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Review of tetrad formalism in general relativity

7.1 introduction

Einstein's equations, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$, relate the energy-momentum tensor of sources with the geometry of space-time. It tells us how matter curves space-time and contains the equations of motion of the matter via Bianchi's identity. It looks deceptively simple: when realizing that we need hundreds of pages to write them out in terms of components of the metric tensor and its derivatives, it comes as no surprise that only very few exact solutions are known that have physical, astronomical or cosmological relevance. Karl Schwarzschild found a spherical symmetric static solution shortly after the birth of general relativity. His solution describes the space-time of non-rotating spherically symmetric matter distributions, for example a non-rotating star or black hole. More relevant to astronomy is the solution that includes the effect of rotation, the Kerr solution, valid for rotating black holes or as an approximation to the exterior solution outside an axially symmetric rotating mass distribution [73]. This was found by Roy Kerr (and independently by Ezra Newman) in 1963: it took almost 50 years just to include angular momentum! This solution is more mysterious than Schwarzschild's in that still, after many years of unsuccessful search, no acceptable interior solution has been found that matches the exterior solution [74] (however, for a different view see [75]). According to Kerr this interior solution will probably never be found [76]. In addition, the maximal analytic extension contains closed timelike curves [77], which form a problem for causality. An exact solution that is important in cosmology is the Friedmann-Lemaitre-Robertson-Walker solution describing a spatially homogeneous and isotropic universe.

Obtaining exact solutions of Einstein's equations has become a field of research in its own. In general relativity, a theory where experiments are extremely difficult, where it is not immediately clear what coordinates mean and where you have to think hard how to define the most basic physical quantities, it is important to think about solutions that have no counterpart in the world around us, the category that contains almost all known exact solutions.
Confronted with the complexity of the field equations of general relativity, there are essentially three options to proceed. We could consider weak fields only and linearize the theory, resulting in a theory that has some similarities with electrodynamics. We could turn to numerics and, not surprisingly in a computer dominated era, numerical relativity has become an important branch of Einstein's theory in the last decade. Finally, we can try to solve the equations analytically with the help of assumptions, like symmetry, that reduces the complexity.

In chapter 8 and chapter 10 we will consider exact solutions. Here we review the mathematical notions that are needed in those chapters. Reference [12] comes closest to our preferences with respect to notation and signature choices and can therefore be best consulted for additional information. In geometrized units with $4\pi G \equiv 1$, Einstein's equations read $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}$. Here, as elsewhere, $c \equiv 1$.

7.2 differential geometric notations

In chapter 1 we considered null tetrads in Minkowski space. In general relativity Minkowski space figures as tangent space to the curved space-time. It is one of our goals in this mathematical intermezzo to introduce the Newman-Penrose formalism, a specific example of the tetrad formalism in curved space-time.

A basis of the tangent space defined by local coordinates (x) of the space-time manifold is $\{\mathbf{e}_\mu\} \equiv \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the dual basis is the collection of one-forms (or linear functionals or covariant vectors) $\{\mathbf{e}^\mu\}$:

$$\mathbf{e}^\mu(\mathbf{e}_\nu) = \mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$$

The following notation is often used: $\mathbf{e}_\mu = \partial_\mu$ and $\mathbf{e}^\mu = dx^\mu$.

Any other n linearly independent tangent vectors $\{\mathbf{e}_\alpha\}$ define a different basis related to the coordinate basis in the following manner:
$\mathbf{e}_\alpha = M_\alpha^{\ \mu}\partial_\mu$.

The dual basis to this is $\{\mathbf{e}^\alpha\}$, $\mathbf{e}^\alpha = M^\alpha_{\ \mu}dx^\mu$.

The matrices $(M_{\alpha}^{\ \mu})$ and $(M^{\alpha}_{\ \mu})$ are inverses of each other, the upper index labeling different elements in one row (the column index). Whenever $\{\mathbf{e}_\alpha\}$ comes from a different basis $(\tilde{x})$, the matrix elements are

$$M_{\alpha}^{\ \mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}.$$

Of course, the aforementioned pointwise definitions are valid throughout the manifold, thus defining vector fields.

7.2.1 tensors

Let T_p denote the tangent space at point p and T_p^* the cotangent space at p. Define

$$\Pi^s_{\ r} = T_p^* \times ... \times T_p^* \times T_p \times ... \times T_p,$$

in which the cotangent space appears r times and the tangent space s times. A tensor of type (r, s) is a multilinear mapping from $\Pi^s_{\ r}$ to $\mathbb{R}$: $\mathbf{T}(\omega^1, ..., \omega^r, \mathbf{V}_1, ..., \mathbf{V}_s) \in \mathbb{R}$. The space of these tensors is denoted

$$T^r_{\ s}(p) = T_p \otimes ... \otimes T_p \otimes T_p^* \otimes ... \otimes T_p^*,$$

in which the tangent space appears r times and the cotangent space s times (note the interchange of r and s). Clearly, the collection $\{\mathbf{e}_{\alpha_1} \otimes ... \otimes \mathbf{e}_{\alpha_r} \otimes \mathbf{e}^{\beta_1} \otimes ... \otimes \mathbf{e}^{\beta_s}\}$ is a basis for $T^r_{\ s}$.

7.2.2 forms

An important subclass of T^r_s consists of antisymmetric tensors. Combined with functions and dual vectors they constitute the class of forms, for which a calculus can be developed that serves as a convenient instrument for calculations in general relativity.

A zero-form, f, is just a function from the manifold into the reals. One-forms are linear combinations of dual basis vectors. A s-form is a totally antisymmetric tensor of type (0, s). We can construct a s-form from any tensor of type (0, s) with aid of the alternating operator $\hat{\mathcal{A}}$:

$$\hat{\mathcal{A}}\mathbf{T}(\mathbf{V}_1, ..., \mathbf{V}_s) = \frac{1}{s!} \sum_{j_1,...,j_s} sgn(j_1, ..., j_s)\mathbf{T}(\mathbf{V}_{j_1}, ..., \mathbf{V}_{j_s}).$$

The $\frac{n!}{s!(n-s)!}$ dimensional vector space of s-forms is denoted $\Lambda^s T_p^*$. A basis for this vector space can be obtained from a basis of T^0_s:

$$\{\hat{A}(\mathbf{e}^{\alpha_1} \otimes ... \otimes \mathbf{e}^{\alpha_s})\} \equiv \{\mathbf{e}^{\alpha_1} \wedge ... \wedge \mathbf{e}^{\alpha_s} \mid \alpha_1 > ... > \alpha_s\}.$$

This defines the exterior or wedge product of one-forms.
The exterior differential operator, d, is defined for forms. For a zero-form f, df is a one-form:

$$df(\mathbf{V}) = df \cdot \mathbf{V} = \mathbf{V}(f).$$

In a local coordinate basis: $df = f_{,\mu} dx^\mu$.

The abstract definition for the other forms comes down to, in a local coordinate basis:

$$d\mathbf{A} = d(A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}) = A_{\mu_1 \ldots \mu_p, \nu} dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}.$$

This rule is consistent with an important fact that is not a result of the foregoing, but part of the abstract definition of exterior differentiation: $d(d\mathbf{A}) = 0$.

7.2.3 covariant differentiation

We define an affine (or Koszul) connection ∇ in such a way that for any vector field $\mathbf{V}$, $\nabla_\mathbf{V}$ maps another vector field $\mathbf{W}$ into the vector field $\nabla_\mathbf{V}\mathbf{W}$. We require the following conditions (with $\mathbf{U}$ an arbitrary vector field):

$$\nabla_{(f\mathbf{V}+g\mathbf{W})}\mathbf{U} = f\nabla_\mathbf{V}\mathbf{U} + g\nabla_\mathbf{W}\mathbf{U}$$

$$\nabla_\mathbf{V}(\mathbf{W} + \mathbf{U}) = \nabla_\mathbf{V}\mathbf{W} + \nabla_\mathbf{V}\mathbf{U}$$

$$\nabla_\mathbf{V} f = \mathbf{V}(f)$$

$$\nabla_\mathbf{V}(f\mathbf{W}) = (\nabla_\mathbf{V} f)\mathbf{W} + f\nabla_\mathbf{V}\mathbf{W}$$

and define $\nabla\mathbf{W}$ to be the following tensor field of type (1,1): $\nabla\mathbf{W}(\mathbf{V}) = \nabla_\mathbf{V}\mathbf{W}$.

For higher order tensors we demand the Leibniz rule to hold.

In terms of an arbitrary dual basis $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}^\alpha\}$, since $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha$ is a vector field:

$$\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \omega^\gamma{}_\alpha(\mathbf{e}_\beta)\mathbf{e}_\gamma \equiv \omega^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma,$$

in which $\omega^{\gamma}_{\ \alpha}$ are one-forms. Thus a connection is specified by 4^2 one-forms $\omega^{\gamma}_{\ \alpha}$ or by 4^3 scalar fields $\omega^{\gamma}_{\ \alpha\beta}$. We write $\Gamma^{\lambda}_{\ \mu\nu}$ for $\omega^{\lambda}_{\ \mu\nu}$ and $W^{\lambda}_{\ ;\mu}$ for $(\nabla_{\partial_\mu}\mathbf{W})^{\lambda}$ in case a local coordinate basis is used. We thus arrive, for example, at the familiar

$$W^{\lambda}_{\ ;\mu} = W^{\lambda}_{\ ,\mu} + \Gamma^{\lambda}_{\ \nu\mu} W^{\nu}. \tag{7.1}$$

We will be concerned with connections that are derived from a metric,

$\Gamma^{\lambda}_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}})$

and for these the torsion tensor vanishes: $T^{\lambda}_{\ \mu\nu} = -(\Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}) = 0.$

The connection coefficients are then called the Christoffel symbols and a vanishing torsion means a symmetric connection.

7.2.4 parallel propagation and geodesics

A vector $\mathbf{V}$ is parallely propagated along a curve c if

$V^{\mu}_{\ ;\nu} \frac{dx^{\nu}(\mathsf{c}(t))}{dt} \delta t = 0.$

A curve is a geodesic when the parallel propagation of its tangent vector is a multiple of itself, the proportionality possibly changing from point to point:

$\frac{d^2x^{\mu}}{dt^2} + \Gamma^{\mu}_{\ \lambda\kappa} \frac{dx^{\lambda}}{dt} \frac{dx^{\kappa}}{dt} = f(t) \frac{dx^{\mu}}{dt}.$

If we reparametrize the geodesic according to $s = \int^{t} dt'' e^{\int^{t''} dt' f(t')}$ this becomes

$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\ \lambda\kappa} \frac{dx^{\lambda}}{ds} \frac{dx^{\kappa}}{ds} = 0.$ In this case s is called an affine parameter.

Written in terms of the 'four velocity' $u^{\mu} = \frac{dx^{\mu}}{ds}$ we do get

$u^{\mu}_{\ ;\lambda} u^{\lambda} = 0.$

7.2.5 Riemann tensor, Ricci tensor, Bianchi identity and more

In general the Riemann tensor, $\mathbf{R}$, is defined by

$$\mathbf{R}(\mathbf{V},\mathbf{W}) = \nabla_{\mathbf{V}}\nabla_{\mathbf{W}} - \nabla_{\mathbf{W}}\nabla_{\mathbf{V}} - \nabla_{[\mathbf{V},\mathbf{W}]}.$$

We will use a basis, and for a coordinate basis this definition boils down to:

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}{}_{\lambda\nu,\mu} - \Gamma^{\kappa}{}_{\lambda\mu,\nu} + \Gamma^{\kappa}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\kappa}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\lambda\mu}.$$

The Ricci tensor is defined by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, and the scalar curvature is defined as $R = R^{\mu}{}_{\mu}$.

Some important formula that can be derived with the help of the definitions stated thus far are:

$$V^{\mu}{}_{;\kappa;\lambda} - V^{\mu}{}_{;\lambda;\kappa} = -R^{\mu}{}_{\nu\kappa\lambda}V^{\nu} \qquad \text{(\textit{Ricci identity}) and obvious generalizations}$$

$$g_{\mu\nu;\lambda} = 0$$

$$R^{\nu}{}_{\lambda\kappa\mu} + R^{\nu}{}_{\kappa\mu\lambda} + R^{\nu}{}_{\mu\lambda\kappa} = 0 \qquad \text{(\textit{cyclic identity})}$$

$$R^{\nu}{}_{\lambda\kappa\mu;\sigma} + R^{\nu}{}_{\lambda\mu\sigma;\kappa} + R^{\nu}{}_{\lambda\sigma\kappa;\mu} = 0 \quad (\text{or } R_{\nu\lambda[\kappa\mu;\sigma]} = 0) \qquad \text{(\textit{Bianchi identity})}$$

$$R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda} = 0,\ R_{\mu\nu\kappa\lambda} + R_{\mu\nu\lambda\kappa} = 0,\ R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu},\ R_{\mu\nu} = R_{\nu\mu}$$

From the symmetries of the Riemann tensor we conclude that it contains 20 independent components, whereas the Ricci tensor contains 10 independent components.

7.2.6 Einstein tensor and Weyl tensor

The Einstein tensor $\mathbf{G}$ is defined as:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R$$

and from the Bianchi identity it follows that

$$G^{\mu}{}_{\nu;\mu} = 0,$$

an identity with important implications for the structure of general relativity. The Weyl tensor $\mathbf{C}$ is defined as follows

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \tfrac{1}{2}(g_{\mu\kappa}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\kappa} - g_{\nu\kappa}R_{\mu\lambda} - g_{\mu\lambda}R_{\nu\kappa}) + \tfrac{1}{6}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})R.$$

It shares all the symmetries of the Riemann tensor, and in addition satisfies $g^{\nu\lambda}C_{\mu\nu\kappa\lambda} = 0$.

As can be seen from the symmetries it satisfies, the number of independent components is 10. The most important property of this tensor is the invariance under conformal transformations of $C^{\mu}_{\ \nu\kappa\lambda}$. In vacuum it coincides with the Riemann tensor.

7.3 tetrad formalism

The tetrad formalism can be used as a mathematical tool by which vector fields with complex functional dependence become the basis vector fields in which all other tensor fields are expressed.
We consider four contravariant vector fields that in every point of the manifold form a basis, called a vierbein or a tetrad:

$$\{\mathbf{e}_{(k)}\} \equiv \{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}, \mathbf{e}_{(4)}\}$$

with $\mathbf{e}_{(k)}^{\ \mu}$ the components along the (coordinate) basis $\{\mathbf{e}_\mu\}$:

$$\mathbf{e}_{(k)} = \mathbf{e}_{(k)}^{\ \mu}\mathbf{e}_\mu.$$

For tetrad and tetrad components we will use round brackets around indices. The discussion somewhat parallels the introduction of tetrads in chapter 1. There is indeed a very elegant spinor alternative to all that follows. Here we opt for the more conventional route, that, except perhaps for the expert, is more intuitive.

Components of the corresponding covariant vector fields are

$$\mathbf{e}_{(k)\mu} = g_{\mu\nu}\mathbf{e}_{(k)}^{\ \nu}.$$

We consider tetrads for which

$\mathbf{g}(\mathbf{e}_{(k)}, \mathbf{e}_{(l)}) = \mathbf{e}_{(k)}^{\ \mu}\mathbf{e}_{(l)\mu} \equiv \eta_{(k)(l)}$ form a constant (symmetric) matrix with inverse $[\eta^{(k)(l)}]$.

The components of the tetrad do form a matrix $[\mathbf{e}_{(k)}^{\ \mu}]$. We define the inverse to be $[\mathbf{e}^{(k)}_{\ \mu}]$:

$\mathbf{e}_{(k)}{}^{\mu}\mathbf{e}^{(l)}{}_{\mu} = \delta^{(l)}{}_{(k)}$ and $\mathbf{e}_{(k)}{}^{\mu}\mathbf{e}^{(k)}{}_{\nu} = \delta^{\mu}_{\nu}$.

Thus the upper index labels different elements in a row (the column index). It is easy to check the following rules:

$\eta_{(k)(l)}\mathbf{e}^{(k)}{}_{\mu} = \mathbf{e}_{(l)\mu}$

$\eta^{(k)(l)}\mathbf{e}_{(k)\mu} = \mathbf{e}^{(l)}{}_{\mu}$

$\mathbf{e}_{(k)\mu}\mathbf{e}^{(k)}{}_{\nu} = g_{\mu\nu}$

$A_{(k)} = \mathbf{e}_{(k)\mu}A^{\mu} = \mathbf{e}_{(k)}{}^{\mu}A_{\mu}$

$A^{(k)} = \eta^{(k)(l)}A_{(l)} = \mathbf{e}^{(k)}{}_{\mu}A^{\mu} = \mathbf{e}^{(k)\mu}A_{\mu}$ are the tetrad components of the vector field $\mathbf{A}$: $\mathbf{A} = A^{(k)}\mathbf{e}_{(k)}$

$A^{\mu} = \mathbf{e}_{(k)}{}^{\mu}A^{(k)} = \mathbf{e}^{(k)\mu}A_{(k)}$

$T_{(k)(l)} = \mathbf{e}_{(k)}{}^{\mu}\mathbf{e}_{(l)}{}^{\nu}T_{\mu\nu} = \mathbf{e}_{(k)}{}^{\mu}T_{\mu(l)}$

etc.

So, for tetrad components η *plays the role of the metric tensor*, and we can change from tetrad indices to tensor indices and vice versa with the help of the matrices formed by the components of the tetrad along the original basis. These matrices now play the role of (M) in the beginning of section 7.2. Tetrads are defined independent of coordinates. Tetrad components of tensors therefore do not change when a coordinate transformation is applied: tetrad components of tensors are scalar fields. Especially when one learned about tensors from the older literature, often still used by physicists, in which a tensor is defined by its transformation properties, this fact has to be remembered carefully.

When the basis $\{\mathbf{e}_{\mu}\}$ is a coordinate basis it is also clear that

$\mathbf{e}_{(k)} = e_{(k)}{}^{\mu}\partial_{\mu}$

$A_{(k),(l)} = \mathbf{e}_{(l)}{}^{\mu}\partial_{\mu}A_{(k)} = ... = A_{(k)|(l)} + \gamma_{(m)(k)(l)}A^{(m)}$ in which

$\gamma_{(m)(k)(l)} = -\gamma_{(k)(m)(l)} \equiv \mathbf{e}_{(m)}{}^{\mu}\mathbf{e}_{(l)}{}^{\nu}\mathbf{e}_{(k)\mu;\nu}$ are called the Ricci rotation coefficients

$A_{(k)|(l)} \equiv \mathbf{e}_{(k)}{}^{\mu}\mathbf{e}_{(l)}{}^{\nu}A_{\mu;\nu}$ is called the intrinsic derivative.

Thus, from comparison with (7.1): *the intrinsic derivative takes over the role* *of covariant derivative*, and

$\gamma^{(m)}{}_{(k)(l)}$ *comes instead of* $\Gamma^{\lambda}_{\mu\nu}$.

Calculation of the Ricci rotation coefficients do not require knowledge of the Christoffel symbols, because of:

$$\gamma_{(k)(l)(m)} = \frac{1}{2}[\lambda_{(k)(l)(m)} + \lambda_{(m)(k)(l)} - \lambda_{(l)(m)(k)}]$$

with $\lambda_{(k)(l)(m)} \equiv [\mathbf{e}_{(l)\mu,\nu} - \mathbf{e}_{(l)\nu,\mu}]\mathbf{e}_{(k)}{}^{\mu}\mathbf{e}_{(m)}{}^{\nu}$ in which only partial, not covariant, derivatives appear.

7.4 Newman-Penrose formalism

The Newman-Penrose formalism [78] results from a particular choice for the matrix $(\eta_{(k)(l)})$, namely the form (1.1). We take the tetrad $\{\mathbf{e}_{(k)}\}$ to be $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ with the first pair being real null vectors and the last pair complex conjugate null vectors. We require the following relations to hold:

$\mathbf{l} \cdot \mathbf{l} = \mathbf{n} \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{m} = \bar{\mathbf{m}} \cdot \bar{\mathbf{m}} = 0$ *null condition*

$\mathbf{l} \cdot \mathbf{m} = \mathbf{l} \cdot \bar{\mathbf{m}} = \mathbf{n} \cdot \mathbf{m} = \mathbf{n} \cdot \bar{\mathbf{m}} = 0$ *orthogonality condition*

$\mathbf{l} \cdot \mathbf{n} = 1 = -\mathbf{m} \cdot \bar{\mathbf{m}}$ *normalization condition*

Then the dual tetrad basis $\{\mathbf{e}^{(k)}\}$ is $\{\mathbf{n}, \mathbf{l}, -\bar{\mathbf{m}}, -\mathbf{m}\}$, and

$$[\eta_{(k)(l)}] = [\eta^{(k)(l)}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.2}$$

When considering directional derivatives, we write $\{D, \Delta, \delta, \delta^*\}$ for $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$.

The Ricci rotation coefficients are now called *spin coefficients*:

$\kappa = \gamma_{(3)(1)(1)}$
$\sigma = \gamma_{(3)(1)(3)}$
$\lambda = \gamma_{(2)(4)(4)}$
$\nu = \gamma_{(2)(4)(2)}$

$\rho = \gamma_{(3)(1)(4)}$
$\mu = \gamma_{(2)(4)(3)}$
$\tau = \gamma_{(3)(1)(2)}$
$\pi = \gamma_{(2)(4)(1)}$
$\epsilon = \frac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)})$
$\gamma = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)})$
$\alpha = \frac{1}{2}(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)})$
$\beta = \frac{1}{2}(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)})$

and the complex conjugates will be arrived at when interchanging the labels (3) and (4).

The equations of the Newman-Penrose formalism consist of the commutation relations for the tetrad seen as derivatives, the Ricci identities, the so called eliminant relations and the Bianchi identities. These equations, some real and some complex, stand in the place of Einstein's equations and will be given in chapter 10, where we try to solve them simultaneously in case the curvature of space-time is due to an electromagnetic Hopf knot.

7.4.1 Weyl tensor

It is a time consuming but otherwise simple exercise to show that the ten independent components of the Weyl tensor in the present formalism can be represented by the complex scalar fields $\Psi_0, ..., \Psi_4$:

$\Psi_0 \equiv -C_{(1)(3)(1)(3)} = -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda l^\mu m^\nu$
$\Psi_1 \equiv -C_{(1)(2)(1)(3)} = -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu m^\nu$
$\Psi_2 \equiv -C_{(1)(3)(4)(2)} = -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda \bar{m}^\mu n^\nu$
$\Psi_3 \equiv -C_{(1)(2)(4)(2)} = -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda \bar{m}^\mu n^l \nu$
$\Psi_4 \equiv -C_{(2)(4)(2)(4)} = -C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu$

Note that there is no need for parentheses around indices of the Ψ's since, besides the fact that they are only defined for the tetrad formalism, the indices do not refer to a decomposition with respect to basis vectors. The explicit form of the Weyl tensor in a coordinate basis in terms of these complex scalar fields is the daunting expression:

$C_{\kappa\lambda\mu\nu} = -(\Psi_2 + \Psi_2^*)[\{l_\kappa n_\lambda l_\mu n_\nu\} + \{m_\kappa \bar{m}_\lambda m_\mu \bar{m}_\nu\}] + (\Psi_2 - \Psi_2^*)\{l_\kappa n_\lambda m_\mu \bar{m}_\nu\} +$
$2Re\{-\Psi_0\{n_\kappa \bar{m}_\lambda n_\mu \bar{m}_\nu\} - \Psi_4\{l_\kappa m_\lambda l_\mu m_\nu\} + \Psi_2\{l_\kappa m_\lambda n_\mu \bar{m}_\nu\} - \Psi_1[\{l_\kappa n_\lambda n_\mu \bar{m}_\nu\} +$
$\{n_\kappa \kappa \bar{m}_\lambda \bar{m}_\mu m_\nu\}] + \Psi_3[\{l_\kappa n_\lambda l_\mu m_\nu\} - \{l_\kappa m_\lambda m_\mu \bar{m}_\nu\}]\}$

in which $\{l_\kappa n_\lambda m_\mu \bar{m}_\nu\}$ stands for

$l_\kappa n_\lambda m_\mu \bar{m}_\nu - l_\kappa n_\lambda \bar{m}_\mu m_\nu - n_\kappa l_\lambda m_\mu \bar{m}_\nu + n_\kappa l_\lambda \bar{m}_\mu m_\nu + m_\kappa \bar{m}_\lambda l_\mu n_\nu - m_\kappa \bar{m}_\lambda n_\mu l_\nu - \bar{m}_\kappa m_\lambda l_\mu n_\nu + \bar{m}_\kappa m_\lambda n_\mu l_\nu$.

This time, simply by looking at the expression above, one convinces oneself of the following form for all the distinct tetrad components of the Weyl tensor in terms of the Ψ's:

$$C_{(1)(3)(1)(4)} = C_{(2)(3)(2)(4)} = C_{(1)(3)(3)(2)} = C_{(1)(4)(4)(2)} = 0$$
$$C_{(1)(3)(3)(4)} = \Psi_1$$
$$C_{(2)(4)(4)(3)} = \Psi_3$$
$$C_{(1)(2)(1)(2)} = C_{(3)(4)(3)(4)} = -(\Psi_2 + \Psi_2^*)$$
$$C_{(1)(2)(3)(4)} = (\Psi_2 - \Psi_2^*)$$

supplemented with the defining equations for the Ψ's, and the complex conjugates of all these (interchanging labels (3) and (4)). The equations that follow for the complex scalar fields, as well as those of the following subsection, as a result of the equations of general relativity will be given in chapter 10.

7.4.2 Ricci tensor

We have seen that the Ricci tensor contains ten independent components. In the Newman-Penrose formalism the tetrad components of this tensor will be represented in terms of four real and three complex scalar fields as follows:

$$\Lambda = \frac{1}{24}R = \frac{1}{12}(R_{(1)(2)} - R_{(3)(4)})$$
$$\Phi_{00} = -\frac{1}{2}R_{(1)(1)}$$
$$\Phi_{11} = -\frac{1}{4}(R_{(1)(2)} + R_{(3)(4)})$$
$$\Phi_{22} = -\frac{1}{2}R_{(2)(2)}$$
$$\Phi_{01} = -\frac{1}{2}R_{(1)(3)} \quad (\text{and } \Phi_{10} = -\frac{1}{2}R_{(1)(4)})$$
$$\Phi_{02} = -\frac{1}{2}R_{(3)(3)} \quad (\text{and } \Phi_{20} = -\frac{1}{2}R_{(4)(4)})$$
$$\Phi_{12} = -\frac{1}{2}R_{(2)(3)} \quad (\text{and } \Phi_{21} = -\frac{1}{2}R_{(2)(4)}).$$

7.4.3 electrodynamical quantities and equations

Einstein's (strong) equivalence principle implies that the equations of physics in the presence of a gravitational field, when written in a freely falling (and non-rotating) coordinate system, have the same form (locally) as in special relativity. This means that we do get the manifest covariant tensor equations in the presence of gravity from the special relativistic form by substituting semicolons for commas. See in this respect equation (7.1). The source-free Maxwell equations in the absence of gravity can, according to (3.11) and (3.12), be written as $F^{\mu}{}_{\nu,\mu} = 0$ and $F_{[\mu\nu,\lambda]} = 0$. Maxwell's equations without source in a curved space-time become therefore $F^{\mu}{}_{\nu;\mu} = 0$ and $F_{[\mu\nu;\lambda]} = 0$.

The six independent components of the electromagnetic field tensor **F** will be represented by three complex fields:

$$\phi_0 \equiv F_{(1)(3)} = F_{\mu\nu} l^\mu m^\nu$$
$$\phi_1 \equiv \tfrac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu)$$
$$\phi_2 \equiv F_{(4)(2)} = F_{\mu\nu} \bar{m}^\mu n^\nu$$

in terms of which Maxwell's equations in curved space-time

$F_{[\mu\nu;\lambda]} = 0,\ F^{\mu}{}_{\nu;\mu} = 0$ (in a coordinate basis) or

$F_{[(k)(l)|(m)]} = 0,\ F^{(m)}{}_{(k)|(m)} = 0$ (in a tetrad basis) reads

$$D\phi_1 - \delta^*\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2$$

$$D\phi_2 - \delta^*\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2$$

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2$$

and in terms of which the nine independent components of the electromagnetic energy-momentum tensor $\mathbf{T}^{e.m.}$ (3.13) will be represented as

$$T_{(1)(1)} = -2\phi_0\phi_0^*$$
$$T_{(1)(2)} + T_{(3)(4)} = -4\phi_1\phi_1^*$$
$$T_{(2)(2)} = -2\phi_2\phi_2^*$$
$$T_{(1)(3)} = -2\phi_0\phi_1^*$$
$$T_{(2)(3)} = -2\phi_1\phi_2^*$$
$$T_{(3)(3)} = -2\phi_0\phi_2^*$$

Whenever the electromagnetic field is the only source for gravity, $\mathbf{T} = \mathbf{T}^{e.m.}$, we find, using the fact that $\mathbf{T}^{e.m.}$ is traceless (see (3.13)), from Einstein's equations, $R_{(k)(l)} = 2T_{(k)(l)}$, that

$$\Phi_{nm} = 2\phi_n\phi_m^* \ \text{ and } \ \Lambda = 0 \tag{7.3}$$

This relates the Ricci tensor with the electromagnetic field that acts as its sole source.

7.4.4 tetrad transformations

We will now consider the effect of changing the tetrad. By considering the relation between the tetrad and the basis vectors coresponding to local geodesic coordinates at a point p, for which $g_{\mu\nu}(p) = \eta_{\mu\nu}(p)$, the Minkowski metric, it is clear that we have as much freedom to change the tetrad as we have in changing these basis vectors such that they remain related to local geodesic coordinates at p. This, of course, means a Lorentz transformation and correspondingly we have six degrees of freedom. If we extend this transformation continuously throughout the manifold, we do get a tetrad transformation depending on six function that does not alter $(\eta_{(k)(l)})$. Direct calculation of inner products will show that these tetrad transformations can be represented as follows.

With a and b arbitrary complex functions, A and θ arbitrary real functions:

$$\text{type 1)} \quad \boxed{\begin{array}{l} \mathbf{l} \to \mathbf{l} \\ \mathbf{n} \to \mathbf{n} + a^*\mathbf{m} + a\bar{\mathbf{m}} + aa^*\mathbf{l} \\ \mathbf{m} \to \mathbf{m} + a\mathbf{l} \\ \bar{\mathbf{m}} \to \bar{\mathbf{m}} + a^*\mathbf{l} \end{array}} \tag{7.4}$$

$$\text{type 2)} \quad \boxed{\begin{array}{l} \mathbf{l} \to \mathbf{l} + b^*\mathbf{m} + b\bar{\mathbf{m}} + bb^*\mathbf{n} \\ \mathbf{n} \to \mathbf{n} \\ \mathbf{m} \to \mathbf{m} + b\mathbf{n} \\ \bar{\mathbf{m}} \to \bar{\mathbf{m}} + b^*\mathbf{n} \end{array}} \tag{7.5}$$

$$\text{type 3)} \quad \boxed{\begin{array}{l} \mathbf{l} \to A^{-1}\mathbf{l} \\ \mathbf{n} \to A\mathbf{n} \\ \mathbf{m} \to e^{i\theta}\mathbf{m} \\ \bar{\mathbf{m}} \to e^{-i\theta}\bar{\mathbf{m}} \end{array}} \tag{7.6}$$

As straightforward calculations show, these tetrad transformations do have an effect on the previous defined quantities in the following way.

type 1)

$\Psi_0 \to \Psi_0$
$\Psi_1 \to \Psi_1 + a^*\Psi_0$
$\Psi_2 \to \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0$
$\Psi_3 \to \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0$
$\Psi_4 \to \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_4$

$\kappa \to \kappa$
$\sigma \to \sigma + a\kappa$
$\lambda \to \lambda + a^*(2\alpha + \pi) + (a^*)^2(\rho + 2\epsilon) + (a^*)^3\kappa + \delta^* a^* + a^* D a^*$
$\rho \to \rho + a^*\kappa$
$\mu \to \mu + a\pi + 2a^*\beta + 2aa^*\epsilon + (a^*)^2\sigma + a(a^*)^2\kappa + \delta a^* + aDa^*$
$\tau \to \tau + a\rho + a^*\sigma + aa^*\kappa$
$\pi \to \pi + 2a^*\epsilon + (a^*)^2\kappa + Da^*$
$\epsilon \to \epsilon + a^*\kappa$
$\gamma \to \gamma + a\alpha + a^*(\beta + \tau) + aa^*(\rho + \epsilon) + (a^*)^2\sigma + a(a^*)^2\kappa$
$\alpha \to \alpha + a^*(\rho + \epsilon) + (a^*)^2\kappa$
$\beta \to \beta + a\epsilon + a^*\sigma + aa^*\kappa$
$\nu \to \nu + a\lambda + a^*(\mu + 2\gamma) + (a^*)^2(\tau + 2\beta) + (a^*)^3\sigma + aa^*(\pi + 2\alpha) + a(a^*)^2(\rho + 2\epsilon) + a(a^*)^3\kappa + (\Delta + a^*\delta + a\delta^* + aa^*D)a^*$

$\phi_0 \to \phi_0$
$\phi_1 \to \phi_1 + a^*\phi_0$
$\phi_2 \to \phi_2 + 2a^*\phi_1 + (a^*)^2\phi_0$

type 2)

With respect to type 1) **l** and **n** have been interchanged, resulting in the transformation

$\Psi_0 \leftrightarrows \Psi_4^*$
$\Psi_1 \leftrightarrows \Psi_3^*$
$\Psi_2 \leftrightarrows \Psi_2^*$
$\phi_0 \leftrightarrows -\phi_2^*$
$\phi_1 \leftrightarrows -\phi_1^*$
$\kappa \leftrightarrows -\nu^*$
$\rho \leftrightarrows -\mu^*$
$\sigma \leftrightarrows -\lambda^*$
$\alpha \leftrightarrows -\beta^*$
$\epsilon \leftrightarrows -\gamma^*$
$\pi \leftrightarrows -\tau^*$

Thus, for example: $\Psi_1 \to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4$, etc.

type 3)

$\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0$
$\Psi_1 \to A^{-1}e^{i\theta}\Psi_1$
$\Psi_2 \to \Psi_2$
$\Psi_3 \to Ae^{-i\theta}\Psi_3$
$\Psi_4 \to A^2e^{-2i\theta}\Psi_4$

$\kappa \to A^{-2}e^{i\theta}\kappa$
$\sigma \to A^{-1}e^{2i\theta}\sigma$
$\rho \to A^{-1}\rho$
$\tau \to e^{i\theta}\tau$
$\pi \to e^{-i\theta}\pi$
$\lambda \to Ae^{-2i\theta}\lambda$
$\mu \to A\mu$
$\nu \to A^2e^{-i\theta}\nu$
$\gamma \to A\gamma - \frac{1}{2}\Delta A + \frac{i}{2}A\Delta\theta$
$\epsilon \to A^{-1}\epsilon - \frac{1}{2}A^{-2}DA + \frac{i}{2}A^{-1}D\theta$
$\alpha \to e^{-i\theta}\alpha + \frac{i}{2}e^{-i\theta}\delta^*\theta - \frac{1}{2}A^{-1}e^{-i\theta}\delta^* A$
$\beta \to e^{i\theta}\beta + \frac{i}{2}e^{i\theta}\delta\theta - \frac{1}{2}A^{-1}e^{i\theta}\delta A$

$\phi_0 \to A^{-1}e^{i\theta}\phi_0$
$\phi_1 \to \phi_1$
$\phi_2 \to Ae^{-i\theta}\phi_2$

7.5 the optical scalars

In section **1.2.1** we introduced the optical scalars rotation, expansion and shear for a null congruence in Minkowski space. Here we review a more general mathematical treatment, not confined to flat space-time. In doing so, we will learn about the geometrical interpretation of at least some of the spin coefficients. In the Newman-Penrose formalism Einstein's equations have been replaced by 43 equations, each of which does not have a physical interpretation individually. Often in solving equations of physics, (physical) intuition or (physical) analogies are helpful. Here, in the present formalism, we have to solve many equations containing symbols of a highly mathematical character, the physics of which is hidden far behind the surface. Ultimately the expert will possibly acquire some intuition ("when doing this or that with this parameter such or so will happen"), but, no doubt, in cases like this it is desirable to have powerful general theorems (see next section) or the relief of having an interpretation for at least some of the symbols.

The change in the basis vector $\mathbf{e}_{(k)}$ between two infinitesimally ($\delta\mathbf{s}-$) separated points is $\delta\mathbf{e}_{(k)\mu} = \mathbf{e}_{(k)\mu;\nu}\delta s^{\nu} = \mathbf{e}^{(l)}{}_{\mu}\gamma_{(l)(k)(m)}\mathbf{e}^{(m)}{}_{\nu}\delta s^{\nu} = -\gamma_{(k)(l)(m)}\mathbf{e}^{(l)}{}_{\mu}\delta s^{(m)}$. This leads to $\delta\mathbf{e}_{(k)}(m) = -\gamma_{(k)(l)(m)}\mathbf{e}^{(l)}$ for the change in $\mathbf{e}_{(k)}$ per unit displacement along the direction (m) . Applied to the vectors from the Newman-Penrose tetrad, the change in $\mathbf{l}$ per unit displacement along $\mathbf{l}$ is:

$$\delta\mathbf{l}(1) = -\gamma_{(1)(k)(1)}\mathbf{e}^{(k)} = -\gamma_{(1)(2)(1)}\mathbf{l}+\gamma_{(1)(3)(1)}\bar{\mathbf{m}}+\gamma_{(1)(4)(1)}\mathbf{m} = (\epsilon+\epsilon^*)\mathbf{l}-\kappa\bar{\mathbf{m}}-\kappa^*\mathbf{m}$$

or: $l_{\mu;\nu}l^{\nu} = (\epsilon+\epsilon^*)l_{\mu} - \kappa\bar{m}_{\mu} - \kappa^* m_{\mu}$.

In the same way we find for $\mathbf{n}$ and $\mathbf{m}$:

$\delta\mathbf{n}(1) = -(\epsilon+\epsilon^*)\mathbf{n} + \pi\mathbf{m} + \pi^*\bar{\mathbf{m}}$, $\delta\mathbf{n}(2) = -(\gamma+\gamma^*)\mathbf{n} + \nu\mathbf{m} + \nu^*\bar{\mathbf{m}}$,

$\delta\mathbf{m}(1) = (\epsilon+\epsilon^*)\mathbf{m} + \pi^*\mathbf{l} - \kappa\mathbf{n}$, etc.

From this and section **7.2.4** we see that the vector field $\mathbf{l}$ is a congruence of null geodesics if and only if $\kappa = 0$ and that these are affinely parametrized when in addition $Re(\epsilon) = 0$. We can apply a tetrad type 3) transformation to achieve $\epsilon = 0$, whenever this is not already the case and this will not change the direction of $\mathbf{l}$ or the fact that $\kappa = 0$.
With $\kappa = \epsilon = 0$, writing out $l_{\mu;\nu}$ in terms of spin coefficients and antisymmetrizing the result, we arrive at

$$l_{[\mu;\nu]} = -(\alpha^*+\beta-\tau)l_{[\mu}\bar{m}_{\nu]} - (\alpha+\beta^*-\tau^*)l_{[\mu}m_{\nu]} + (\rho-\rho^*)\bar{m}_{[\mu}m_{\nu]}$$

and thus

$$l_{[\mu;\nu}l_{\kappa]} = (\rho-\rho^*)\bar{m}_{[\mu}m_{\nu}l_{\kappa]}.$$

These two equations tell us that whenever $\rho = \rho^*$, $\mathbf{l}$ will be *proportional* to the gradient of a scalar field (we say: the congruence of the null geodesics is hyper-surface orthogonal) and that when in addition $\alpha^*+\beta = \tau$, $\mathbf{l}$ will be *equal* to the gradient of this scalar field. We also find (again, $\kappa = \epsilon = 0$)

$$\tfrac{1}{2}l^{\mu}{}_{;\mu} = -\tfrac{1}{2}(\rho+\rho^*) \equiv \Theta$$

$$\tfrac{1}{2}l_{[\mu;\nu]}l^{\mu;\nu} = -\tfrac{1}{4}(\rho-\rho^*)^2 \equiv \omega^2$$

$\frac{1}{2}l_{(\mu;\nu)}l^{\mu;\nu} = \Theta^2 + |\sigma|^2$.

Θ, ω and σ are called the optical scalars (expansion, rotation and shear; these names indicate what happens with bundles of light-rays as they travel through a gravitational field for example). The significance of $\Theta = -Re(\rho), \omega = Im(\rho)$ and σ can also be grasped from the change in $\mathbf{l}$ in the orthogonal direction $\mathbf{m}$:

$\delta\mathbf{l}(3) = (\alpha^* + \beta)\mathbf{l} - \rho^*\mathbf{m} - \sigma\bar{\mathbf{m}}$.

When $\kappa = 0 = \epsilon$ we do find the following propagation equations (variation along **l**) from the form the Riemann tensor takes in the present formalism (see also section **10.3**):

$D\sigma = \sigma(\rho + \rho^*) + \Psi_0 = -2\Theta\sigma + \Psi_0$

$D\rho = (\rho^2 + |\sigma|^2) + \Phi_{00}$ (or $D\omega = -2\Theta\omega$ and $D\Theta = \omega^2 - \Theta^2 - |\sigma|^2 - \Phi_{00}$).

7.6 Petrov classification and Goldberg-Sachs theorem

In chapter 6 it was already mentioned that the spinor equivalent of the Weyl tensor is completely symmetric and from chapter 1 we know that as a result the Weyl spinor defines four principal null directions via its principal spinors. These null directions are used to distinguish gravitational fields according to the Petrov classification. In contrast with electrodynamics, where an analoguous classification is possible since the Maxwell spinor is symmetric (see section **3.4**), the invariant classification of the Weyl tensor is very important. Many powerful theorems exist that relate certain properties of the field with its Petrov type, one example of which will be given below: the Goldberg-Sachs theorem.
Many exact solutions have been rediscovered again and again. Perhaps, even likely, our chapter 10, if successful is an example of this. The problem of proving or disproving that two solutions are the same can be very hard, but the complexity is reduced by the invariant classification of fields, combined with the powerful theorems available.

The Petrov classification is as follows (PND = principal null directions):

Type I: four distinct PND
Type II: two PND coincide
Type D: two degenerate PND
Type III: three PND coincide
Type N: four PND coincide

When at least two principal null directions coincide, the space-time is called algebraically special. By considering the tetrad transformations from section **7.4.4** it is not difficult to show that there exists a tetrad such that:

Type I: only $\Psi_0 = \Psi_4 = 0$
Type II: only Ψ_2 and $\Psi_3 \neq 0$
Type D: only $\Psi_2 \neq 0$
Type III: only $\Psi_3 \neq 0$
Type N: only $\Psi_4 \neq 0$

There are many forms of the theorem that goes under the name Goldberg-Sachs [79]. Even the lemmas or corollaries are sometimes presented as the Goldberg-Sachs theorem. Here we list indiscriminately two of these, the first of which will be used in chapter 10:

If a gravitational field contains a shear-free geodesic null congruence $\mathbf{l}$ (so, $\kappa = 0 = \sigma$) and $\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$, then the field is algebraically special (so, $\Psi_0 = 0 = \Psi_1$) with $\mathbf{l}$ a degenerate principal null direction.

A vacuum metric is algebraically special if and only if it contains a shear-free geodesic null congruence.

Similar in spirit is the Mariot-Robinson theorem [65]:

An arbitrary space-time admits a geodesic shear-free null congruence if and only if it admits an electromagnetic null field satisfying Maxwell's equations in it

and its corollary:

The Weyl tensor of Einstein-Maxwell fields with an electromagnetic null field is algebraically special.