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Twistors and electromagnetic knots

4.1 electromagnetic field with Hopf-structure

In 1989 A.F. Rañada published an article [2] in which he showed that solutions to source free Maxwell's equations can be obtained from functions $\phi, \theta : S^3 \to S^2$ that satisfy a duality relation. By considering the Hopf map, he arrived at an electromagnetic field where the magnetic field lines are the closed curves C from chapter 2. The electric field is everywhere perpendicular to the magnetic field and the electric field lines also form a Hopf fibration. We will refer to such a configuration as an electromagnetic knot [1,4].
Here we will shortly summarize that part of Rañada's paper that will be of importance to us in the sequel, supplemented with explicit calculations for the electromagnetic field.

Start with two complex functions, $\phi(t, \mathbf{r})$ and $\theta(t, \mathbf{r})$, defined on the three-sphere. Let

$$f_{\mu\nu}(\chi) \equiv \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \chi^* \partial_\nu \chi - \partial_\nu \chi^* \partial_\mu \chi}{(1+\chi^* \chi)^2}.$$

Define $F_{\mu\nu} = f_{\mu\nu}(\phi)$ and $G_{\mu\nu} = f_{\mu\nu}(\theta)$ and suppose that F is the dual of G:

$$F_{\mu\nu} = (*G)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}.$$

When we apply ∂_β to $F_{\gamma\delta}$ we get sums of terms symmetrical in β and γ or δ. Thus $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0$. For the same reason we have $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta G_{\gamma\delta} = 0$.

Therefore, F and G satisfy the homogeneous Maxwell equations, equation (3.12). But $0 = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\partial^{\beta}G^{\gamma\delta} = \partial^{\beta}(*G)_{\alpha\beta} = \partial^{\beta}F_{\alpha\beta}$. Thus F also satisfies the source-free inhomogeneous Maxwell equation, equation (3.11). Interchange of the role of F and G in this argument shows that G also satisfies all of the (source-free) Maxwell equations.

Rewriting Maxwell's equations in the Lagrange formalism, using the duality condition as a constraint, shows that it is only necessary to have the functions ϕ and θ for $t = 0$, because the duality relation, which takes the form (or better: that implies) $(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0$ now, remains valid once it is for $t = 0$. See the appendix to this chapter for more details.

So all we need to make this prescription work are two functions, $\phi, \theta : S^3 \to S^2$ that satisfy $(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0$. The first function that comes to mind is the Hopf map $H(z^1, z^2) = (\pi_N^2)^{-1}(\frac{z^1}{z^2})$ (2.1). If we stereographically project (from N) S^3 onto $\mathbb{R}^3$ and S^2 onto $\mathbb{R}^2$, this function becomes $\phi_H(\mathbf{r}) = \phi_H(x, y, z) = \frac{2(x+iy)}{2z+i(r^2-1)}$. This follows from $\pi_N^3(z^1, z^2) = \pi_N^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}) \equiv (x, y, z)$ and $\frac{z^1}{z^2} = \frac{x^1+ix^2}{x^3+ix^4} = \frac{x+iy}{z+i\frac{x^4}{1-x^4}}$. With $r^2 = x^2 + y^2 + z^2$ we have $r^2 - 1 = \frac{2x^4}{1-x^4}$ since $||(x^1, x^2, x^3, x^4)|| = 1$. A straightforward calculation shows that $\mathbf{B}(t = 0, \mathbf{r}) \propto \nabla\phi_H^* \times \nabla\phi_H \propto \begin{pmatrix} 2(xz - y) \\ 2(x + yz) \\ 1 - x^2 - y^2 + z^2 \end{pmatrix}$. As expected the magnetic field lines coincide with the integral curves C (2.2) from chapter 2. We still need a function $\theta(0, \mathbf{r})$ that solves the duality relation. It is natural to try another Hopf structure, rotated with respect to the previous one. The level curves of the function $\theta_H(x, y, z) = \phi_H^*(y, z, x)$ certainly exhibit a Hopf structure (level curves of ϕ^* coincide with those of ϕ , but different numbers are attached to the same curves). This θ_H gives $\mathbf{E}(t = 0, \mathbf{r}) \propto \nabla\theta_H^* \times \nabla\theta_H \propto \begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix}$ and it follows that it solves the duality relation $(\nabla\phi_H^* \times \nabla\phi_h) \cdot (\nabla\theta_H^* \times \nabla\theta_H) = 0$. Therefore we finally have a solution to Maxwell's equations in vacuum in which at $t = 0$ the electric and magnetic field lines are perpendicular Hopf fibrations. In particular, any two electric (or magnetic) field lines are (possibly degenerate) circles that are linked once. Although the smooth time development according to Maxwell's equations deforms the circles into intricate closed curves, the linking property is not broken.

4.2 relation between non-null twistors and electromagnetic knots

A detailed calculation shows that the electric and magnetic field at $t = 0$ are given by the following expressions:

$$\mathbf{E}(0,\mathbf{r}) = \tfrac{1}{(1+r^2)^3}\begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix},$$

$$\mathbf{B}(0,\mathbf{r}) = \tfrac{1}{(1+r^2)^3}\begin{pmatrix} 2(xz - y) \\ 2(x + yz) \\ 1 - x^2 - y^2 + z^2 \end{pmatrix}.$$

As can be expected, the expression for both fields for arbitrary t are rather intricate, but if we combine these to form the Riemann-Silberstein vector we have a simple expression:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \tfrac{1}{((t-i)^2-r^2)^3}\begin{pmatrix} (x - iz)^2 - (t - i + y)^2 \\ 2(x - iz)(t - i + y) \\ i(x - iz)^2 + i(t - i + y)^2 \end{pmatrix}$$

(again, the speed of light has been be put equal to 1). It is easy to check that the field is null.
However, in the sequel we will often use a rotated version of this field configuration, a(n active) rotation around the x-axis, such that the direction of the z-axis goes over into the direction of the y-axis:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3}\begin{pmatrix} (x - iy)^2 - (t - i - z)^2 \\ i(x - iy)^2 + i(t - i - z)^2 \\ -2(x - iy)(t - i - z) \end{pmatrix} \tag{4.1}$$

From this it is immediately clear that the separate expressions for the fields become simple only for $t = 0$, the instant of time when the electric-, magnetic- and also the Poynting vector field all have the structure of a Hopf fibration. The general formula for the Poynting vector in terms of the Riemann-Silberstein vector

$$\mathbf{S} = \tfrac{-i}{2}\mathbf{F}^{RS*} \times \mathbf{F}^{RS}$$

gives a tractable expression for all times:

$$\mathbf{S} = \frac{(1 + x^2 + y^2 + (t - z)^2)}{((t^2 - r^2)^2 + 1 + 2(t^2 + r^2))^3}\begin{pmatrix} 2(x(t - z) + y) \\ 2(y(t - z) - x) \\ x^2 + y^2 - (t - z)^2 - 1 \end{pmatrix} \tag{4.2}$$

This shows that the *structure* (i.e. a picture of the field lines) of this vector field moves undistorted along the z-axis with the speed of light (see figure 4.1). This closely resembles the geometrical picture of a non-null twistor described at the end of chapter 1. We now present a deeper correspondence between a non-null twistor and the knotted electromagnetic field, in which the Poynting vector plays a central role. But first we note that equation (4.1) implies that the electric field can be written as a combination of two vector fields that both have the structure of a Hopf fibration that moves undistorted along the z-axis. A different combination of these two fields yield the magnetic field (with $f = f(x^\mu)$):

$$\mathbf{E}((t,\mathbf{r}) = Re(f)\begin{pmatrix}-1-x^2+y^2+(t-z)^2\\ -2((t-z)+xy)\\ 2(-y+x(t-z))\end{pmatrix} - Im(f)\begin{pmatrix}2(xy-(t-z))\\ 1-x^2+y^2-(t-z)^2\\ -2(x+y(t-z))\end{pmatrix} \tag{4.3}$$

$$\mathbf{B}((t,\mathbf{r}) = Re(f)\begin{pmatrix}2(xy-(t-z))\\ 1-x^2+y^2-(t-z)^2\\ -2(x+y(t-z))\end{pmatrix} + Im(f)\begin{pmatrix}-1-x^2+y^2+(t-z)^2\\ -2((t-z)+xy)\\ 2(-y+x(t-z))\end{pmatrix}. \tag{4.4}$$

We therefore have three orthogonal Hopf fibrations moving with the speed of light along the z-axis, one of which corresponds with the Poynting vector, and different space and time dependent combinations of the other two make up the electric and magnetic field.

The Riemann-Silberstein vector (4.1) can also be written as a combination of five time independent vector fields with time dependent coefficients:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}\{\mathbf{E}_0 + i\mathbf{B}_0 - \frac{4t}{(1+r^2)}\mathbf{A}_0 + i\frac{4t}{(1+r^2)}\mathbf{C}_0 + \frac{t^2}{(1+r^2)^3}\mathbf{W}\}. \tag{4.5}$$

In this expression $\mathbf{A}$ is the vector potential, $\mathbf{B} = \nabla\times\mathbf{A}$, and $\mathbf{C}$ is the vector potential for the electric field (since $\nabla\,.\,\mathbf{E} = 0$), $\mathbf{E} = \nabla\times\mathbf{C}$:

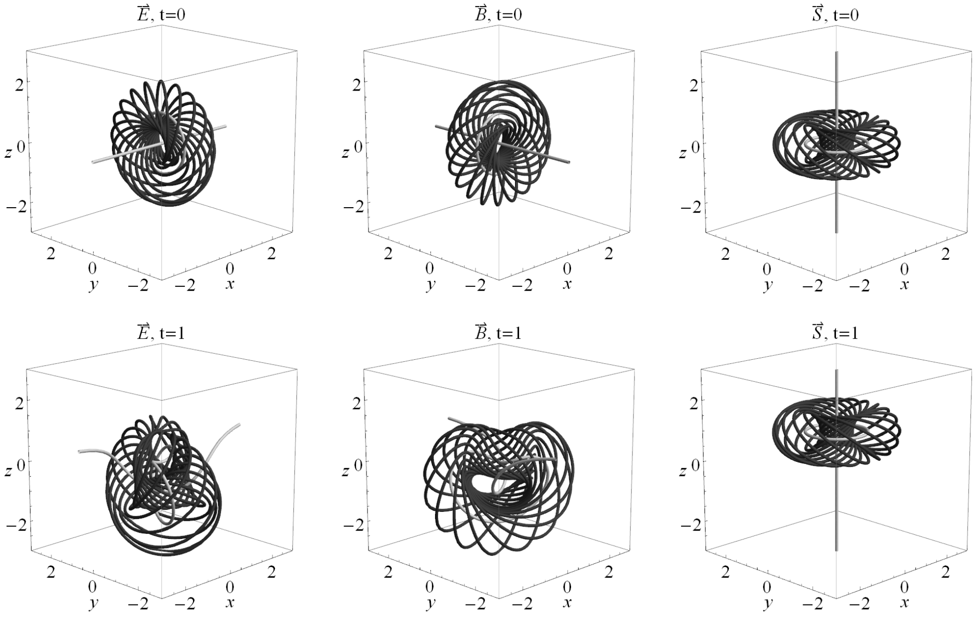

Figure 4.1: Some electric, magnetic and Poynting vector field lines of an electromagnetic knot for t=0 and t=1. At t=0 the (electric, magnetic or Poynting) field lines are singly linked circles on nested tori that fill all of space. At t=0 the structure of all three fields is identical up to rotation. Linking is conserved, but the overall structure of the electric and magnetic field lines gets strongly distorted, whereas the structure of the Poynting vector is independent of time: it moves along the central axis with the speed of light. (picture produced by A. Wickes)

$$\mathbf{A}_0 = \mathbf{A}_{t=0} = \tfrac{1}{2(1+r^2)^2}\begin{pmatrix} z \\ 1 \\ -x \end{pmatrix},\ \mathbf{C}_0 = \mathbf{C}_{t=0} = \tfrac{1}{2(1+r^2)^2}\begin{pmatrix} -1 \\ z \\ -y \end{pmatrix} \text{ and } \mathbf{W} = \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}.$$

Alternatively, $\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}\{\mathbf{F}^{RS}_{t=0} + i\frac{4t}{(1+r^2)}\mathbf{D}_{t=0} + \frac{t^2}{(1+r^2)^3}\mathbf{W}\}$,

in which $\mathbf{D} = \mathbf{C} + i\mathbf{A}$ is the vector potential for $\mathbf{F}^{RS}$. Note that $\mathbf{W}$ corresponds to a constant electromagnetic field, on which we have to say more in chapter 5.

One of the first things one learns when studying twistor theory [3, 27] is that there exists an optical analogy for a null twistor. A null twistor corresponds to a null geodesic in Minkowski space, hence to a light ray. No such direct analogy is given for a non-null twistor, although we have seen that its geometrical picture resembles the Poynting vector of an electromagnetic knot. We proceed

to show that this analogy is not merely a pictorial one.
As was shown in chapter 1, the geometrical picture corresponding to the non-null twistor $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ is defined to be the congruence in M^4 build from all null geodesics (straight lines) that correspond to null twistors $\mathcal{Z}$ orthogonal to $\mathcal{A}$: $\{\mathcal{Z}^\alpha | \bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha = 0\}$. Since $\mathcal{Z}$ must be null, it can be written as $(\mathcal{Z}^\alpha) = (iX^{AY'}\bar{\Pi}_{Y'}, \bar{\Pi}_{X'})$ with $(X^{AY'})$ given by equation (1.8) and $(t, x, y, z) \in M^4 \subset M^4_{\mathbb{C}}$. From $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ it follows that $(\bar{\mathcal{A}}_\alpha) = (0, 1, 0, \frac{1}{\sqrt{2}})$. The orthogonality condition can now be written as $(x - iy)\bar{\Pi}_{1'} + (-i + t - z)\bar{\Pi}_{2'} = 0$ and is solved by

$$(\Pi_1, \Pi_2) = f(x^\mu)(-t + z - i, x + iy), \tag{4.6}$$

in which $f(x^\mu)$ is an arbitrary scalar function that does not affect the integral curves of the corresponding vector field, known as a Robinson congruence, $k^\mu = \sigma^\mu{}_{AX'}\Pi^A\bar{\Pi}^{X'}$. Note the change from $(\bar{\Pi}_{1'}, \bar{\Pi}_{2'})$ to (Π_1, Π_2). According to Kerr's theorem (1.10) this vector field is geodesic and shear-free (in the notation of chapter 1: $w + Yv + iY \equiv 0$ here). These two properties also follow directly from $\Pi^A\Pi^B\nabla_{AX'}\Pi_B = 0$ (1.9), which is easy to check for considered spinor field, or from (1.16) with $f(\mathcal{Z}^\alpha) = \bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha$. According to a theorem by Robinson [28] it is possible to construct a null electromagnetic field from any geodesic shear free null congruence: a scalar field $\psi(t, x, y, z)$ exists such that the spinor equivalent of the anti-self-dual part of the electromagnetic field tensor is given by

$$\Phi_{AB} = e^\psi \Pi_A \Pi_B. \tag{4.7}$$

(On hindsight, looking back at the discussion following (3.16) in combination with (1.9), we could ourselves have come up with Robinson's theorem, or at least (and somewhat less presumptuous), this theorem should come as no surprise. However, Robinson wrote [28] in 1961. Not much of what is needed for the theorem was known then, and besides, spinors were not used. Even just checking [28] is not so easy a task.)
We now show that an explicit solution can be obtained in the form of an electromagnetic knot. From $F_i^{RS*} = \mathcal{F}_{0i} = \sigma_0{}^{AX'}\sigma_i{}^{BY'}\mathcal{F}_{AX'BY'} = 2\sigma_0{}^{AX'}\sigma_i{}^{BY'}\phi_{AB}\epsilon_{X'Y'}$ and using $(\sigma_0{}^{AX'}) = (\sigma^0{}_{AX'})$, $(\sigma_1{}^{AX'}) = (\sigma^1{}_{AX'})$, $(\sigma_2{}^{AX'}) = -(\sigma^2{}_{AX'})$ and $(\sigma_3{}^{AX'}) = (\sigma^3{}_{AX'})$ together with equation (1.4), we find

$$\mathbf{F}^{RS*} = \begin{pmatrix} \Phi_{00} - \Phi_{11} \\ i(\Phi_{00} + \Phi_{11}) \\ -2\Phi_{01} \end{pmatrix}. \quad (4.8)$$

This leads to the following relation between the Poynting vector corresponding to Φ_{AB} and the vector field $(k^0, \mathbf{k})$ that is related to the Robinson congruence:

$$\mathbf{S} = 2e^{\psi}e^{\psi^c}k^0\mathbf{k} \quad (4.9)$$

$$k^\alpha = \sigma^\alpha{}_{AX'}\Pi^A\bar{\Pi}^{X'}. \quad (4.10)$$

Here we already see that the structure of an electromagnetic knot emerges, since $\mathbf{k}$ is a Hopf fibration that moves with the speed of light along the z-axis. For null electromagnetic fields and given Poynting vector, the only freedom left for the electric and magnetic field is a local duality transformation:

$$\mathbf{F}^{RS\prime} = e^{i\alpha}\mathbf{F}^{RS} \quad (4.11)$$

in which $\alpha(x^\mu)$ stisfies

$$\nabla\alpha = \beta(x^\mu)\mathbf{S} \quad (4.12)$$

and

$$\partial_t\alpha = -\beta(x^\mu)||\mathbf{S}|| \quad (4.13)$$

for $\beta(x^\mu)$ an arbitrary function. This follows from writing out Maxwell's equations for $\mathbf{F}^{RS\prime}$ and for $\mathbf{F}^{RS}$ (3.9). For $\beta = 0$, α is constant and the duality transformation is global. For β constant, α satisfies the homogeneous wave equation as follows from the for null fields valid equation $\nabla \cdot \mathbf{S} = -\partial_t||\mathbf{S}||$.
The general formula for the energy density, ϵ, in terms of the Riemann-Silberstein

vector

$\epsilon = \frac{1}{2}\mathbf{F}^{RS*} \cdot \mathbf{F}^{RS}$

when combined with equation (4.7) leads to

$\epsilon = 2e^{\psi}e^{\psi^c}(k^0)^2.$

Together with equation (4.9), this gives

$(E_{field}, \mathbf{P}_{field}) = 2\int d\mathbf{r} e^{\psi}e^{\psi^c}k^0(k^0, \mathbf{k})$

which for source-free fields is an (energy-momentum) four-vector [29].
Combining equations (4.6), (4.7) and (4.8) leads directly to the electromagnetic field that corresponds to the twistor $(\mathcal{A}^\alpha)$:

$$\mathbf{F}^{RS} = g(x^\mu)\begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix}$$

in which $g(x^\mu)$ is a function built from $f(x^\mu)$ and ψ. When compared with equation (4.1), we see that with the choice

$g(x^\mu) = \frac{1}{((t-i)^2 - r^2)^3},$

we indeed do get a solution to Maxwell's equations. The mathematical relation between a non-null twistor and an electromagnetic knot is hereby established. However, we still do need to investigate the freedom left in the choice of $g(x^\mu)$.

4.2.1 relation is not unique

Any solution of Maxwell's equations with the property of having a Poynting vector that has the structure of a Hopf fibration that moves undistorted with the speed of light along the central axis will be referred to as an electromagnetic Hopf knot. From equations (4.6), (4.9) and (4.10), it is clear that any choice of $g(x^\mu)$ that is allowed by Maxwell's equations corresponds to such a knot. We have seen that among the solutions is the particular knot given by equation (4.1), say $\mathbf{F}^{part.}$ and corresponding $\mathbf{S}^{part.}$ (4.2). From equations (4.6)-(4.10) we see that for any other solution there exist functions $\tilde{\psi}$ and $\tilde{f}$ such that

$\mathbf{F}^{RS} = h(x^\mu)\mathbf{F}^{part.}, \quad \mathbf{S} = h(x^\mu)h^*(x^\mu)\mathbf{S}^{part.}$ and $h(x^\mu) = e^{\tilde{\psi}}(\tilde{f})^2.$

Again, we see explicitly that all solutions are electromagnetic Hopf knots.

Those that have $\mathbf{S} = \mathbf{S}^{part.}$ can be obtained from the particular knot by a local duality transformation, see equations (4.11), (4.12) and (4.13). The case $|h(x^\mu)| \neq 1$ might lead to different solutions, but not to different structures, since at every point the Poynting vector is multiplied by a positive number and (because $\mathbf{E}\cdot\mathbf{B} = 0$ and $||\mathbf{E}|| = ||\mathbf{B}||$) the only new consequence is that at each point $||\mathbf{E}||$ gets multiplied by the square root of this number.
Write $h = |h|e^{i\alpha}$ and consider Maxwell's equations for $\mathbf{F}^{RS} = h(x^\mu)\mathbf{F}^{part.}$:
$\nabla\cdot\mathbf{F}^{RS} = 0$ gives

$$\nabla|h|\cdot\mathbf{E}^{part.} - |h|\nabla\alpha\cdot\mathbf{B}^{part.} = 0 \tag{4.14}$$

$$\nabla|h|\cdot\mathbf{B}^{part.} + |h|\nabla\alpha\cdot\mathbf{E}^{part.} = 0 \tag{4.15}$$

$\nabla\times\mathbf{F}^{RS} = i\partial_t\mathbf{F}^{RS}$ gives

$$\nabla|h|\times\mathbf{E}^{part.} - |h|\nabla\alpha\times\mathbf{B}^{part.} = -\mathbf{B}^{part.}\partial_t|h| - |h|\mathbf{E}^{part.}\partial_t\alpha \tag{4.16}$$

$$\nabla|h|\times\mathbf{B}^{part.} + |h|\nabla\alpha\times\mathbf{E}^{part.} = \mathbf{E}^{part.}\partial_t|h| - |h|\mathbf{B}^{part.}\partial_t\alpha. \tag{4.17}$$

Now suppose α satisfies equations (4.12) and (4.13) for some β and $\mathbf{S} = \mathbf{S}^{part.}$. The equations (4.14) - (4.17) then become

$\nabla|h|\cdot\mathbf{E}^{part.} = 0$, $\nabla|h|\cdot\mathbf{B}^{part.} = 0$, $\nabla|h|\times\mathbf{E}^{part.} = -\mathbf{B}^{part.}\partial_t|h|$ and $\nabla|h|\times\mathbf{B}^{part.} = \mathbf{E}^{part.}\partial_t|h|$.

The first two equations imply $\nabla|h| = \gamma(x^\mu)\mathbf{S}^{part.}$ for some function $\gamma(x^\mu)$, and this combined with the last two equations give $\partial_t|h| = -\gamma(x^\mu)||\mathbf{S}^{part.}||$. The norm function $|h|$ thus satisfies the same equations as the phase function α, except for a (possibly) different function multiplying the Poynting vector.
We proceed to show that this exhausts the possibilities for solving Maxwell's equations with given Poynting vector $\mathbf{S} = h(x^\mu)h^*(x^\mu)\mathbf{S}^{part.}$. Suppose $\alpha(x^\mu) = \alpha_E(x^\mu) + \alpha_B(x^\mu) + \alpha_S(x^\mu)$ with $\nabla\alpha_S = s(x^\mu)\mathbf{S}^{part.}$, $\nabla\alpha_E = e(x^\mu)\mathbf{E}^{part.}$, $\nabla\alpha_B = b(x^\mu)\mathbf{B}^{part.}$ and e, b not both equal to 0. From equation (4.16) it follows that $(\nabla\alpha\times\mathbf{B}^{part.})\cdot\mathbf{E}^{part.} = ||\mathbf{S}^{part.}||\partial_t\alpha$ and, given the assumption for α, this leads to $\partial_t\alpha = -s(x^\mu)||\mathbf{S}^{part.}||$. This equation is used in that part of equation (4.16) that is perpendicular to $\mathbf{B}^{part.}$ to obtain $e(x^\mu) = 0$ and in that part

of equation (4.17) that is perpendicular to $\mathbf{E}^{part.}$ to obtain $b(x^\mu) = 0$. This contradicts the assumption.
The requirements on $|h|$ and α define an equivalence relation and partitions the electromagnetic Hopf knots into equivalence classes. A class is defined by the structure (i.e. integral curves) of the Poynting vector and again within this class an equivalence relation is defined by the condition on $\alpha(x^\mu)$. This subclass is defined by the Poynting vector, not only its structure but also its norm. This result is obtained as follows. For $\mathbf{F}' \sim \mathbf{F} \Leftrightarrow \mathbf{F}' = ge^{i\beta}\mathbf{F}$, reflexivity is achieved by $g = 1$ and $\beta = 0$, symmetry by $g \to \frac{1}{g}$ and $\beta \to -\beta$, and multiplication of norm functions and addition of phases supplies transitivity.

The question arises whether there exist a local duality transformation satisfying (4.12) and (4.13) such that the electric and magnetic field lines also move undistorted in time, like the structure of the Poynting vector. This will be the case if there exists a function $h(x^\mu)$ such that $\frac{h(x^\mu)}{((t-i)^2-r^2)^3} \equiv G(x^\mu)$ is a real function (see (4.3) and (4.4)), and $|h(x^\mu)|$ and $\alpha \equiv Arg(h(x^\mu))$ both satisfy equations (4.12) and (4.13) with a function $\beta = \beta_1$ and $\beta = \beta_2$ respectively. This leads to:

from the condition on $|h(x^\mu)|$:

$$((t^2+1-r^2)^2+4r^2)\partial_t|G| + 6|G|(t^2+1-r^2)t = -\frac{\beta_1}{\sqrt{((t^2+1-r^2)^2+4r^2)}}||\mathbf{S}||$$

and

$$((t^2+1-r^2)^2+4r^2)\nabla|G| - 6|G|(t^2-1-r^2)\mathbf{r} = \frac{\beta_1}{\sqrt{((t^2+1-r^2)^2+4r^2)}}\mathbf{S}$$

from the condition on α, split into real and imaginary parts:

$$(t^2-1-r^2)\partial_t G + 6tG = -(2t\beta_2 + \frac{\beta_1(t^2-1-r^2)}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G||\mathbf{S}||,$$

$$2t\partial_t G + 6G = ((t^2-1-r^2)\beta_2 - \frac{2t\beta_1}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G||\mathbf{S}||$$

and

$$(t^2-1-r^2)\nabla G - 6G\mathbf{r} = (2t\beta_2 + \frac{\beta_1(t^2-1-r^2)}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G\mathbf{S},$$

$$2t\nabla G = -((t^2-1-r^2)\beta_2 - \frac{2t\beta_1}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G\mathbf{S}.$$

These are six real equations for three real functions, β_1, β_2 and G. From the structure of the Poynting vector we can conclude that $G = 0$ does not appear, so either $|G| = G$ or $|G| = -G$ everywhere. From the three equations containing time derivatives or the three vector equations, it is easy to show by elimination of β_2 that the only possibility is $|G| = G$. Elimination of β_1 from the vector equations now leads to a contradiction. So the "best" we can achieve in this respect is the decomposition of (4.3) and (4.4).

4.2.2 changing the twistor

A different choice for the non-null twistor $\mathcal{A}^\alpha$ will lead to a different class of electromagnetic Hopf knots. It is instructive to calculate the electromagnetic field that corresponds to the arbitrary non-null twistor $\mathcal{A}^\alpha = (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}, c, d)$. The analogue of (4.6) is

$$(\Pi_1, \Pi_2) = f(x^\mu)(c(x-iy) + d(t-z) + ib, -c(t+z) - d(x+iy) - ia)$$

which leads to

$$\Phi_{00} = e^\psi f^2[c(x-iy) + d(t-z) + ib]^2 \tag{4.18}$$

$$\Phi_{11} = e^\psi f^2[c(t+z) + d(x+iy) + ia]^2 \tag{4.19}$$

$$\Phi_{01} = -e^\psi f^2[c(x-iy) + d(t-z) + ib][c(t+z) + d(x+iy) + ia]. \tag{4.20}$$

From these, remembering (4.8), interesting properties can be read off.
For example, $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$ and $b \in \mathbb{R}\backslash\{0\}$ leads to the same solution space as before, but with x^μ replaced by $(x')^\mu = \frac{1}{b}x^\mu$. For $b > 0$ this amounts to an expansion ($b > 1$) or contraction ($b \in \langle 0, 1\rangle$) of previous solutions. For $b < 0$ there is an additional parity change and time reversal. All of these will be referred to as dilations. This interpretation of b remains valid for $c \neq 0$, but not when $a \neq 0$ unless a and b change both with the same factor. In general, for a non-null ($Re(b) \neq 0$) twistor $\mathcal{A}^\alpha = (a, b, 0, 1)$, we find that $Re(b)$ is related to dilations whenever $\frac{a}{b}$ is constant, $Im(b)$ to time translations or, equivalently, z-translations and that $Re(a)$ is related to y-translations and $Im(a)$ to translations in x direction.
When we interchange the components of the spinors that constitute the twistor and negate a, $(a, b, 0, 1) \to (b, -a, 1, 0)$, we find that the new solutions will be obtained from the old ones by a conformal inversion, $x^\mu \to \frac{x^\mu}{x_\nu x^\nu}$ (the general rule is $\mathbf{F}^{RS*} \to -\mathbf{F}^{RS}(t \to -t)$, which for Hopf knots amounts to a conformal inversion, see chapter 5). The interpretation of a and b is exactly as before.
The class of solutions of Maxwell's equations that is obtained for $a = b = c = d = 1$ is:

$$\mathbf{E} = Re(g(x^\mu)) \begin{pmatrix} 2(y - z(t+x)) \\ -2(yz + (t+x)) \\ (t+x)^2 + y^2 - z^2 - 1 \end{pmatrix} - Im(g(x^\mu)) \begin{pmatrix} 2(z + y(t+x)) \\ 1 + y^2 - z^2 - (t+x)^2 \\ 2(yz - (t+x)) \end{pmatrix},$$

$$\mathbf{B} = Re(g(x^\mu)) \begin{pmatrix} 2(z + y(t+x)) \\ 1 + y^2 - z^2 - (t+x)^2 \\ 2(yz - (t+x)) \end{pmatrix} + Im(g(x^\mu)) \begin{pmatrix} 2(y - z(t+x)) \\ -2(yz + (t+x)) \\ (t+x)^2 + y^2 - z^2 - 1 \end{pmatrix},$$

$$\mathbf{S} = (1 + (t+x)^2 + y^2 + z^2)(g(x^\mu))(g(x^\mu))^* \begin{pmatrix} 1 + (t+x)^2 - y^2 - z^2 \\ 2(y(t+x) - z) \\ 2(y + z(t+x)) \end{pmatrix}.$$

It can be seen that a representative of this class is given by a rotated version of (4.1), the axis of rotation is the y-axis and it is such that the +z-axis rotates to the +x-axis.

4.3 the method applied to a null twistor

We have seen two routes to visualize a twistor. The first is valid for a null twistor and gives a ray of light, the second for a non-null twistor and gives a class of electromagnetic Hopf knots. What happens if we use the second method for a null twistor? Applying this method to $(\mathcal{A}^\alpha) = (0, 0, 0, 1)$ leads to the class of

$$\mathbf{F}^{RS} = \frac{1}{(t^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t-z)^2 \\ i(x - iy)^2 + i(t-z)^2 \\ -2(x - iy)(t - z) \end{pmatrix}, \tag{4.21}$$

with corresponding Poynting vector

$$\mathbf{S} \propto \begin{pmatrix} 2x(t-z) \\ 2y(t-z) \\ x^2 + y^2 - (t-z)^2 \end{pmatrix}. \tag{4.22}$$

Changing this null twistor into the non-null $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$ amounts to applying an imaginary time translation $t \to t - ib$ to the field (4.21), as can be seen from (4.18)-(4.20). Since for $b = 1$, $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ corresponds to (4.1) this implies that an electromagnetic Hopf knot is obtained from (4.21) by the transformation $t \to t - i$, as can be verified by direct comparison. For this reason

we shall refer to the null congruence corresponding to (4.21) as a degenerate Robinson congruence in the sequel.
The null twistor $(\mathcal{A}^\alpha) = (0, 1, 0, 0)$ leads to (the class of) plane waves travelling in the -z direction (see the examples at the end of chapter 3), and $(\mathcal{A}^\alpha) = (1, 0, 0, 0)$ to plane waves propagating in the z direction. As will be shown in chapter 5, a conformal inversion of this last field in the limit $k = 0$ (thus a constant electromagnetic field) leads to (4.21). Therefore we have the surprising sequence:

$$\mathbf{E} = \hat{x}, \mathbf{B} = \hat{y} \rightarrow \text{(4.21)} \rightarrow \text{(4.1)}, \tag{4.23}$$

where the first arrow indicates a conformal inversion and the second an imaginary time translation. $(\mathcal{A}^\alpha) = (1, 1, 0, 0)$ leads to a plane wave propagating in the x direction.
As a final example, the null twistor $(\mathcal{A}^\alpha) = (0, 0, 1, 1)$ gives the class of solutions for which the $\frac{\pi}{2}$ rotated version (with y-axis as rotation axis) of (4.21) is a representative.
The light rays that correspond to these null twistors via the first method coincide with the unique direction defined by the Poynting vector of the electromagnetic fields that correspond to these twistors via the second method. From equations (4.9), (4.10) and (1.14) it follows that this is true in general, and the second method can therefore be considered as a generalization of the first.

4.4 shift in the complex domain

By considering Maxwell's equations for the Riemann-Silberstein vector (3.9), complex numbers have been introduced in electrodynamics. On various occasions we have seen that it is a convenient way to combine electric and magnetic fields into one vector. The complex structure here acts as a bookkeeping device, to distinguish between electric and magnetic parts. The complex time shift in the previous paragraph hints at another possible role for a complex structure. The source-free Maxwell's equations for the Riemann-Silberstein vector are invariant under all the transformations that have been obtained in the last two paragraphs (dilation, (complex) translation, rotation, conformal inversion). Let us consider these equations as being defined not only in M^4, but in $M^4_{\mathbb{C}}$. It is tempting to assume that all the aforementioned electromagnetic fields that correspond to different kinds of twistors are part of one (super)solution to these Maxwell equations in $M^4_{\mathbb{C}}$. Depending on where we agree to locate the real slice M^4 in $M^4_{\mathbb{C}}$ (and this really is a matter of choice) we do get one of the aforementioned solutions on this slice. For example, for a particular choice of real space-time we have solution (4.21). If now we shift the choice of real slice to a different part of

$M^4_{\mathbb{C}}$ related to the previous choice by $t \to t - i$, we get the solution (4.1) on this space-time. Pushing the real slice still further into the imaginary time direction, $t \to t - ib$ ($b > 1$) we get the expanded version of (4.1).
A careful analysis is needed to show whether this assumption can stand the test of mathematical scrutiny, and to disclose the possible relationship with work of E. Newman [9,30–34] and of A. Burinskii [35,36].

A somewhat related question is the following. Dilations are part of the conformal group $C(1,3)$. The transformations in twistor space that leave invariant the inner product belong to $SU(2,2)$ which is a fourfold cover of the conformal group. Does a $SU(2,2)$ transformation of a twistor always correspond to a particular conformal transformation of the corresponding electromagnetic field? Of course, the norm of $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ is in general not the same as that of $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$, but this last twistor leads to the same class of solutions as some constant times this twistor. The constant can be adjusted so that the two twistors are related by a $SU(2,2)$ transformation.

4.5 conclusion and final remarks

In this chapter several forms to represent an electromagnetic Hopf knot have been discussed. Using a theorem by Robinson, we presented a correspondence between a non-null twistor and a class of solutions of the source-free Maxwell equations. Based on the relation (4.9) the class is characterized by the structure of the Poynting vector. All solutions within this class are electromagnetic Hopf knots. All knots that constitute one class can be parametrized by a complex function $h(x^\mu) = |h|e^{i\alpha}$ in which both $\alpha(x^\mu)$ and $|h(x^\mu)|$ satisfy equations (4.12) and (4.13): a restricted local duality transformation. A different twistor will correspond to a different class of knots only when the twistor is non-null. The method can be applied to a null twistor, but will lead to a class of solutions different from electromagnetic Hopf knots.

The main results of this chapter were published in [4]. Later, in [37], B.H. Trishin discussed part of present calculations from a different perspective. He used the Newman-Penrose formalism, the Kerr theorem and null coordinates as will be discussed in chapter 8, to arrive at the general solution to our equation (4.7), translated into the Riemann-Silberstein vector as:

$$\mathbf{F}^{RS} = (1 + \bar{w}\partial_u Y + v\partial_w Y) f(u + \bar{w}Y, w + vY) \begin{pmatrix} 1 - Y^2 \\ i(1 + Y^2) \\ -2Y \end{pmatrix},$$

in which f is an arbitrary holomorphic function. The notation is that of chapter 8. It is interesting to note that when we use the congruence that corresponds to the Schwarzschild solution of Einsteins equations (according to chapter 8 we then have $Y = -\frac{z+r}{x-it}$) we do get the following expressions for the electric, magnetic and Poynting vector field:

$$\mathbf{E} \propto \begin{pmatrix} -(y^2+z^2+zr) \\ xy \\ x(z+r) \end{pmatrix}, \quad \mathbf{B} \propto \begin{pmatrix} -xy \\ x^2+z^2+zr \\ -y(z+r) \end{pmatrix} \quad \text{and} \quad \mathbf{S} \propto \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The electromagnetic torus knots described in [38] (published october 2013) all have the same Poynting vector as the electromagnetic Hopf knot presented in this chapter. These torus knots thus belong to the same equivalence class as the Hopf knot. For every torus knot, there exist therefore a local duality transformation (4.11), (4.12) and (4.13) relating it to the Hopf knot. This argument might be used to show that in the case of torus knots all the field lines are closed. In [38] this is shown only for the "core field lines" that correspond to degenerate tori.
It is also of some interest to note that the β appearing in [38] and describing part of the Bateman construction for generating null electromagnetic fields, corresponds to Y in the method of Debney, Kerr and Schild as presented here in chapter 8. Does this give a clue of how to arrive at exact solutions of Einstein equations that can be described as gravitational torus knots?

Appendix

From $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k \Leftrightarrow B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$ we have $\mathbf{E}\cdot\mathbf{B} = E_iB_i = -\frac{1}{2}\epsilon_{ijk}F_{0i}F_{jk}$. For $F_{\mu\nu} = f_{\mu\nu}(\phi)$ we see that $F_{0i}F_{jk}$ contains only factors symmetrical in i and j or k:

$\mathbf{E}\cdot\mathbf{B} = 0$

Since $\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$ we have $G_{\mu\nu} = -(*F)_{\mu\nu}$ and so $G_{0i} = -B_i$ and $G_{ij} = -\epsilon_{ijk}E_k \Leftrightarrow E_i = -\frac{1}{2}\epsilon_{ijk}G_{jk}$.

Using $E_i = F_{0i}$ we get

$\mathbf{E}(0,\mathbf{r}) = \frac{\sqrt{a}}{2\pi i}\frac{\partial_0\phi^*(0,\mathbf{r})\nabla\phi(0,\mathbf{r})-\partial_0\phi(0,\mathbf{r})\nabla\phi^*(0,\mathbf{r})}{(1+\phi^*(0,\mathbf{r})\phi(0,\mathbf{r}))^2}$

and from $E_i = -\frac{1}{2}\epsilon_{ijk}G_{jk}$ we get

$\mathbf{E}(0,\mathbf{r}) = -\frac{1}{2}\epsilon_{ijk}\frac{\sqrt{a}}{2\pi i}\frac{\partial_j\theta^*(0,\mathbf{r})\partial_k\theta(0,\mathbf{r})-\partial_j\theta(0,\mathbf{r})\partial_k\theta^*(0,\mathbf{r})}{(1+\theta^*(0,\mathbf{r})\theta(0,\mathbf{r}))^2}.$

Combining these expressions makes it possible to calculate $\partial_0\phi$ and a similar argument shows the possibility of calculating $\partial_0\theta$ from $\phi(0,\mathbf{r})$ and $\theta(0,\mathbf{r})$.

The points in $\mathbb{R}^3$ for which ϕ takes a constant value consists of the intersection between two surfaces, one for which the real part of ϕ is constant and the other for which the imaginary part of ϕ is constant. Very special cases aside, this intersection is a one dimensional curve. The magnetic field lines coincide with these level curves as follows from the following consideration. We have $\nabla\phi\cdot\mathbf{B} = (\partial_i\phi)B_i = -\frac{1}{2}\epsilon_{ijk}(\partial_i\phi)F_{jk} = 0$, since $(\partial_i\phi)F_{jk}$ contains only terms that are symmetrical in i and j or k. For the same reasons we also have $\nabla\phi^*\cdot\mathbf{B} = 0$, and thus $\mathbf{B} \propto \nabla\phi^* \times \nabla\phi$. A similar argument shows that the electric field lines coincide with the level curves of θ. We now see that the duality condition implies $(\nabla\phi^* \times \nabla\phi)\cdot(\nabla\theta^* \times \nabla\theta) = 0$.