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Title: The Robinson congruence in electrodynamics and general relativity
Issue Date: 2014-03-25

6

Linked field configurations and the Penrose transform

This chapter marks the transition from electrodynamics to general relativity. Here we 'transform' an electromagnetic Hopf knot into a gravitational Hopf knot. For this we need the twistor functions leading via the Penrose transform to respective knots. In a sense these functions are to be considered the same and were, in present context, first considered in [11]. An article on the main results of this chapter, J. Swearngin, A. Thompson, A.Wickes, J.W. Dalhuisen, and D. Bouwmeester, Linked Gravitational Radiation, arXiv preprint arXiv:1302.1431 (2013) has been submitted for publication.

6.1 introduction

In chapter 4 we used Robinson's theorem to show that there is a correspondence between a non-null twistor and an electromagnetic Hopf knot. Robinson's theorem associates a null electromagnetic field to every geodesic shear free null congruence. Here we will consider the Penrose transform, a contour integral formula that associates solutions to the zero rest mass free field equations with twistor functions that satisfy certain conditions. Since vacuum Maxwell's equations form a particular instance of a zero rest mass equation, it must be possible to arrive at an electromagnetic Hopf knot from a specific twistor function. Having found this twistor function, it can be used in a modified form to generate solutions to linearized Einstein's equations, which forms another example of a zero rest mass equation. Although this strategy could have been used with respect to a generalized Robinson's theorem [60], its solutions depend on solving a set of difficult partial differential equations. When using the contour integral formula the solution comes 'automatically'.

From the discussion of the Kerr theorem in chapter 1 we know that every analytic geodesic shear-free null congruence is related to a holomorphic twistor function that is homogeneous of some degree (1.16). Combined with Robinson's theorem we thus have a relation between homogeneous holomorphic twistor functions and solutions to the zero rest mass equation. The precise relation is encoded in the following contour integral, known as the Penrose transform [27]:

$$\bar{\phi}_{X'...Y'}(x^\mu) = \frac{1}{2\pi i}\oint \bar{\pi}_{X'}...\bar{\pi}_{Y'}\rho_x f(\mathcal{Z}^\alpha)\bar{\pi}_{Z'}d\bar{\pi}^{Z'} \tag{6.1}$$

or

$$\psi_{A...B}(x^\mu) = \frac{1}{2\pi i}\oint \rho_x \frac{\partial}{\partial\omega^A}...\frac{\partial}{\partial\omega^B} f(\mathcal{Z}^\alpha)\bar{\pi}_{Z'}d\bar{\pi}^{Z'} \tag{6.2}$$

where the symbol ρ_x means that the twistors are restricted to correspond to the line through $x = (x^\mu)$:

$(\mathcal{Z}^\alpha) = (\omega^A, \bar{\pi}_{X'})$, $\quad \omega^A = iX^{AX'}\bar{\pi}_{X'}$ and $X^{AX'}$ corresponds to x (1.13).

The integrals in equations (6.1) and (6.2) make sense only when the twistor function has the correct degree of homogeneity (see also the remark concerning projective twistor space in chapter 1). In equation (6.1) this means -n-2, with n the number of indices in $X'...Y'$ and in (6.2) this means n-2, now with n the number of indices in $A...B$.
That the fields $\phi_{X'...Y'}$ and $\psi_{A...B}$ satisfy the zero rest mass free field equation follows by noting that $\nabla_A{}^{X'} = \epsilon^{X'Y'}\frac{\partial}{\partial X^{AY'}}$ and $\frac{\partial}{\partial X^{AY'}}\rho_x f(\mathcal{Z}^\alpha) = i\bar{\pi}_{Y'}\rho_x\frac{\partial f}{\partial\omega^A}$: the resulting expression is a product of an anti-symmetric quantity in two indices and a symmetric quantity with respect to the same indices. The n=0 case leads to the wave equation $\Box\phi = 0$.
From equations (6.1) and (6.2) it follows that there is not a single twistor function that corresponds to a particular solution to the zero rest mass equation, but a whole class of twistor functions. This touches upon the subject of sheaf cohomology [61], that will not be needed for our purposes.

6.2 examples

Fields that result from the Penrose transform of twistor functions that have the form $\frac{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)^p (\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^q}{(\bar{\mathcal{C}}_\alpha \mathcal{Z}^\alpha)^r (\bar{\mathcal{D}}_\beta \mathcal{Z}^\beta)^s}$ $(p, q, r, s \in \mathbb{N} \cup \{0\})$, are called elementary states.
We will consider three examples, n=0, n=1 and n=2, in which in the first example we use the simplest possible twistor function in (6.1) leading to a non-trivial elementary state. In the two following examples we then alter the twistor function minimally, by raising one of the powers in the denominator, in order to satisfy the homogeneity requirements. In a quantum context this could be seen as a spin or helicity raising operation. In a sense the resulting fields can be considered as generalizations of each other.

6.2.1 wave equation

(n=0) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)}$

$\bar{\mathcal{A}}_\alpha = (A_A, \bar{C}^{X'})$ and $\bar{\mathcal{B}}_\alpha = (B_A, \bar{D}^{X'})$ are fixed dual twistors.

This leads to the integral $\bar{\phi}(x^\mu) = \frac{1}{2\pi i} \oint \frac{1}{(\bar{\alpha}^{X'} \bar{\pi}_{X'})(\bar{\beta}^{Y'} \bar{\pi}_{Y'})} \bar{\pi}_{Z'} d\bar{\pi}^{Z'}$,

where $\bar{\alpha}^{X'} = iA_A X^{AX'} + \bar{C}^{X'}$ and $\bar{\beta}^{Y'} = iB_A X^{AY'} + \bar{D}^{Y'}$.

Suppose that $\bar{\mathcal{A}}_\alpha$ and $\bar{\mathcal{B}}_\alpha$ are such that $\bar{\alpha}^{X'}$ and $\bar{\beta}^{X'}$ provide a basis in (conjugate) spin-space. The poles in the integral are then different and we can write $\bar{\pi}_{Z'} = \bar{\alpha}_{Z'} + z\bar{\beta}_{Z'}$ and $d\bar{\pi}^{Z'} = \bar{\beta}^{Z'} dz$ for some complex z. Choosing the contour to enclose only the pole with respect to $\bar{\mathcal{A}}_\alpha$ we now have

$\bar{\phi}(x^\mu) = \frac{1}{2\pi i} \oint \frac{dz}{(\bar{\alpha}^{X'} \bar{\beta}_{X'})z} = \frac{1}{\bar{\alpha}^{X'} \bar{\beta}_{X'}} = \frac{2}{A_A B^A (x^\mu - y^\mu)(x_\mu - y_\mu)}.$

Where the last equality follows if we assume in addition that there exists a $y \in M^4_{\mathbb{C}}$ corresponding to $Y^{AX'}$ such that

$\bar{C}^{X'} = -iY^{AX'} A_A$ and $\bar{D}^{X'} = -iY^{AX'} B_A$.

(Compare the discussion of the incidence relation (1.12); note that above equation implies $C^A = i\bar{Y}^{AX'} \bar{A}_{X'}$.)
This assumption is equivalent to $\mathcal{A}^\alpha \bar{\mathcal{B}}_\alpha = iB_A(\bar{Y}^{AX'} - Y^{AX'})\bar{A}_{X'}$ and in case the two twistors are null, this equals 0 and then it implies that the corresponding geodesics have one point in M^4 in common.
We can solve the two equations to obtain an expression for $Y^{AX'}$:

$Y^{AX'} = i\frac{A^A\bar{D}^{X'} - B^A\bar{C}^{X'}}{A^A B_A}$

In the calculation of the last expression of $\bar{\phi}$ the following identity is useful:

$A_B B^A - A^A B_B = \delta_B{}^A A_C B^C.$

6.2.2 source-free electrodynamics

(n=1) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^3}$

We now have to compute

$\bar{\phi}_{X'Y'}(x^\mu) = \frac{1}{2\pi i}\oint \bar{\pi}_{X'}\bar{\pi}_{Y'}\rho_x \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^3}\bar{\pi}_{Z'} d\bar{\pi}^{Z'}$,

and with the same assumptions for $\mathcal{A}^\alpha$ and $\mathcal{B}^\alpha$ as in the previous example this leads to

$\bar{\phi}_{X'Y'}(x^\mu) = \frac{1}{2\pi i}\oint(\bar{\alpha}_{X'} + \bar{\beta}_{X'}z)(\bar{\alpha}_{Y'} + \bar{\beta}_{Y'}z)\frac{1}{z(\bar{\alpha}^{Z'}\bar{\beta}_{Z'})^3}dz = (\frac{2}{A_A B^A (x^\mu - y^\mu)(x_\mu - y_\mu)})^3 \bar{\alpha}_{X'}\bar{\alpha}_{Y'}$,

or, for comparison with equation (4.7):

$\phi_{AB}(x^\mu) = (\frac{2}{\bar{A}_{X'}\bar{B}^{X'}(x^\mu - \bar{y}^\mu)(x_\mu - \bar{y}_\mu)})^3 \alpha_A \alpha_B.$

If we now take $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ we get $(\alpha_A) = \frac{-i}{\sqrt{2}}(-t + z - i, x + iy)$. Comparison with equation (4.6) shows that the solution belongs to the same class of electromagnetic Hopf knots as the one given in equation (4.1), which will be reproduced exactly for $(\mathcal{B}^\alpha) = (\frac{-2}{\sqrt{2}}, \frac{b}{\sqrt{2}}, -2, b)$ and arbitrary b. With present choices of $(\mathcal{A}^\alpha)$ and $(\mathcal{B}^\alpha)$ we have $(y^\mu) = (i, 0, 0, 0)$, $\bar{A}_{X'}\bar{B}^{X'} = 2$ and $\bar{\alpha}_{X'}\bar{\beta}^{X'}(x^\mu) \neq 0$, so that indeed $\bar{\alpha}^{X'}$ and $\bar{\beta}^{X'}$ do form a basis for all (x^μ).
A possible choice for b would be -2 and we know from chapter 4 that in this case $\mathcal{B}$ corresponds via Robinson's theorem with the class of which a rotated version of equation (4.1) is a representative.

Starting with the twistor $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$, in chapter 4 we had to solve partial differential equations for $g(x^\mu)$ in order to arrive at a solution of Maxwell's equations. Here we start with the same twistor $\mathcal{A}$ and have to find another twistor $\mathcal{B}$ algebraically. In chapter 4 we could solve the problem easily because we knew what to look for, otherwise it would have been a hard exercise. The Penrose transform makes this exercise much simpler.

In the following example we will use $(\mathcal{A}^\alpha) = (i, 1, i\sqrt{2}, \sqrt{2})$ and therefore we note here that a representative of the class that Robinson's theorem associates with this twistor is given by a rotated version of the field (4.1) (rotation around x-axis such that the z-axis turns into the y-axis):

$$\mathbf{F}^{RS}(t, \mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3} \begin{pmatrix} (x-iz)^2 - (t-i+y)^2 \\ 2(x-iz)(t-i+y) \\ i(x-iz)^2 + i(t-i+y)^2 \end{pmatrix}. \tag{6.3}$$

When combined with $(\mathcal{B}^\alpha) = (\frac{1}{2})^{\frac{2}{3}}(1, i, \sqrt{2}, i\sqrt{2})$ the Penrose transform also leads to this field. The interchange of $\mathcal{A}$ and $\mathcal{B}$ would lead to a 180^0 rotation around the z-axis of the field (6.3).

Present discussion shows that an electromagnetic Hopf knot is to be considered as what is known in the twistor literature as an elementary state.

6.2.3 linearized Einstein equation

(n=2) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^5}$

In chapter 3 we showed how the spinor form of Maxwell's equations can be derived. A similar but much more complicated analysis applied to Einstein's equations lead to the following result [62]. The spinor form of the Riemann tensor is (a short mathematical introduction to the tensors used here is given in section 7.2)

$R_{AW'BX'CY'DZ'} =$

$\Phi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \bar{\Phi}_{W'X'Y'Z'}\epsilon_{AB}\epsilon_{CD} +$
$\Psi_{ABY'Z'}\epsilon_{W'X'}\epsilon_{CD} + \bar{\Psi}_{CDW'X'}\epsilon_{AB}\epsilon_{Y'Z'} +$
$2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{W'Z'}\epsilon_{X'Y'})$,

in which

$\Phi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \bar{\Phi}_{W'X'Y'Z'}\epsilon_{AB}\epsilon_{CD}$ corresponds to the Weyl tensor $C_{\kappa\lambda\mu\nu}$,

$-2\Psi_{ABW'X'}$ corresponds to the trace free Ricci tensor $R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$ and 24Λ corresponds to the Ricci scalar R.

From this we see that in vacuum the Einstein field equations are simply $\Psi_{ABW'X'} = 0 = \Lambda$. The Bianchi identity in vacuum gives $\nabla^{AX'}\Phi_{ABCD} = 0$. It is important to note that in a curved space-time the form (1.4) for the Infeld van der Waerden symbols do not solve the requirement (1.3). We first need the metric before we can find a suitable representation for the symbols. Therefore we cannot solve the vacuum Bianchi identity for Φ_{ABCD} without further assumptions. But, in the weak field limit (or linearized Einstein theory) we have $\nabla^{AX'}\phi_{ABCD} = 0$ for the linearized Weyl spinor ϕ_{ABCD} and the usual Minkowski space form of the differential operators. Conversely, in vacuum, a solution to this zero rest mass field equation defines a linearized Weyl tensor [63].

In electrodynamics, the sources are represented by the four-current (j^μ) and the field degrees of freedom by the Faraday tensor $(F^{\mu\nu})$. The analogues in general relativity are the energy-momentum tensor $(T^{\mu\nu})$ for the sources and the Weyl tensor (C^{abcd}) for the field. The Weyl tensor can be regarded as the Riemann tensor with the influences of the sources "taken out of it". So, when classifying gravitational fields, it is natural to look at the Weyl tensor, not the Riemann tensor. The Weyl spinor, the spinor equivalent of the Weyl tensor, is completely symmetric. Its principal spinors define four principal null directions that are used to classify fields.

There exists a striking algebraic similarity between the electric $(E_i \equiv F_{0i})$ and magnetic $(B_i \equiv *F_{0i})$ part of a Faraday tensor in Maxwell theory and the electric part $(E_{ij} \equiv C_{i0j0})$ and magnetic part $(B_{ij} \equiv - * C_{i0j0})$ of the Weyl tensor in general relativity [64,65] .
Since $E_{ij}\xi^j$ measures the relative (tidal) acceleration between to points separated by a small vector $\boldsymbol{\xi}$, we can interpret the electric part of the Weyl tensor as a tidal field. The matrix (E_{ij}) is traceless and symmetric, and can therefore be characterized by its eigenvalues and eigenvectors, the integral curves of which are called tendex lines [66] and can be considered the gravitational analogue of electric field lines. An extended object placed in the field E_{ij} will be stretched in the direction of the lines corresponding to positive eigenvalues and compressed along the curves corresponding to negative eigenvalues. The strength of this effect is related to the eigenvalue.

The interpretation of the magnetic part of the Weyl tensor is as a frame-drag field (possibly a misnomer, see [67,68]). A gyroscope at the tip of the vector $\boldsymbol{\xi}$ will precess with angular velocity $B_{ij}\xi^j$ relative to inertial frames at the tail. The matrix (B_{ij}) is also traceless and symmetric and the integral curves of the eigenvectors are now called vortex lines [66].

Continuing now the n=2 example, with the help of the previous examples

it can be seen immediately that $\phi_{ABCD}(x^\mu) \propto \alpha_A\alpha_B\alpha_C\alpha_D$. There is only one fourfold degenerate principal null direction. In the Petrov classification of gravitational fields this corresponds to a type N field (see section **7.6**).
For $(\mathcal{A}^\alpha) = (i, 1, i\sqrt{2}, \sqrt{2})$ and $(\mathcal{B}^\alpha) = (\frac{1}{2})^{\frac{2}{3}}(1, i, \sqrt{2}, i\sqrt{2})$ the calculation leads to the eigenvalues $-\mathsf{a}, 0$ and a for both the electric part and the magnetic part of the Weyl tensor. The corresponding (eigen) vector fields will be denoted by $\mathbf{E}_-, \mathbf{E}_0, \mathbf{E}_+$ and $\mathbf{B}_-, \mathbf{B}_0, \mathbf{B}_+$ respectively, and we will use the Riemann-Silberstein like combination $\mathbf{F}_E \equiv \mathbf{E}_- + i\mathbf{E}_+$ and $\mathbf{F}_B \equiv \mathbf{B}_- + i\mathbf{B}_+$.

A detailed calculation shows:

$$\mathsf{a}(x^\mu) = \frac{(1+x^2+(y+t)^2+z^2)^2}{2^{\frac{2}{3}}((t^2-r^2)^2+1+2(t^2+r^2))^{\frac{5}{2}}},$$

$$\mathbf{E}_0(x^\mu) = \mathbf{B}_0(x^\mu) \propto \begin{pmatrix} 2(x(t+y)+z) \\ 1+(t+y)^2-x^2-z^2 \\ 2(z(t+y)-x) \end{pmatrix}$$

and

$$\mathbf{F}_E(x^\mu) = e^{i\frac{\pi}{4}}\mathbf{F}_B(x^\mu) \propto e^{i\theta}\mathbf{F}^{RS}_{(6.3)}(x^\mu)$$

Here, $\mathbf{F}^{RS}_{(6.3)}$ is given by (6.3) and $\theta(x^\mu) = Arg(\sqrt{-(t-i)^2+r^2})$. Both proportionality factors are real constants, not functions and can simultaneously be made equal to unity by multiplying the twistor $(\mathcal{B}^\alpha)$ with a suitable factor. This then changes the eigenvalue field accordingly.

We thus have a solution to linearized Einstein's equations in which five Hopf knots appear. Two perpendicular tendex (or tidal) knots and two perpendicular vortex (or frame drag) knots. These two sets are rotated 45° with respect to each other. Perpendicular to these sets we have another Hopf structure, at every point defining the direction where no stretching, no compression and no precession occurs. Only this last structure moves without distortion.
At $t = 0$ the two tendex Hopf knots (one indicating compression, the other stretching) are exactly aligned with the electric and magnetic fields of the electromagnetic Hopf knot (6.3). At other times the difference, apart from the proportionality constant, is a local duality transformation between the corresponding Riemann-Silberstein vectors. Both elementary states were obtained from the Penrose transform using the same twistors $(\mathcal{A}^\alpha)$ and $(\mathcal{B}^\alpha)$ in a slightly different function, to meet the requirements of homogeneity. Note that the fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are proportional to the Poynting vector corresponding to the field (6.3). In analogy with electrodynamics, in [64] a super Poynting vector is defined that in present context is aligned with $\mathbf{E}_0(x^\mu)$ and $\mathbf{B}_0(x^\mu)$. It is thus possible to

describe this gravitational Hopf knot or "linked gravitational radiation" in terms of the electromagnetic Hopf knot [10].

Note that $\theta(x^\mu) = \arctan\{\frac{1}{2t}(t^2 - r^2 - 1 + \sqrt{(t^2-r^2)^2+1+2(t^2+r^2)})\}$ and does not satisfy the generalized wave equation (4.12) and (4.13): as expected $\mathbf{F}_E$ is not a solution of Maxwell's equations.

Although in general relativity the interpretation of quantities that appear in equations has to be done very carefully due to diffeomorphism invariance of the theory [58, 69], it is important to realize that according to a valid interpretation of *linearized* general relativity the physical fields can be considered as Lorentz tensors on Minkowski (flat) space-time [68]. A nice feature, since every physicists has gained some intuition in this respect. This changes dramatically when considering the full nonlinear equations of Einstein's theory, and some of the problems that this gives rise to will be discussed in later chapters.
Since in linearized general relativity the fields can be considered as fields in Minkowski space, it is still possible to construct the Riemann-Silberstein like structures appearing in present section, in the obvious way. In the full theory however one should be more careful. Now one needs for its definition a time-like congruence, or, what amounts to the same, two real null congruences.

As a final remark, we mention that besides the aforementioned similarity between the Faraday tensor in electrodynamics and the Weyl tensor in general relativity, there is another electromagnetic analogy in *linearized* general relativity [70]. Under certain restrictions this analogy can for instance be used to predict the motion of test particles in a gravitational field, based on the knowledge of the corresponding problem in electrodynamics. With the help of this analogy it is however not possible to simply transform a solution of Maxwell's equations into a solution of linearized Einstein equations. It is therefore of no help in our endeavour to 'generalize' an ectromagnetic Hopf knot to a gravitational one.
A lucid account of this and the previous analogy, including a discussion of their limitations, is to be found in [71,72].