Author: Dalhuisen, Jan Willem

Title: The Robinson congruence in electrodynamics and general relativity

Issue Date: 2014-03-25

3 Electrodynamics

Among all the great theories of physics, the one completed by Maxwell about a century and a half ago takes a very special place. Maxwell's equations show that the speed of light is the same in all inertial reference frames. Whereas many tried to resolve this "problem" Einstein derived special relativity from it. It serves as a paradigm in gauge field theory and it is here that almost everyone starts learning about that. Besides being aesthetically very appealing, it is very hard to overrate its technological significance or its value for society in general (try to imagine a world without knowledge of electricity). Accordingly, the effort by engineers and scientists to solve the central equations of this theory, to understand and make use of these solutions has been huge. The more surprising it is that even now amazing new solutions are being found. [1, 2]. In the next chapter we will have a closer look at one such solution in which the Hopf fibration plays an important role.

In this chapter we want to rewrite Maxwell's equations in the language of spinors that was introduced in chapter 1.

3.1 Maxwell equations

In nondimensionalized (Heaviside-Lorentz, c=1) units Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = \rho \tag{3.1}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{3.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3.4}$$

The electromagnetic field $(\mathbf{E}, \mathbf{B})$ can be characterized in terms of potentials ϕ and $\mathbf{A}$ as follows:

$\mathbf{B} = \nabla \times \mathbf{A}$
$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$.

Associated with $(\mathbf{E}, \mathbf{B})$ are the energy density w, energy flux (Poynting vector) $\mathbf{S}$, momentum density $\mathbf{p}$ and stress tensor σ^{ij} given by:

$$w = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \tag{3.5}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} \tag{3.6}$$

$$\mathbf{p} = \mathbf{E} \times \mathbf{B} \tag{3.7}$$

$$\sigma^{ij} = -E^i E^j - B^i B^j + \delta^{ij} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}. \tag{3.8}$$

Here, as elsewhere, $i, j, k \in [1, 2, 3]$ and Greek indices will indicate an element from $[0, 1, 2, 3]$, except when it is used for a twistor (see section **1.3**).
An electromagnetic field that satisfies $|\mathbf{E}| = |\mathbf{B}|$ and $\mathbf{E} \cdot \mathbf{B} = 0$ is called null.

3.2 Riemann-Silberstein vector

A convenient way to rewrite Maxwell's equations is with the help of the Riemann-Silberstein vector $\mathbf{F}^{RS} = \mathbf{E} + i\mathbf{B}$. It's complex conjugate will be denoted by $\mathbf{F}^{RS*}$. Maxwell's equations become

$$\nabla \cdot \mathbf{F}^{RS} = \rho, \qquad \partial_t \mathbf{F}^{RS} = -i\nabla \times \mathbf{F}^{RS} - \mathbf{j}. \tag{3.9}$$

The Poynting vector and energy density of the electromagnetic field can be written as

$\mathbf{S} = \frac{i}{2}\mathbf{F}^{RS} \times \mathbf{F}^{RS*}$ and $w = \frac{1}{2}\mathbf{F}^{RS} \cdot \mathbf{F}^{RS*}$.

A null electromagnetic field satisfies $\mathbf{F}^{RS} \cdot \mathbf{F}^{RS} = 0$.

3.3 Faraday tensor

Maxwell's equations can be written in manifest Lorentz covariant form with the help of the electromagnetic field tensor (or Faraday tensor):

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3.10}$$

and the (four-)current density $(\rho, \mathbf{j})$ as follows:

$$\partial_\nu F^{\mu\nu} = -j^\mu \tag{3.11}$$

and

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{3.12}$$

In terms of the (four-)potential $(\phi, \mathbf{A})$ we can write $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Other forms of the electromagnetic field tensor are:

$$(F^\mu{}_\nu) = (F^{\mu\beta}\eta_{\beta\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

$$(F_{\mu\nu}) = (\eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

$$((*F)^{\mu\nu}) = (\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

It follows that $F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$ and $(*F)^{\mu\nu}F_{\mu\nu} = -\mathbf{E}\cdot\mathbf{B}$ and therefore we have for a null electromagnetic field: $F_{\mu\nu}F^{\mu\nu} = 0 = F_{\mu\nu}(*F)^{\mu\nu}$.

The energy density, energy flux, momentum density and stress tensor ((3.5) - (3.8)) are combined in the symmetrical energy-momentum tensor of the electromagnetic field

$$T^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \begin{pmatrix} w & \mathbf{S} \\ \mathbf{S} & \boldsymbol{\sigma} \end{pmatrix} \tag{3.13}$$

Note that (3.12) does not contain 4^3 independent equations, but only 4 as becomes clear by rewriting this equation in the form $\epsilon_{\alpha\beta\gamma\delta}\partial^\beta F^{\gamma\delta} = 0$. This in turn can be written in terms of the dual of $F^{\mu\nu}$: $\partial^\beta(*F)_{\alpha\beta} = 0$, or

$$\partial_\nu(*F)^{\mu\nu} = 0$$

and this is like the source-free form of (3.11). Therefore, with the definition

$$\mathcal{F}^{\mu\nu} = F^{\mu\nu} + i(*F)^{\mu\nu}$$

all of Maxwell's equations (3.1) - (3.4) can be written succinctly as

$$\partial_\nu \mathcal{F}^{\mu\nu} = -j^\mu. \tag{3.14}$$

In terms of the Riemann-Silberstein vector we have

$$(\mathcal{F}^{\mu\nu}) = \begin{pmatrix} 0 & -F_x^{RS*} & -F_y^{RS*} & -F_z^{RS*} \\ F_x^{RS*} & 0 & -iF_z^{RS*} & iF_y^{RS*} \\ F_y^{RS*} & iF_z^{RS*} & 0 & -iF_x^{RS*} \\ F_z^{RS*} & -iF_y^{RS*} & iF_x^{RS*} & 0 \end{pmatrix}. \tag{3.15}$$

3.4 Maxwell spinor

Since the electromagnetic field tensor is anti-symmetric, we know from (1.7) that its spinor equivalent can be written as

$$F_{AX'BY'} = \phi_{AB}\epsilon_{X'Y'} + \bar{\phi}_{X'Y'}\epsilon_{AB}, \quad \phi_{AB} = \tfrac{1}{2}F_{AR'B}{}^{R'} = \phi_{BA}.$$

This ϕ_{AB} is called Maxwell spinor. From the formula for the dual of an anti-symmetric tensor (1.8) we find for the spinor equivalent of $\mathcal{F}_{\mu\nu}$:

$$\tfrac{1}{2}\mathcal{F}_{AX'BY'} = \phi_{AB}\epsilon_{X'Y'}$$

and for Maxwell's equations (3.14) without source term (after some index manipulation):

$$\nabla^{AX'}\phi_{AB} = 0. \tag{3.16}$$

We see that when ϕ_{AB} is a solution, $f\phi_{AB}$ is also a solution whenever $\alpha^A\nabla_{AX'}f = 0$. It is a simple exercise to show that for a null electromagnetic field the principal spinors and thus the principal null directions of ϕ_{AB} coincide: $\phi_{AB} = \alpha_A\alpha_B$ and therefore $\phi_{AB}\phi^{AB} = 0$. In this case equation (3.16) implies $\alpha^A\alpha^B\nabla_{AX'}\alpha_B = 0$, as can be seen by substitution and contraction with α^B. Thus the congruence defined by α is geodesic and shear free (see (1.9)).

In anticipation of the Petrov classification in general relativity (section **7.6**) it may be remarked here that the Maxwell spinor gives rise to a simple classification of electromagnetic fields. For this we use the two principal null directions (PND) of the symmetric Maxwell spinor:

regular electromagnetic field: the PND are different,
null electromagnetic field: the PND coincide.

From section **1.2** we know that the principal null directions of the Maxwell spinor correspond to the eigenvectors of the Faraday tensor. The two eigenvalues turn out to be $\lambda_\pm = \pm\sqrt{-\frac{1}{4}(|\mathbf{B}|^2 - |\mathbf{E}|^2) - \frac{i}{2}\mathbf{E}\,\mathbf{.}\,\mathbf{B}}$. As can be seen using equation (4.8) from chapter 4, this is also the solution for to the following equation: ${\phi_A}^B\beta_B = \lambda_\pm\beta_A$. Combining previous results, we see that:

field is null $\leftrightarrow$ $|\mathbf{B}| = |\mathbf{E}|,\ \mathbf{E}\,\mathbf{.}\,\mathbf{B} = 0 \ \leftrightarrow\ \lambda_\pm = 0 \ \leftrightarrow\ \phi_{AB} = \alpha_A\alpha_B.$

The first arrow is the definition of section **3.1**, the second arrow is a result from this section, and the last arrow follows from section **1.2**.

3.5 simple examples of Maxwell spinors

We now give some simple examples in which the dependence on the space-time coordinates only appears in a common factor. Except in the last example, all fields are circularly polarized.

plane wave in -z direction

$(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$:

This is a solution of (3.16) if $f(x^\mu)$ satisfies $\partial_t f = \partial_z f$ and $(\partial_x + i\partial_y)f = 0$. For continuous f the last equation implies that it is holomorphic in $x + iy$. The first equation is satisfied whenever f is a function of the combination $t + z$. For $f(x^\mu) = e^{ik_\mu x^\mu}$ and $(k_\mu) = (k, 0, 0, k)$ with $k > 0$ the electromagnetic field is a right circularly polarized plane wave travelling in the -z direction. Using $f(x^\mu) = e^{-ik_\mu x^\mu}$ with the same (k_μ) leads to a left circularly polarized plane wave propagating in the same direction.

plane wave in +z direction

When instead of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we use $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ the function $f(x^\mu)$ must be anti-holomorphic (a function of the combination $x - iy$ and not $x + iy$) and with respect to t and z it can only depend on the combination $t - z$. Using the same $f(x^\mu)$ as before, but now with $(k_\mu) = (k, 0, 0, -k)$, leads to plane waves travelling in the +z direction. The polarization interpretation is the same as before.

plane wave in -x direction

$(\phi_{AB}(x^\mu)) = f(x^\mu) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

Now $f(x^\mu)$ has to satisfy $(\partial_t - \partial_z - \partial_x + i\partial_y)f = 0$ and $(-\partial_x - i\partial_y + \partial_t + \partial_z)f = 0$, which is equivalent to $\partial_t f = \partial_x f$ and $(\partial_z - i\partial_y)f = 0$. So $f(x^\mu)$ is a function of $z - iy$ and $t + x$. Again, we can use the same $f(x^\mu)$ as before, now with $(k_\mu) = (k, k, 0, 0)$ and we have a plane wave travelling in the -x direction with amplitude twice as big as before. The interpretation of polarization is the same as in the previous cases.

plane wave in +x direction

When instead of using $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ we use $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ the function $f(x^\mu)$ depends on $z - iy$ and $t - x$. Again using the same $f(x^\mu)$ leads to propagation in the x direction and polarizations as before.

plane wave in $\pm$ y direction

$(\phi_{AB}(x^\mu)) = f(x^\mu) \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$:

Now f is a function of $x + iz$ and $t + y$. The same f as before with $(k_\mu) = (k, 0, k, 0)$ leads to plane waves propagating in the -y direction and polarizations as before. Complex conjugation of the matrix entries leads to propagation in the y direction.

constant fields

Note that $(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$ are only possible for constant f and leads to a constant electric field without magnetic field and a constant magnetic field without an electric field respectively.