reissner-Nordström solutions by putting $a = 0$ in the choices from above. Here we are interested in other choices of the parameters. However, for comparison we first give the results of the present procedure for the Schwarzschild and Kerr solutions.

Schwarzschild solution, $\Phi = 0 = \Psi,\ q = 0,\ p = \frac{1}{\sqrt{2}} = c$:

$$\rho = \sqrt{2}\frac{z+r}{x^2+y^2},\ h = \frac{m}{2}\frac{(x^2+y^2)^2}{(r+z)^2 r^3},\ r^2 \equiv x^2+y^2+z^2$$

$$(\mathbf{e}^\mu) = \frac{\sqrt{2}(r+z)}{x^2+y^2}\begin{pmatrix} r \\ x \\ y \\ z \end{pmatrix}$$

Note that the combination $h\mathbf{e}_\mu\mathbf{e}_\nu$ that appears in the metric does not contain the common factor in $(\mathbf{e}^\mu)$, in agreement with the sperical symmetry of the solution. The form of the expansion, -$Re(\rho) = \rho$ here, seems to contradict this symmetry. However, the spin coefficients depend on the chosen tetrad. A type 3) transformation with A=$\sqrt{2}\frac{z+r}{r(x^2+y^2)}$ leads to a manifest spherical symmetric expansion.

Kerr solution, $\Phi = iaY,\ \Psi = 0,\ q = 0,\ p = \frac{1}{\sqrt{2}} = c$:

$$\rho = \sqrt{2}\frac{z+\tilde{r}}{x^2+y^2}\frac{\tilde{r}^2}{\tilde{r}^2+iaz},\ h = \frac{m}{2}\frac{(x^2+y^2)^2}{(\tilde{r}+z)^2}\frac{\tilde{r}}{\tilde{r}^4+a^2z^2}\ , \tilde{r}^4 + (a^2 - x^2 - y^2 - z^2)\tilde{r}^2 - a^2z^2 = 0$$

$$(\mathbf{e}^\mu) = \frac{\sqrt{2}(\tilde{r}+z)}{x^2+y^2}\begin{pmatrix} \tilde{r} \\ \tilde{r}\frac{x\tilde{r}+ay}{\tilde{r}^2+a^2} \\ \tilde{r}\frac{y\tilde{r}-ax}{\tilde{r}^2+a^2} \\ z \end{pmatrix}$$

Note the difference between $\tilde{r}$ (an ellipsoidal radial coordinate) and r. In the limit $a \to 0$ we have $\tilde{r} \to r$, resulting in the Schwarzschild solution. Apart

from the flat space-time part, the metric now contains the factor $\frac{2m\tilde{r}^3}{\tilde{r}^4+a^2z^2}$, shared by all coefficients. It it clear that the angular momentum is directed in the z-direction.
The coordinate transformation $u = t + \tilde{r},\ e^{i\phi}\sin\theta = (x+iy)(\tilde{r}-ia)^{-1},\ z = \tilde{r}\cos\theta$ leads to the form of the Kerr metric as it appeared for the first time in the literature [80]:

$$ds^2(= g_{\mu\nu}dx^\mu dx^\nu) = [1 - \frac{2m\tilde{r}}{\tilde{r}^2+a^2\cos^2\theta}](du + a\sin^2\theta d\phi)^2 - 2(du + a\sin^2\theta d\phi)(d\tilde{r} + a\sin^2\theta d\phi) - (\tilde{r}^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2).$$

The limit $a \to 0$ of this expression gives the Schwarzschild metric in the familiar advanced Eddington-Finkelstein coordinates. For $\tilde{r} = 0$ and $\theta = \frac{\pi}{2}$ there is a curvature singularity (not just a coordinate singularity) that has the geometry of a ring *in auxiliary Minkowski space*.

8.3.1 Robinson congruence

$\Phi = \frac{i}{\sqrt{2}}Y,\ \ \Psi = 0,\ \ p = 0 = q,\ \ c = 1$

These choices lead to $P = 1,\ \ F = (\frac{i}{\sqrt{2}} + v)Y + w,$ thus $F \equiv 0 \iff Y = \frac{-\sqrt{2}w}{i+\sqrt{2}v}$.

In terms of the auxiliary coordinates $\{t, x, y, z\}$ we have:

$Y = \frac{x+iy}{-t+z-i},\quad \rho = \frac{\sqrt{2}}{1+(t-z)^2}(-t+z+i),\quad h = m\sqrt{2}(\frac{-t+z}{1+(t-z)^2})$ and

$$(\mathrm{e}^\mu) = \frac{1}{\sqrt{2}(1+(t-z)^2)}\begin{pmatrix} 1+x^2+y^2+(t-z)^2 \\ 2(x(t-z)+y) \\ 2(y(t-z)-x) \\ x^2+y^2-1-(t-z)^2 \end{pmatrix}. \tag{8.3}$$

We recognise the Robinson congruence or Hopf fibration in this form, see equations (4.2), (4.9) and (4.10). Alternatively, the corresponding spinor field (Π_1, Π_2) with $\frac{\Pi_2}{\Pi_1} = Y$ is $(\Pi_1, \Pi_2) = f(x^\mu)(-t+z-i, x+iy)$ which is exactly equation (4.6), which led to the Robinson congruence.

In all the previous appearances of the Hopf fibration we were able to give a physical interpretation of the integral curves of the vector field: as electric or magnetic field lines, lines related to energy flow ((super-)Poynting vector),

tendex or vortex lines. But now the vector field appears twice in the metric without obvious interpretation. In addition, we do not have a clear and global distinction between time and space here, as was the case in past examples, where fields could be considered to be defined on flat space-time. The vector field in (8.3) is still to be considered as a Robinson congruence, but a projection on a time-slice in order to arrive at the structure of a Hopf fibration cannot be done, except in auxiliary space.
Of course, the congruence is built from null geodesics and these are possible photon paths or light rays. But in every point there are possible photon paths in all directions.

If we change Ψ from zero to constant e, the difference appears only in the expressions for h and the electromagnetic field. There is an extra term in h due to the contribution of the electromagnetic energy density to the curvature of space-time: $\frac{-e^2}{1+(z-t)^2}$. The electromagnetic field is represented as $\phi_0 = 0 = \phi_2$, $\phi_1 = e\frac{(z-t+i)^2}{(1+(z-t)^2)^2}$. This expression is not very illuminating, but again a Robinson congruence pops up, since the corresponding electromagnetic field can also be obtained from a one form α: $F_{\mu\nu}dx^\mu \wedge dx^\nu = d\alpha$, in which

$\alpha = e\frac{2\sqrt{2}(z-t)}{1+(z-t)^2}\mathbf{e}$.

8.3.2 degenerate Robinson congruence

$\Phi = 0 = \Psi, \;\; p = 0 = q, \;\; c = 1$

This leads to $Y = \frac{x+iy}{z-t}$, $\rho = \frac{\sqrt{2}}{z-t}$, $h = \frac{m\sqrt{2}}{z-t}$ and

$$(\mathbf{e}^\mu) = \frac{1}{\sqrt{2}(t-z)^2}\begin{pmatrix} x^2+y^2+(t-z)^2 \\ 2x(t-z) \\ 2y(t-z) \\ x^2+y^2-(t-z)^2 \end{pmatrix},$$

in which we recognise a degenerate Robinson congruence, see (4.22).

Putting $\Psi = e$, a constant, leads to an additional term in h: $-\frac{e^2}{(z-t)^2}$, and an electromagnetic field that can be represented as in example (**8.3.1**), with $\alpha = e\frac{2\sqrt{2}}{z-t}\mathbf{e}$.

8.4 conclusion and final remarks

In this chapter we used a method developed by Debney, Kerr and Schild to show that the Robinson congruence also appears in exact solutions of the full Einstein equations. However, within the confines of classical general relativity it is fair to say that in contrast with previous cases we do not know whether a physical interpretation is possible. An attempt at an interpretation along the lines of [35, 36, 81, 82] could be worthwhile.
An interesting possibility suggests itself when comparing the solution of section **8.3.1** with the solution in chapter 6. Could the latter be the linearized version of the former? If yes, can the interpretation of the curves of the Robinson congruence in the solution of the linearized theory in some sense be taken over to the exact solution in the full theory?

It has not yet been investigated whether the solution in section **8.3.1** belongs to the class of solutions for which no general solution is known [83] or perhaps to the class of which only a few solutions are known [84], or neither of these possibilities.

It may further be noted that from [85] we may conclude the Petrov type-D character of at least the vacuum solutions in sections **8.3.1** and **8.3.2**, as well as the fact that these solutions must contain singularities not confined to a bounded region. This makes it unlikely that they will bring any changes to the observation in [86]: 'But the hope of finding metrics amongst the solutions [. . .] which describe the radiation field of a physically meaningful matter distribution has not been realized.'

As in chapter 4, there is a complex shift related to the transition degenerate Robinson congruence → Robinson congruence. The relation between the two shifts has not been investigated. Again, there is a possible connection with work done by E. Newman [31, 33].

Finally, it is of interest to note that in the case of rotation-free Kerr-Schild metrics there is a geometrical interpretation for (e^{μ}) and other quantities appearing in the present procedure [87] (also described in [88]). For this interpretation to work for the degenerate Robinson congruence (rotation-free, since ρ is a real function) we need the unphysical assumption of a massive particle travelling along the z-axis with the speed of light in order to have the correct retarded distance [88]. In addition, other quantities do not fit into the scheme presented in said references.