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Title: The Robinson congruence in electrodynamics and general relativity
Issue Date: 2014-03-25

5
Conformal inversion

5.1 introduction

It is well known that Maxwell's equations are invariant under Poincare transformations $\mathcal{P}(1,3)$. Less familiar is the fact that $\mathcal{P}(1,3)$ is part of a larger group, the conformal group $\mathcal{C}(1,3)$, under which the source free Maxwell equations are also invariant, as was discovered by Bateman and Cunningham in 1909 [39–42]. This extra symmetry does not lead to new (unknown) conservation laws, since in the case of vacuum Maxwell equations the generators of the conformal group can be expressed analytically in terms of the elements of the Poincare algebra, all of whose conserved quantities are already known [43, 44].
Apart from electrodynamics, conformal transformations play an important role in general relativity. They are also used in a reformulation of Brans-Dicke theory as an alternative to general relativity, in which the mass acquires a space-time dependent variation [45, 46]. This variation depends on the conformal factor Ω from the definition of conformally related metrics: $g' = \Omega^2 g$.
It is noteworthy to remark that in contrast with Poincare symmetry $\mathcal{P}(1,3)$, requiring conformal symmetry $\mathcal{C}(1,3)$ for a (spin 1) vector field uniquely leads to Maxwells equations [44].
Besides the Poincare transformations, the conformal group consists of dilations and special conformal transformations. Here we investigate what can be learned from this lesser known symmetry.

5.2 special conformal transformations and conformal inversion

When applying Lie's approach to symmetries of differential equations (see appendix **A**) to source free electrodynamics, you will arrive at the maximal invariance group of Maxwell equations in vacuum [44]. The elements that make up this group are duality transformations, dilations, Poincare transformations and special conformal transformations. The first three of these are very familiar, therefore we concentrate on the special conformal group. We would like to warn the reader with more than average knowledge about the conformal group that it is a mistake to interpret a special conformal transformation as a change from an inertial system to an uniformly accelerating coordinate system [43,47–49], as used to be done early in the history of conformal transformations and is still being done every now and then (see for example [50]).

A special conformal transformation ($\mathcal{SCT}(b)$) is given by

$$x^\mu \rightarrow \tilde{x}^\mu = \frac{x^\mu + b^\mu(x_\nu x^\nu)}{1+2(b_\nu x^\nu)+(x_\nu x^\nu)(b_\nu b^\nu)}$$

and can be build from a conformal inversion ($\mathcal{CI}(a)$) $x^\mu \rightarrow \tilde{x}^\mu = a\frac{x^\mu}{x_\nu x^\nu}$ and a translation ($\mathcal{T}(c)$) $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + c^\mu$ as follows:

$$\mathcal{SCT}(b) = \mathcal{CI}(1) \circ \mathcal{T}(b) \circ \mathcal{CI}(1).$$

In deriving this result, use has been made of lowering indices on transformed coordinates. This is done with the original flat space-time metric tensor, η: the transformation is thus not to be regarded as a coordinate transformation! (See the sequel for more on this.) Note that $\mathcal{SCT}(b) \circ \mathcal{SCT}(a) = \mathcal{SCT}(a+b)$.

Dilations $\mathcal{D}(a) : x^\mu \rightarrow ax^\mu$ can be obtained in the same way, but with different parameters: $\mathcal{D}(a) = \mathcal{CI}(a) \circ \mathcal{T}(0) \circ \mathcal{CI}(1) = \mathcal{CI}(a) \circ \mathcal{CI}(1)$. We conclude that any conformal transformation can be built from Poincare transformations and conformal inversions, therefore we limit our discussion to $\mathcal{CI} = \mathcal{CI}(1)$, which can be regarded as a length inversion operator [43] and is in itself a conformal transformation: Maxwell's equations in vacuum are invariant under $\mathcal{CI}$.
Note that conformal inversion is ill-defined for points on the light-cone. A general discussion of the conformal group should therefore include M^4_C, the compactification of Minkowski space. However, for our discussion it will suffice to consider M^4.

5.3 prescription for obtaining new fields from old ones

It is of no use to consider a $\mathcal{SCT}$ or a $\mathcal{CI}$ as a coordinate transformation (as is often done in a different context), since Maxwell equations would not be valid in the new system. According to the approach of Lie, the physical content of the conformal symmetry of vacuum electrodynamics is that every solution can be transformed into another different solution. In order to arrive at the new transformed solution from a given one, we proceed as follows.

We start with Minkowski space M^4, coordinates (x) and metric tensor $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, a copy hereof $\tilde{M}^4$, coordinates $(\tilde{x})$ and metric tensor $\tilde{\eta} = \eta$ and a mapping between these manifolds $f : (x) \to (\tilde{x})$ given by $\tilde{x}^\mu = \frac{x^\mu}{(x \cdot x)}$. We use the notation $x \cdot x$ for $x_\nu x^\nu$. Because $\tilde{\eta} = \eta$ we have for the inverse $f^{-1} : (\tilde{x}) \to (x)$ with $x^\mu = \frac{\tilde{x}^\mu}{(\tilde{x} \cdot \tilde{x})}$.
We use f to pull back tensor fields to M^4. In this way the metric tensor $\tilde{\eta}$ is mapped to $g_{\mu\nu}(x) := (f^* \tilde{\eta})_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x) \frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x) \tilde{\eta}_{\alpha\beta}$. Using $\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \frac{\delta^\alpha_\mu}{x \cdot x} - 2\frac{x^\alpha x_\mu}{(x \cdot x)^2}$ we get $g_{\mu\nu}(x) = \frac{1}{(x \cdot x)^2} \eta_{\mu\nu}$ and this shows that, by definition, f is a conformal map.
We will regard tensor fields in M^4 and therefore use η to raise and lower indices, not g. We proceed with pulling back a Faraday tensor (3.10) (satisfying Maxwell equations) $(\tilde{F}_{\mu\nu})$ from $\tilde{M}^4$.

$F_{\mu\nu}(x) := (f^* \tilde{F})_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x) \frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x) \tilde{F}_{\alpha\beta}(\tilde{x}(x))$ and

$F^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} \frac{\partial \tilde{x}^\sigma}{\partial x^\alpha} \frac{\partial \tilde{x}^\kappa}{\partial x^\beta} \eta_{\sigma\lambda}\eta_{\kappa\rho} \tilde{F}^{\lambda\rho}(\tilde{x}(x)) = |\frac{\partial(\tilde{x})}{\partial(x)}| \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} \frac{\partial x^\nu}{\partial \tilde{x}^\rho} \tilde{F}^{\lambda\rho}(\tilde{x}(x))$.

See appendix **B** for details of the calculation. These are tensor fields on M^4. Since $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$, the question arises wether F is a Faraday tensor on M^4:

$\partial_\lambda F_{\mu\nu}(x) + \partial_\nu F_{\lambda\mu}(x) + \partial_\mu F_{\nu\lambda}(x) = 0$ (3.12) follows easily by reshuffling of indices, systematic cancellation of terms and using the fact that $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$. The other pair of equations (3.11) is more intricate, but the result is $\partial_\mu F^{\mu\nu}(x) = 0$ (see appendix **C**). So indeed, the pull back of a Faraday tensor is again a Faraday tensor when the mapping is a conformal inversion. With this result we end up with a prescription for constructing new Maxwell fields (vacuum solutions to Maxwell equations) from known ones:

In Minkowski space M^4, coordinates (x) and metric tensor $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

we define functions $f^\mu(x) = \frac{x^\mu}{(x \cdot x)}$.

If F is a Faraday tensor, so:

$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$ and $\partial_\mu F^{\mu\nu} = 0$ $(F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta})$

then F' is also a Faraday tensor, in which:

$F'_{\mu\nu}(x) = (\partial_\mu f^\alpha)(\partial_\nu f^\beta)F_{\alpha\beta}(\frac{x}{x \cdot x})$ and $F'^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta}F'_{\alpha\beta}(x)$.

A similar strategy to obtain new solutions from old with the help of conformal transformations exists in general relativity, see for a general discussion [51] and more specifically the references therein, such as [52].

5.4 transformation of Riemann-Silberstein vector

In electromagnetic calculations it can be advantageous to work with the Riemann-Silberstein vector $\mathbf{F}^{RS} = \mathbf{E} + i\mathbf{B}$. Instead of the Faraday tensor, F, we should then use the anti-self- or self-dual form, $F^{(asd)}$ or $F^{(sd)}$, of it:

$F^{(asd)}_{\mu\nu} := (F_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho})$ and $F^{(sd)}_{\mu\nu} := (F_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho})$

Note that $F^{(asd)}$ coincides with $\mathcal{F}$ from chapter 3 (3.15).
Because $F'_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F_{\alpha\beta}(\frac{x}{x \cdot x})$ and $F'^{\mu\nu}(x) = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x})$ it is not immediately clear how to calculate $F^{(asd)\prime}$ from $F^{(asd)}$. But with help from appendix **B**(4) and **B**(5) it can be shown that in the case of conformal inversions (*but not in general!*)

$F'^{\mu\nu}(x) = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial \tilde{x}^\nu}{\partial x^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x})$.

Now we have: $F'^{(asd)}_{\mu\nu} = (F'_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F'^{\lambda\rho}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F_{\alpha\beta}(\frac{x}{x.x}) + \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\kappa\sigma}(\frac{x}{x.x})$,

and this leads to (see appendix **D**):

$F'^{(asd)}_{\mu\nu}(x) = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial\tilde{x}^\beta}{\partial x^\nu}(x)F^{(sd)}_{\alpha\beta}(\frac{x}{x\cdot x}),$

so the self-dual is transformed into an anti-self-dual form.

And thus: $F'^{\rho\sigma}_{(asd)}(x) = \eta^{\rho\mu}\eta^{\sigma\nu}F'^{(asd)}_{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial\tilde{x}^\beta}{\partial x^\nu}(x)F^{(sd)}_{\alpha\beta} = \frac{\partial\tilde{x}_\alpha}{\partial x_\rho}(x)\frac{\partial\tilde{x}_\beta}{\partial x_\sigma}(x)F^{\alpha\beta}_{(sd)} = \frac{\partial\tilde{x}^\rho}{\partial x^\alpha}(x)\frac{\partial\tilde{x}^\sigma}{\partial x^\beta}(x)F^{\alpha\beta}_{(sd)} = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\rho}{\partial\tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial\tilde{x}^\beta}F^{\alpha\beta}_{(sd)}(\frac{x}{x\cdot x})$

From this the transformation of the Riemann-Silberstein vector follows:

$$\boxed{\mathbf{F}^{RS\prime}(x) = \frac{1}{(t^2-r^2)^3}[-(t^2-r^2)\mathbf{F}^{RS*}(\frac{x}{x\cdot x}) + 2\mathbf{r}\times(\mathbf{r}\times\mathbf{F}^{RS*}(\frac{x}{x\cdot x})) - 2it\mathbf{r}\times\mathbf{F}^{RS*}(\frac{x}{x\cdot x})]}$$

where we have used $x = (x^\mu) = (t, \mathbf{r})$. From this it can be shown that applying the transformation twice leads back to the original input field: $\mathcal{CI}(\mathcal{CI}(\mathbf{E},\mathbf{B})) = (\mathcal{CI})^2(\mathbf{E},\mathbf{B}) = (\mathbf{E},\mathbf{B})$. This, of course, also follows immediately from $x''^\mu = \frac{x'^\mu}{x'\cdot x'} = (x\cdot x)x'^\mu = x^\mu$. In looking for eigenstates of this conformal inversion operation, the only possible eigenvalues are +1 and -1. More generally we have: $\mathcal{CI}(b)\circ\mathcal{CI}(a) = \mathcal{D}(\frac{b}{a}) = \mathcal{CI}(\frac{1}{a})\circ\mathcal{CI}(\frac{1}{b})$.

From the transformation formula it follows that the Poynting vector transforms in a complicated way, but it can be expressed as:

$\mathbf{S}' = f_1(x^\mu)\mathbf{S}(\frac{x}{x\cdot x}) + f_2(x^\mu)\mathbf{r}.$

The precise form of the functions f_1 and f_2 is not very illuminating.

The nice formula for the transformation of the Riemann-Silberstein vector under conformal inversion is to be contrasted with the complicated transformation of this vector under a special conformal transformation. In [44] it is given as (adapted to our notation and combined to form the RS-vector):

$\mathbf{F}^{RS\prime}(x^\mu) = (1 - 2b^\mu\tilde{x}_\mu + b_\nu b^\nu\tilde{x}_\mu\tilde{x}^\mu)\{(b_\mu\tilde{x}^\mu - 1)^2\mathbf{F}^{RS}(\tilde{x}) + 2(b^\mu\tilde{x}_\mu - 1)(-ib_0\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x}) + i\tilde{t}\mathbf{b}\times\mathbf{F}^{RS}(\tilde{x}) - \mathbf{b}(\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x})) + \tilde{\mathbf{r}}(\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x}))) + \mathbf{b}\times\tilde{\mathbf{r}}(-i\tilde{t}(\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x})) + ib_0(\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x})) + \mathbf{b}\cdot\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x})) + (\mathbf{b}\tilde{t} - \tilde{\mathbf{r}}b_0)(-i\mathbf{b}\cdot\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x}) - \tilde{t}\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x}) + b_0\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x}))\}$

Note that in this formula, $\tilde{x}$ still has to be expressed as function of x. For $(b^\mu) = (b, 0, 0, 0)$ this becomes the relatively simple

$\mathbf{F}^{RS\prime}(x) =$

$\frac{1}{(1+2bt+b^2(t^2-r^2))^3}\{(1+bt)^2\mathbf{F}^{RS}(\frac{x}{x\cdot x}) - b^2\mathbf{r}(\mathbf{r}\cdot\mathbf{F}^{RS}(\frac{x}{x\cdot x})) + 2ib(1+bt)\mathbf{r}\times\mathbf{F}^{RS}(\frac{x}{x\cdot x})\}.$

Since $\mathcal{SCT}(b,\mathbf{0}) = \mathcal{CI}(1)\circ\mathcal{T}(b,\mathbf{0})\circ\mathcal{CI}(1)$ and the conformal inversion of

an electromagnetic Hopf knot is again a Hopf knot (see example (**5.5.2**) below), the sequence (4.23) from chapter 4 implies that a special conformal transformation with $(b^\mu) = (b, \mathbf{0})$ applied to the field $\{\mathbf{E} = \hat{x}, \mathbf{B} = \hat{y}\}$ gives an electromagnetic Hopf knot when b is imaginary.
Substitution of $\mathbf{F}^{RS} = \hat{x} + i\hat{y}$ and $b = i$ leads to

$$\mathbf{F}^{RS\prime} = \tfrac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} x(x+iy) - (t-i)^2 - 2z(t-i) \\ y(x+iy) - i(t-i)^2 - 2iz(t-i) \\ (x+iy)(z+2(t-i)) \end{pmatrix}$$

whereas $\mathcal{CI}(1) \circ \mathcal{T}(i, \mathbf{0}) \circ \mathcal{CI}(1)$ applied to $\mathbf{F}^{RS} = \hat{x} + i\hat{y}$ gives (see examples (**5.5.1**) and (**5.5.2**) below)

$$\mathbf{F}^{RS\prime} = \tfrac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} (x+iy)^2 - (t-i+z)^2 \\ -i(x+iy)^2 - i(t-i+z)^2 \\ 2(x+iy)(t-i+z) \end{pmatrix}.$$

The difference between these is

$\tfrac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} iy(x+iy) - z^2 \\ -ix(x+iy) - iz^2 \\ z(x+iy) \end{pmatrix}$, which can be written as
$\tfrac{-b^2}{(1+2bt+b^2(t^2-r^2))^3} \mathbf{r} \times (\mathbf{r} \times \mathbf{F}^{RS})$.

We conclude that this last term, the general form of which will be

$$(1 - 2b^\mu \tilde{x}_\mu + b_\nu b^\nu \tilde{x}_\mu \tilde{x}^\mu)(\mathbf{b}\tilde{t} - \tilde{\mathbf{r}} b_0) \times (b_0 \tilde{\mathbf{r}} \times \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})),$$

is missing from the expression given in [44].

With the missing term included, the transformation formula for $\mathcal{SCT}(b, \mathbf{0})$ becomes:

$$\mathbf{F}^{RS\prime}(x) = \tfrac{1}{(1+2bt+b^2(t^2-r^2))^3} \times$$

$$\{(1+bt)^2 \mathbf{F}^{RS}(\tfrac{x}{x \cdot x}) - b^2(\mathbf{r}(\mathbf{r} \cdot \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})) + \mathbf{r} \times (\mathbf{r} \times \mathbf{F}^{RS})) + 2ib(1+bt)\mathbf{r} \times \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})\},$$

the correctness of which can be checked with application of $\mathcal{SCT}(b, \mathbf{0}) = \mathcal{CI}(1) \circ \mathcal{T}(b, \mathbf{0}) \circ \mathcal{CI}(1)$ to other examples and various b.

An interesting application is the use of this formula to a constant electric field $\mathbf{E} = \hat{x}$:

$$\mathcal{SCT}(i, \mathbf{0}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\tfrac{1}{2}\{\mathbf{F}_H + \mathbf{F}_H^*(t \to -t)\},$$

in which $\mathbf{F}_H$ is the Riemann-Silberstein vector of the Hopf knot (4.1). Using example (**5.5.2**) below we infer that the special conformal transformation with parameter $(i, \mathbf{0})$ of a constant electric field gives an eigenstate of the conformal inversion operation with eigenvalue -1. It also implies that applying an imaginary time translation $t \to t - i$ to the conformal inversion of a constant electric field in the x-direction gives a superposition of two Hopf knots that, except for a minus sign, are conformal inverted states of each other, $\frac{1}{2}\{\mathbf{F}_H + \mathbf{F}^*_H(t \to -t)\}$. At $t = 0$ this field is given by:

$$\mathbf{E}(0,\mathbf{r}) = \frac{1}{(1+r^2)^3}\begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix}, \quad \mathbf{B}(0,\mathbf{r}) = \mathbf{0}.$$

The electric field is a Hopf fibration, identical to the one in $\mathbf{F}_H(t = 0)$. For arbitrary t, the field can be expressed with the same definitions as in (4.5) as

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}\{\mathbf{E}_0 + i\frac{4t}{1+r^2}\mathbf{C}_0 + \frac{t^2}{(1+r^2)^3}Re(\mathbf{W})\}.$$

The same exercise with a constant magnetic field $\mathbf{B} = \hat{y}$ gives an eigenstate of conformal inversion with eigenvalue +1, $\frac{1}{2}\{\mathbf{F}_H - \mathbf{F}^*_H(t \to -t)\}$. At $t = 0$ this field is:

$$\mathbf{E}(0,\mathbf{r}) = \mathbf{0}, \;\; \mathbf{B}(0,\mathbf{r}) = \begin{pmatrix} 2(xy + z) \\ 1 - x^2 + y^2 - z^2 \\ -2(x - yz) \end{pmatrix},$$

so the magnetic field is the same as that of $\mathbf{F}_H(t = 0)$. For arbitrary t we now have

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}i\{\mathbf{B}_0 + i\frac{4t}{1+r^2}\mathbf{A}_0 + \frac{t^2}{(1+r^2)^3}Im(\mathbf{W})\}.$$

In (4.5) we defined five vector fields of which only four had a clear physical interpretation. It is only now that we can say something about the interpretation of the other, $\mathbf{W}$:

$$\mathcal{SCT}(i,\mathbf{0})(\mathbf{W}) = \mathbf{F}_H,$$

the electromagnetic Hopf knot that was used to define $\mathbf{W}$.

5.5 examples of conformal inversion

The transformation formula for the Riemann-Silberstein vector will be used to investigate the conformal inversion of some specific electromagnetic fields.

5.5.1 constant electric and magnetic field

$$\mathbf{F}^{RS} = (\hat{x} + i\hat{y}) = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.$$

This leads to

$$\mathbf{F}^{RS\prime} = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-z)^2 \\ i(x-iy)^2 + i(t-z)^2 \\ -2(x-iy)(t-z) \end{pmatrix}, \text{ or}$$

$$\mathbf{E}' = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} x^2 - y^2 - (z-t)^2 \\ 2yx \\ -2x(t-z) \end{pmatrix}, \qquad \mathbf{B}' = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} -2xy \\ x^2 - y^2 + (t-z)^2 \\ 2y(t-z) \end{pmatrix}.$$

This field has been considered in chapter 4, equation (4.21) (degenerate Robinson congruence). Applying the imaginary time translation $t \to t - i$ to this Riemann-Silberstein vector gives the electromagnetic Hopf knot of the next example.

5.5.2 electromagnetic Hopf knot

$$\mathbf{F}^{RS} = \frac{1}{((t-i)^2-r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix}.$$

A lengthy calculation (see appendix **E**) results in: $\mathbf{F}^{RS\prime} = -\mathbf{F}^{RS*}(t \to -t)$.
Notice that this can also be seen as a rotation of $180°$ around the x-axis: $\mathbf{F}^{RS\prime} = -Rot_{\hat{x}}(\pi)\{\mathbf{F}^{RS}\}$.
Thus the conformal inversion of an electromagnetic Hopf knot is a different electromagnetic Hopf knot. The sum and difference of these two knots is obviously an eigenstate of conformal inversion.

5.5.3 plane wave

Example (**5.5.1**) is the $k \downarrow 0$ limit of the plane wave:

$$\mathbf{F}^{RS} = e^{ik.x} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \ (k^{\mu}) = (k, 0, 0, k),\ k > 0.$$

The transformed field is a local duality transformation of the field obtained in (**5.5.1**):

$$\mathbf{F}^{RS\prime} = \frac{e^{i\alpha}}{(t^2-r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-z)^2 \\ i(x-iy)^2 + i(t-z)^2 \\ -2(x-iy)(t-z) \end{pmatrix},$$

in which $\alpha = k\frac{t-z}{t^2-r^2}$ satisfies a "generalization" of the homogeneous wave equation, (4.12) and (4.13), with $\beta(x^{\mu}) = k\frac{(t^2-r^2)^4}{(t-z)^2+x^2+y^2}$.

Of course, the field we started with in (**5.5.3**) is itself a local duality transformation of the field we started with in (**5.5.1**) and in this case α satisfies the homogeneous wave equation (see the discussion of equations (4.12) and (4.13)).

The field of a charge distribution satisfies vacuum Maxwell's equations except on the distribution's world tube. In this case we can still use the transformation formula, the resulting field of which will be the field due to a different charge distribution. Clearly, the transformation of the charge distribution is defined such that Maxwell's equations including sources are invariant under present transformation. For a conformal inversion this does not necessarily lead to a physical acceptable charge distribution. Here are a few examples.

5.5.4 Coulomb field

$$\mathbf{F}^{RS}(x) = \frac{q}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\mathbf{F}^{RS*}\left(\frac{x}{x.x}\right) = q\frac{|t^2-r^2|^3}{(t^2-r^2)r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{q(t^2-r^2)^2 \mathrm{sgn}(t^2-r^2)}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives the dichotomic $\mathbf{E}'(x) = -q\frac{\mathrm{sgn}(t^2-r^2)}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{B}' = \mathbf{0}$.

This amounts to changing the sign of the charge of the particle within the particle's light cone ($t^2 > r^2$), or, disregarding the solution for $t < 0$, it is the field of a point particle that turns into its anti-particle at $t = 0$, the effect of which spreads out with the speed of light.
But a better way to cope with the twofold character of the transformed solution might be the following. Since the only purpose of the transformation is to find new solutions to Maxwell's equations and the transformation at hand is ill-defined on the light cone, it is allowed to change the sign of the result in any of the three parts in which the light-cone splits space-time. The resulting field will still be a solution, with the possible exception of the light-cone, where it might be ill-defined. If we apply this strategy we end up with the following physically acceptable possibilities, of which the first two can be extended smoothly across the light-cone.

- the transformed field is the same as the input field
- the transformation has the effect of charge conjugation $q \to -q$
- at $t = 0$ the charge of the particle changes sign, which has a spreading effect.

5.5.5 uniformly moving charged particle

(through origin, velocity v, direction $\hat{x}$)

$\mathbf{F}^{RS}(x) = \frac{q}{A\|\mathbf{r}-\mathbf{v}t\|^3}\begin{pmatrix} x - vt \\ y - ivz \\ z + ivy \end{pmatrix}$,

in which $A = \gamma^2(1 - \beta^2 sin^2\psi)^{\frac{3}{2}}$ and $\psi = cos^{-1}(\mathbf{n}.\frac{\mathbf{v}}{v})$ and $\mathbf{n}$ a unit vector from the charge to fieldpoint ($\mathbf{r}$) [29].

This gives $\mathbf{F}^{RS\prime}(x) = -\mathrm{sgn}(t^2 - r^2)\mathbf{F}^{RS}(x)$,

and again, this amounts to trading the particle for its antiparticle within the light cone (centered at the coordinate origin) and no change outside it. Exactly the same considerations concerning sign changes in part of the solution apply as in (**5.5.4**).

5.5.6 charged particle with uniform acceleration

(acceleration g, direction $\hat{z}$)
The case for uniform acceleration is much more difficult to handle, especially due to retardation effects. This touches the very interesting and conceptually difficult subject of radiation reaction for charges in uniform acceleration. A lot has been written about it for over a century, also in the context of general relativity. Some of the frequently cited authors are Bondi and Gold (1955) [53], Fulton and Rohrlich (1960) [54], DeWitt and Brehme (1960) [55], Boulware (1980) [56] and Parrott (2001) [57]. Any two authors agree on some points and disagree on others. Part of the problem is the interpretation of coordinates and related with this the correct definition of physical measurable quantities like energy.
Here we follow Lyle (Uniformly Accelerating Charged Particles: A Threat to the Equivalence Principle [58]), and take his solution (eq. (15.178)-(15.182)) adapted to our notations and conventions as our input field:

$$\mathbf{F}^{RS}(x) = \frac{q}{4\pi}\frac{g\theta(z+t)}{R^3}\begin{pmatrix} -xz+iyt \\ -yz-ixt \\ -\frac{1}{2}(z^2-t^2-x^2-y^2-\frac{1}{g^2}) \end{pmatrix} + \frac{q}{4\pi}\frac{2g^2\delta(z+t)}{1+g^2(x^2+y^2)}\begin{pmatrix} -x-iy \\ -y+ix \\ 0 \end{pmatrix},$$

with θ the Heaviside step function, δ Dirac's function, g the (constant) acceleration of the particle (taken to be in the +z direction), and $R := \frac{g}{2}[(\frac{1}{g^2}+x^2+y^2+z^2-t^2)^2 - 4(z^2-t^2)\frac{1}{g^2}]^{\frac{1}{2}}$.
The charge moves along the hyperbola $(\frac{1}{g}\cosh(g\tau), 0, 0, \frac{1}{g}\sinh(g\tau))$, in which the parameter τ is eigen time. It is important to note that the step function has been "put in by hand" in order not to violate causality principles. The Dirac function is needed for the field to be a solution of Maxwell's equations on all of Minkowski space.

For the calculation of the output field we need $\mathbf{F}^{RS*}(\frac{x}{x\cdot x})$:

We first calculate the transformed prefactor (skipping all the details) of the θ part

$R^3(\frac{x}{x\cdot x}) = \frac{g^3}{8}\frac{1}{|t^2-r^2|^3}[(\frac{r^2-t^2}{g^2}+1)^2 - \frac{4}{g^2}(z^2-t^2)]^{\frac{3}{2}}$ and note that this equals the original prefactor with g^{-1} substituted for g and an additional prefactor $\frac{1}{|t^2-r^2|^3}$:

$R^3(\frac{x}{x.x}) = \frac{1}{|t^2-r^2|^3}\{R(g \to g^{-1})\}^3.$

This is used to show that $\mathbf{E}'_\theta(x) = q\frac{\text{sgn}(t^2-r^2)}{4\pi}\frac{g^{-1}\theta(\frac{z+t}{t^2-r^2})}{(R(g^{-1}))^3}\begin{pmatrix} xz \\ yz \\ \frac{1}{2}(z^2-t^2-x^2-y^2-g^2) \end{pmatrix}$

and thus:

for $t^2 > r^2$: $\mathbf{E}'_\theta(x) = -\mathbf{E}_\theta(x; g \to g^{-1})$.

Note that in the region $z + t > 0$ and $t^2 - r^2 > 0$ (future pointing light cone) this amounts to changing the direction of the z-axis (besides the change in acceleration):

$\mathbf{E}'_\theta(x) = \mathbf{E}_\theta(\hat{z} \to -\hat{z}; g \to g^{-1})$.

For $t^2 < r^2$ we get a + sign instead of a - sign, but now the field is nonzero only for $z + t < 0$. It can be written as:

$\mathbf{E}'_\theta(x) = -\mathbf{E}_\theta(t \to -t, \hat{\mathbf{r}} \to -\hat{\mathbf{r}}; g \to g^{-1})$

A similar calculation leads to the transformed $\mathbf{B}$-fields:

$$\mathbf{B}'_\theta(x) = q\frac{\text{sgn}(t^2-r^2)}{4\pi}\frac{g^{-1}\theta(\frac{z+t}{t^2-r^2})}{(R(g^{-1}))^3}\begin{pmatrix}-yt\\ xt\\ 0\end{pmatrix}$$

and thus:

for $t^2 > r^2$: $\mathbf{B}'_\theta(x) = -\mathbf{B}_\theta(x; g \to g^{-1})$

for $t^2 < r^2$ we get a + sign instead of a - sign and it is nonzero only for $z + t < 0$.

The transformation of the δ-part:

As before, first the prefactor:

We need the following:

$$\delta(\tfrac{z+t}{t^2-r^2}) = \delta(\tfrac{z+t}{(t-z)(z+t)-(x^2+y^2)}) = (x^2+y^2)\delta(z+t)$$

and frequently use the fact that $f(t^2 - r^2)\delta(z + t) = f(-(x^2 + y^2))\delta(z + t)$

in order to arrive at:

$$\frac{2g^2}{1+g^2(x^2+y^2)}\delta(z+t) \to \frac{2g^2(x^2+y^2)^2}{g^2+(x^2+y^2)}\delta(z+t)$$

and:

$$\mathbf{E}'_\delta(x) = \frac{q}{4\pi}\delta(z+t)\frac{2g^2}{(x^2+y^2)(g^2+x^2+y^2)}\begin{pmatrix}-x\\ -y\\ 0\end{pmatrix},$$

$$\mathbf{B}'_\delta(x) = \frac{q}{4\pi}\delta(z+t)\frac{2g^2}{(x^2+y^2)(g^2+x^2+y^2)}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}$$

This expression is unlike the θ part: it is not a simple replacement $g \to g^{-1}$. But these δ fields are defined only on a hypersurface of measure zero. They were in the first place defined in such a way that the combined θ and δ fields are a solution of Maxwell equations on the hypersurface where the δ field is nonzero, because the θ field alone is only a solution in Minkowski space with this hypersurface left out.

Note: if we really would like an expression for the δ field like that of the θ field, it would be:

$$\mathbf{E}'_\delta(x) = \frac{g^2}{x^2+y^2}\mathbf{E}_\delta(x; g \to g^{-1}) \;, \quad \mathbf{B}'_\delta(x) = \frac{g^2}{x^2+y^2}\mathbf{B}_\delta(x; g \to g^{-1}).$$

Even if we disregard the δ part of the field, the resulting field is physically unacceptable. It would be due to an unrealistic charge distribution, with charges running backwards in time. Of course, this is due to the fact that conformal inversion can not be considered a valid coordinate transformation. Like before, artificial changes in the transformed field will result in acceptable solutions.

5.6 conclusion

We started this chapter in order to find out what can be learned from a symmetry in the equations of source free electrodynamics that is rarely mentioned and seems not to be investigated on a large scale: invariance under conformal transformations. We picked out an ingredient, conformal inversion, of this symmetry that together with well known Poincare transformations can be used to build all conformal transformations.

After studying some examples of conformal inversion we are left with the somewhat surprising result that a Hopf knot turns into a different, but simply related Hopf knot under this transformation. However, it remains difficult to attach meaning to the transformed fields in general. For the time being, perhaps the best we can do is to subscribe the conclusion of Parrott [59], "This [conformal invariance] rather remarkable mathematical fact seems to have no obvious physical interpretation.", and consider the nice transformation formula for the Riemann-Silberstein vector only as a way to generate new solutions from old ones.

Appendix A method of Lie

Starting point is the linear differential equation $\hat{L}(x^\mu, \partial_\mu)\psi(x^\mu) = 0$, in which $\hat{L}$ is a linear operator and ψ stands for $(\psi_1, ..., \psi_r)$. First order differential operators, $\hat{Q}_A = \xi^\mu_A(x,\psi)\partial_\mu + \eta^k_A(x,\psi)\frac{\partial}{\partial\psi_k}$, are sought that satisfy the invariance condition $\hat{L}\hat{Q}_A\psi(x) = 0$ and form a Lie algebra $[\hat{Q}_A, \hat{Q}_B] = C_{ABC}\hat{Q}_C$. The solution leads to the invariance algebra for the differential equation, with help of which the symmetry group can be constructed [44].
This construction clearly shows that related to a symmetry, new solutions can be obtained from old ones. It is also clear that there exists many more symmetries not obtainable by this method.

Appendix B frequently used equalities

(1) $\frac{\partial f}{\partial x_\alpha} = \frac{\partial f}{\partial x^\beta}\frac{\partial x^\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(\eta^{\beta\sigma}x_\sigma)\frac{\partial f}{\partial x^\beta} = \eta^{\beta\alpha}\frac{\partial f}{\partial x^\beta}$

This relation would not be valid if raising and lowering is done by g.

(2) $|\frac{\partial(\tilde{x})}{\partial(x)}| = |\frac{1}{(x.x)^4}| = |\frac{\partial(x)}{\partial(\tilde{x})}|^{-1}$

This is a long but straightforward calculation.
The Jacobian of the transformation equals $\frac{-1}{(x.x)^4}$. We use the convention $|\frac{\partial(\tilde{x})}{\partial(x)}| =$ absolute value of the Jacobian.

(3) $\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\eta_{\alpha\beta} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\eta_{\mu\nu}$ and $\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\eta_{\alpha\beta} = |\frac{\partial(x)}{\partial(\tilde{x})}|^{\frac{1}{2}}\eta_{\mu\nu}$

The second follows from the first by noting the functional similarity between f and f^{-1}.

(4) $\frac{\partial\tilde{x}_\mu}{\partial x_\nu} = \eta_{\mu\lambda}\frac{\partial\tilde{x}^\lambda}{\partial x^\rho}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\lambda}\eta_{\alpha\beta}\frac{\partial\tilde{x}^\lambda}{\partial x^\rho}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\delta^\beta_\rho\eta_{\alpha\beta}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\delta^\nu_\alpha = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\nu}{\partial\tilde{x}^\mu}$

Use has been made of (1) and (3).

(5) For conformal inversions: $\frac{\partial\tilde{x}_\mu}{\partial x_\nu} = \frac{\partial\tilde{x}^\nu}{\partial x^\mu}$

Appendix C is F a Faraday tensor?

We need

(1) $\partial_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = \partial_\mu(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2\frac{\tilde{x}^\mu \tilde{x}_\lambda}{(\tilde{x}.\tilde{x})^2}) = \partial_\mu(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2x^\mu \eta_{\lambda\alpha} x^\alpha) = \partial_\lambda \frac{1}{\tilde{x}.\tilde{x}} - 8\eta_{\lambda\alpha}x^\alpha - 2x^\mu \eta_{\lambda\alpha}\delta^\alpha_\mu = \partial_\lambda(x.x) - 10\eta_{\lambda\alpha}x^\alpha = 2x_\lambda - 10\eta_{\lambda\alpha}x^\alpha = -8\eta_{\lambda\alpha}x^\alpha$

(2) $\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}(\partial_\mu \frac{\partial x^\nu}{\partial \tilde{x}^\rho})\tilde{F}^{\lambda\rho} = \frac{\partial^2 x^\nu}{\partial \tilde{x}^\lambda \partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho} = 0$

(3) $\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\partial_\mu \tilde{F}^{\lambda\rho} = \partial_{\tilde{\lambda}}\tilde{F}^{\lambda\rho} = 0$

Thus:

$\partial_\mu(|\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x))) = (\partial_\mu|\frac{\partial(\tilde{x})}{\partial(x)}|)\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x)) + |\frac{\partial(\tilde{x})}{\partial(x)}|(-8\eta_{\lambda\alpha}x^\alpha)\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x)) + 0 + 0$

The first part of the first term is:

$(\partial_\mu|\frac{\partial(\tilde{x})}{\partial(x)}|)\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = (\partial_\mu(x.x)^{-4})\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = \frac{-8x_\mu}{(x.x)^5}\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = -8(x.x)^{-5}\eta_{\mu\alpha}x^\alpha(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2\frac{\tilde{x}^\mu \tilde{x}_\lambda}{(\tilde{x}.\tilde{x})^2}) = -8(x.x)^{-4}(\eta_{\lambda\alpha}x^\alpha - 2x^\mu \tilde{x}_\lambda x^\alpha \eta_{\mu\alpha})$

The last term in brackets is $-2x^\mu \tilde{x}_\lambda x^\alpha \eta_{\mu\alpha} = -2\eta_{\lambda\alpha}x^\alpha$, and this makes the first term (in the complete expression) cancel the second in the complete expression.

Appendix D transformation of self-dual form

We have: $F'^{(sd)}_{\mu\nu} = (f'_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}f'^{\lambda\rho}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)f_{\alpha\beta}(\tilde{x}) - \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}f^{\kappa\sigma}(\tilde{x})$.

Now $\frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\epsilon_{\alpha\beta\mu\nu} = -\epsilon_{\kappa\sigma\alpha\beta}|\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} = -\epsilon_{\kappa\sigma\alpha\beta}\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)$.

The second equality follows from appendix **B**(4)-(5) and the first comes from the following property of the ϵ symbol:

$M^\alpha_a M^\beta_b M^\gamma_c M^\delta_d \epsilon_{\alpha\beta\gamma\delta} = \epsilon_{abcd}(det(M))$

and thus $M_a^\alpha M_b^\beta \epsilon_{\alpha\beta\lambda\rho} = \epsilon_{abcd}(M^{-1})_\lambda^c (M^{-1})_\rho^d (det(M))$.

The required equality follows from taking $M_\beta^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}$ (so $det(M) = \frac{-1}{(x.x)^4} = -|\frac{\partial(\tilde{x})}{\partial(x)}|$).

Appendix E conformal inversion of electromagnetic Hopf knot

Here we will use the transformation formula for the (anti-) self-dual form of the electromagnetic field tensor.
We have:

$$(F_{\mu\nu}^{(asd)}(x)) = \begin{pmatrix} 0 & E_x + iB_x & E_y + iB_y & E_z + iB_z \\ -E_x - iB_x & 0 & -B_z + iE_z & B_y - iE_y \\ -E_y - iB_y & B_z - iE_z & 0 & -B_x + iE_x \\ -E_z - iB_z & -B_y + iE_y & B_x - iE_x & 0 \end{pmatrix}^* =$$

$$\frac{1}{((t+i)^2-r^2)^3} \begin{pmatrix} 0 & a^2 - d^2 & -ia^2 - id^2 & -2ad \\ -a^2 + d^2 & 0 & 2iad & a^2 + d^2 \\ ia^2 + id^2 & -2iad & 0 & -ia^2 + id^2 \\ 2ad & -a^2 - d^2 & ia^2 - id^2 & 0 \end{pmatrix}$$

in which $a \equiv x + iy$ and $d \equiv t - z + i$.

Thus: $$(F_{\mu\nu}^{(asd)}(\frac{x}{t^2-r^2})) = \frac{-(t^2-r^2)}{((t-i)^2-r^2)^3} \begin{pmatrix} 0 & a^2 - b^2 & -ia^2 - ib^2 & -2ab \\ -a^2 + b^2 & 0 & 2iab & a^2 + b^2 \\ ia^2 + ib^2 & -2iab & 0 & -ia^2 + ib^2 \\ 2ab & -a^2 - b^2 & ia^2 - ib^2 & 0 \end{pmatrix}$$

in which $a \equiv x + iy$ and $b \equiv t - z + i(t^2 - r^2)$.

Using the transformation formula, we now find:

$F_{0i}'^{(sd)}(x) = \frac{F_{0i}^{(asd)}(\frac{x}{t^2-r^2})}{(t^2-r^2)^2} + 2\frac{x^i x^j}{(t^2-r^2)^3} F_{0j}^{(asd)}(\frac{x}{t^2-r^2}) - 2\frac{tx^\alpha}{(t^2-r^2)^3} F_{\alpha i}^{(asd)}(\frac{x}{t^2-r^2})$
$= \frac{1}{(t^2-r^2)^3}\{(t^2 - r^2) - 2t^2\} F_{0i}^{(asd)}(\frac{x}{t^2-r^2}) + 2\frac{x^i x^j}{(t^2-r^2)^3} F_{0j}^{(asd)}(\frac{x}{t^2-r^2}) - 2\frac{tx^j}{(t^2-r^2)^3} F_{ji}^{(asd)}(\frac{x}{t^2-r^2})$
$= \frac{-1}{(t^2-r^2)^3}\{t^2 + r^2 - 2(x^i)^2\} F_{0i}^{(asd)}(\frac{x}{t^2-r^2}) + \{2\frac{x^i x^j}{(t^2-r^2)^3} F_{0j}^{(asd)}(\frac{x}{t^2-r^2}) -$

$2\frac{tx^j}{(t^2-r^2)^3}F^{(asd)}_{ji}(\frac{x}{t^2-r^2})\}_{j\neq i}$

Substituting from above we find for the x component of the RS vector:

$F'^{(sd)}_{01}(x) = \frac{1}{(t^2-r^2)^3}(t^2-x^2+y^2+z^2)\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{(x+iy)^2-(t-z+i(t^2-r^2))^2\} - 2\frac{xy}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-i(x+iy)^2-i(t-z+i(t^2-r^2))^2\} - 2\frac{xz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{ty}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2i(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{tz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-(x+iy)^2-(t-z+i(t^2-r^2))^2\}$

$\equiv \frac{A}{(t^2-r^2)^2((t-i)^2-r^2)^3}$

in which:

$A = (t^2-x^2+y^2+z^2)\{(x+iy)^2-(t-z+i(t^2-r^2))^2\} + 2ixy\{(x+iy)^2+(t-z+i(t^2-r^2))^2\} + 4xz\{(x+iy)(t-z+i(t^2-r^2))\} - 4ity\{(x+iy)(t-z+i(t^2-r^2))\} - 2tz\{(x+iy)^2+(t-z+i(t^2-r^2))^2\}$

$= (-(t+z)^2+(x+iy)^2)\{t-z+i(t^2-r^2)\}^2 + ((t-z)^2-(x-iy)^2)(x+iy)^2 + 4(xz-ity)(x+iy)(t-z+i(t^2-r^2))$

(Now use $(xz-ity) = \frac{1}{2}\{(x-iy)(t+z)-(x+iy)(t-z)\}$)

$= (-(t+z)^2+(x+iy)^2)\{t-z+i(t^2-r^2)\}^2 + ((t-z)^2-(x-iy)^2)(x+iy)^2 + 2\{(x-iy)(t+z)-(x+iy)(t-z)\}(x+iy)(t-z+i(t^2-r^2))$

$= \{-(t+z)^2+(x+iy)^2\}\{(t-z)+i(t^2-r^2)\}^2 + \{(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(t+z)(x+iy)\{(t-z)+i(t^2-r^2)\} - 2(x+iy)^2(t-z)\{(t-z)+i(t^2-r^2)\}$

$= \{-(t+z)^2+(x+iy)^2\}\{(t-z)+i(t^2-r^2)\}^2 + \{-(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\} - 2(x+iy)^2(t-z)i(t^2-r^2)$

$= -(t+z)^2\{(t-z)+i(t^2-r^2)\}^2 + (x+iy)^2\{(t-z)^2-(t^2-r^2)^2\} + \{-(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\}$

$= -(t+z)^2\{(t-z)+i(t^2-r^2)\}^2 - (x+iy)^2(t^2-r^2)^2 - (x-iy)^2(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\}$

$= -(t+z)^2\{(t-z)^2-(t^2-r^2)^2+2i(t-z)(t^2-r^2)\} - (x+iy)^2\{(t^2-r^2)^2+(x-iy)^2\} + 2(x-iy)(x+iy)(t+z)(t-z) + 2i(x-iy)(x+iy)(t+z)(t^2-r^2)$

$= -(t+z)^2\{(t-z)^2-(t^2-r^2)^2\} - (x+iy)^2\{(t^2-r^2)^2+(x-iy)^2\} + 2(x-iy)(x+iy)(t+z)(t-z) - 2i(t+z)(t^2-r^2)^2$

$= (t^2-r^2)^2\{(t+z)^2-(x+iy)^2-2i(t+z)\}-(t-z)^2(t+z)^2-(x+iy)^2(x-iy)^2+2(x^2+y^2)(t^2-z^2)$

$= (t^2-r^2)^2\{(t+z)^2-(x+iy)^2-2i(t+z)\}-(t^2-z^2)^2-(x^2+y^2)^2+2(x^2+y^2)(t^2-z^2)$

$= (t^2-r^2)^2\{(t+z)^2-(x+iy)^2-2i(t+z)\}-\{(t^2-z^2)-(x^2+y^2)\}^2$

$= (t^2-r^2)^2\{-1+(t+z)^2-(x+iy)^2-2i(t+z)\}$

And thus:

$(\mathbf{F}^{RS\prime})_x(x) = \frac{1}{((t-i)^2-r^2)^3}\{-1+(t+z)^2-x^2+y^2-2i(t+z+xy)\}$

$= \frac{-1}{((t-i)^2-r^2)^3}\{(x+iy)^2-(t-i+z)^2\}$

$= -(\mathbf{F}^{RS}(t\to -t))^*_x$

The y component is rather simple now:

$F'^{(sd)}_{02}(x) = \frac{1}{(t^2-r^2)^3}(t^2-y^2+x^2+z^2)\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-i(x+iy)^2-i(t-z+i(t^2-r^2))^2\}-2\frac{xy}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{(x+iy)^2-(t-z+i(t^2-r^2))^2\}-2\frac{yz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2(x+iy)(t-z+i(t^2-r^2))\}+2\frac{tx}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{2i(x+iy)(t-z+i(t^2-r^2))\}+2\frac{tz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{i(x+iy)^2-i(t-z+i(t^2-r^2))^2\}$

Notice: $F'^{(sd)}_{02}(x) = iF'^{(sd)}_{01}(x\to -y, y\to x)$ as can be seen from the expression for $F'^{(sd)}_{01}$ above.

So: $F'^{(sd)}_{02}(x) = \frac{-i}{((t-i)^2-r^2)^3}\{(-y+ix)^2-(t-i+z)^2\} = \frac{-1}{((t-i)^2-r^2)^3}\{-i(x+iy)^2-i(t-i+z)^2\}$

$= -(\mathbf{F}^{RS}(t\to -t))^*_y.$

What remains is the z component (in less detail than before):

With the details of $F'^{(sd)}_{01}$ fresh in mind it is not difficult to see the first step:

$F'^{(sd)}_{03}(x) = \frac{B}{(t^2-r^2)^2((t-i)^2-r^2)^3}$

in which:

$B = (t^2 + x^2 + y^2 - z^2)\{-2(x + iy)(t - z + i(t^2 - r^2))\} - 2xz\{(x + iy)^2 - (t - z + i(t^2 - r^2)^2\} + 2iyz\{(x + iy)^2 + (t - z + i(t^2 - r^2))^2\} + 2tx\{(x + iy)^2 + (t - z + i(t^2 - r^2))^2\} + 2ity\{-(x + iy)^2 + (t - z + i(t^2 - r^2))^2\}$

$= \{t - z + i(t^2 - r^2)\}^2 2(x + iy)(t + z) + (x + iy)^2 2(x - iy)(t - z) - 2(x + iy)(t - z + i(t^2 - r^2))(t^2 + x^2 + y^2 - z^2)$

$= \{(t - z)^2 - (t^2 - r^2)^2 + 2i(t - z)(t^2 - r^2)\}2(x + iy)(t + z) + (x + iy)^2 2(x - iy)(t - z) - 2(x + iy)(t - z)(t^2 - z^2 + x^2 + y^2) - 2(x + iy)i(t^2 - r^2)(t^2 - z^2 + x^2 + y^2)$

$= 2(x+iy)\{-(t^2-r^2)^2(t+z)+2i(t-z)(t+z)(t^2-r^2)-i(t^2-r^2)(t^2-z^2+x^2+y^2)\}$

$= 2(x + iy)\{-(t^2 - r^2)^2(t + z) + i(t^2 - r^2)\{t^2 - z^2 - x^2 - y^2\}\}$

$= -2(x + iy)(t^2 - r^2)^2(t - i + z)$

Thus: $F_{03}^{\prime(sd)}(x) = \frac{-1}{((t-i)^2-r^2)^3} 2(x + iy)(t - i + z)$ and this agrees with the previous results. Combining the components:

$\mathbf{F}^{RS\prime}(x) = -\{\mathbf{F}^{RS}(t \to -t)\}^*$

Q.E.D.