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The Hopf fibration

Next we present a detailed and elementary study of the leading part in this thesis, the Hopf fibration. Amazingly, this somewhat contrived structure is relevant in various situations in physics.

2.1 stereographic projection

Let S^2 be the two-sphere in $\mathbb{R}^3$: $S^2 = \{x \in \mathbb{R}^3 | \; ||x|| = 1\}$, $N = (0,0,1)$ the north-pole of S^2 and $S = (0,0,-1)$ its south-pole. $||x||$ denotes the Euclidian norm of x. Define S^2_S to be the two-sphere without its north-pole: $S^2_S = S^2 \setminus N$, and similarly $S^2_N = S^2 \setminus S$.
The stereographic projection from the north-pole, π^2_N, maps points from S^2_S to $\mathbb{R}^2$, seen as the plane through the equator of S^2, in the following way.
Let (x^1, x^2, x^3) be a point in S^2. The line through N and this point is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} x^1 \\ x^2 \\ x^3 - 1 \end{pmatrix}.$$

This line intersects the equator-plane in the point where $\lambda = \frac{1}{1-x^3}$: $(\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3}, 0)$.
Now, π^2_N is defined as $\pi^2_N(x^1, x^2, x^3) = (\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3})$. Projection from N projects points from the northern hemisphere outside the equator in the equator-plane and points from the southern-hemisphere inside the equator. On the equator π^2_N is the identity mapping. (Strictly speaking this is incorrect, since π^2_N maps points from one space into another.)

Stereographic projection from the south-pole can be done in a similar way: $\pi_S^2 : S_N^2 \to \mathbb{R}^2$ given by $\pi_S^2(x^1,x^2,x^3) = (\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3})$. The image of the northern-hemisphere is now inside the equator and the image of the southern-hemisphere is outside the equator. On the equator π_S^2 is the identity.

In physics one often encounters continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ that approach the same value, c, in whatever direction, once the distance from the origin in $\mathbb{R}^2$ is large enough. In this case it is possible to add to $\mathbb{R}^2$ one point, ∞, and define $f(\infty) = c$. Thus f can be extended to a function $f : S^2 \to \mathbb{R}$ in a natural way. The same is true for a function $g : \mathbb{R}^3 \to \mathbb{R}$ that under similar conditions can be extended to a function $g : S^3 \to \mathbb{R}$. The domain of the function f and g has been compactified by one point: S^2 is the one point compactification of $\mathbb{R}^2$, S^3 the one point compactification of $\mathbb{R}^3$.
When we consider the inverse of the projection π_N^2, we see that here we do have a mapping from $\mathbb{R}^2$ that, although its values are not in $\mathbb{R}$, approach the same point in S^2 (when S_S^2 is considered as part of S^2) whenever we proceed away from the origin. It is then natural to add to the above definition of π_N^2 the rule $\pi_N^2(0,0,1) \equiv \infty$. We then have $\pi_N^2 : S^2 \to \mathbb{R}^2 \cup \{\infty\} \equiv \mathbb{R}_C^2$. Similar considerations hold for π_S^2.

An equivalent definition of the projection follows from the identification of $\mathbb{R}^2$ with $\mathbb{C}$ via $(x^1,x^2) \leftrightarrow x^1 + ix^2$. This gives $\pi_N^2 : S^2 \to \mathbb{C}_C$ as $\pi_N^2(x^1,x^2,x^3) = \frac{x^1+ix^2}{1-x^3}$, and we can add $\pi_N^2(0,0,1) = \infty$. Furthermore $\pi_S^2 : S^2 \to \mathbb{C}_C$ as $\pi_S^2(x^1,x^2,x^3) = \frac{x^1+ix^2}{1+x^3}$, and $\pi_N^2(0,0,-1) = \infty$.

The inverse of the projection is easily obtained by noting that the line through N and $(x^1,x^2,0)$ is given by

$$\begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda \begin{pmatrix}x^1\\x^2\\-1\end{pmatrix}.$$

This line intersects the sphere when $(\lambda x^1)^2 + (\lambda x^2)^2 + (1-\lambda)^2 = 1$. The solution for λ is $\frac{2}{1+(x^1)^2+(x^2)^2}$, so:

$$(\pi_N^2)^{-1}(x^1,x^2) = (\tfrac{2x^1}{1+(x^1)^2+(x^2)^2}, \tfrac{2x^2}{1+(x^1)^2+(x^2)^2}, \tfrac{(x^1)^2+(x^2)^2-1}{1+(x^1)^2+(x^2)^2}).$$

We can take $(\pi_N^2)^{-1}(\infty) = (0,0,1)$ to be included in this formula. With the identification $(x^1,x^2) \leftrightarrow x^1 + ix^2 \equiv z$ and $\bar{z} = x^1 - ix^2$ we have:

$$(\pi_N^2)^{-1}(z) = (\tfrac{z+\bar{z}}{1+z\bar{z}}, \tfrac{z-\bar{z}}{i(1+z\bar{z})}, \tfrac{z\bar{z}-1}{1+z\bar{z}}).$$

Similarly:

$(\pi_S^2)^{-1}(x^1, x^2) = (\frac{2x^1}{1+(x^1)^2+(x^2)^2}, \frac{2x^2}{1+(x^1)^2+(x^2)^2}, \frac{1-((x^1)^2+(x^2)^2)}{1+(x^1)^2+(x^2)^2})$, or

$(\pi_S^2)^{-1}(z) = (\frac{z+\bar{z}}{1+z\bar{z}}, \frac{z-\bar{z}}{i(1+z\bar{z})}, \frac{1-z\bar{z}}{1+z\bar{z}})$.

The "z-form" makes calculations very simple, for example $\pi_S^2((\pi_N^2)^{-1}(z)) = \frac{1}{\bar{z}}$.

Very little changes in the calculations and definitions above when we consider the stereographic projection in one dimension higher. Let S^3 be the three sphere in $\mathbb{R}^4$: $S^3 = \{x \in \mathbb{R}^4 | \ ||x|| = 1\}$, $N = (0,0,0,1)$ and $S = (0,0,0,-1)$. The projection from N is:

$\pi_N^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4})$.

The stereographic projection from S is:

$\pi_S^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1+x^4}, \frac{x^2}{1+x^4}, \frac{x^3}{1+x^4})$.

Extension of the definition and compactification of spaces work as before, for example $\pi_N^3(0,0,0,1) = \infty$ and $\pi_N^3 : S^3 \to \mathbb{R}_C^3$. There is however a difference when we would like to consider $\mathbb{C}$ instead of $\mathbb{R}^2$. This cannot be done for the target-space, but for the domain it can: $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 | \ ||z^1||^2 + ||z^2||^2 = 1\}$. We identify z^1 with $x^1 + ix^2$ and z^2 with $x^3 + ix^4$ and define $\pi_N^3 : \mathbb{C}^2 \to \mathbb{R}_C^3$ as $\pi_N^3(z^1, z^2) \equiv \pi_N^3(x^1, x^2, x^3, x^4)$.

2.2 the Hopf map

We are now in a position to define a very special mapping, discovered by Heinz Hopf in 1931 [19, 20], from S^3 to S^2. This map, the Hopf map, has played an important role in physics in seemingly unrelated situations ranging from qubits to Taub-NUT spaces in general relativity [21] and it will figure prominently in this thesis:

$$H : S^3 \to S^2, \ H(z^1, z^2) = (\pi_N^2)^{-1}(\frac{z^1}{z^2}). \tag{2.1}$$

Since for an arbitrary point $p \in S^2$ it is always possible to find $(z^1, z^2) \in S^3$ such that $\pi_N^2(p) = \frac{z^1}{z^2}$ and the definition of H involves the ratio $\frac{z^1}{z^2}$ only, the pre-image of (the arbitrary point) p is given by $\{(e^{i\phi}z^1, e^{i\phi}z^2) \in \mathbb{C}^2 \mid \phi \in [0, 2\pi]\}$. See figure 2.1 for this and the following discussion. We now investigate the stereographic projection of this pre-image.

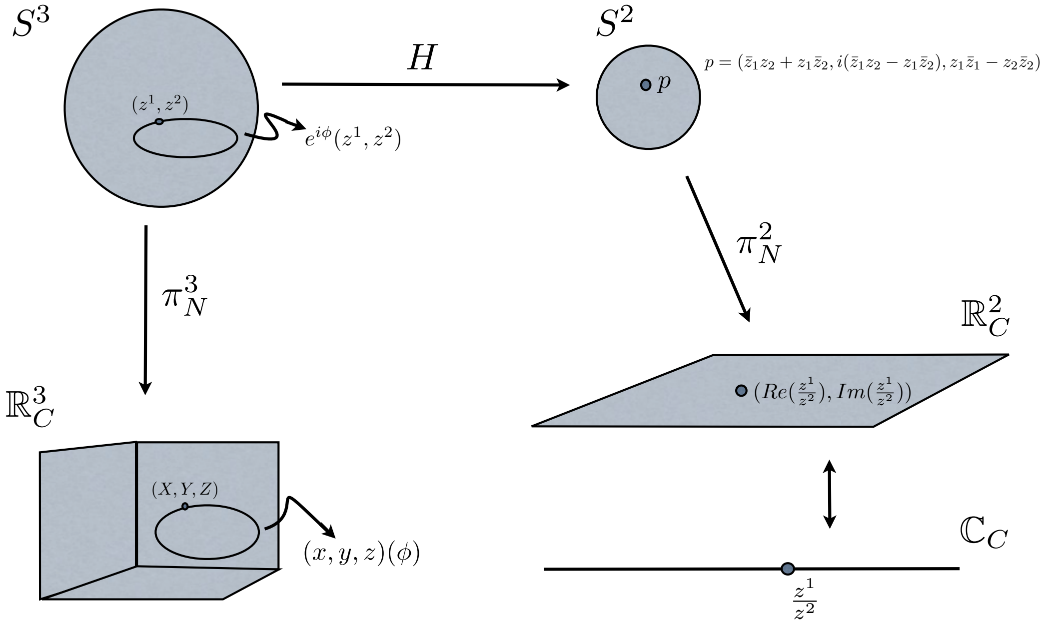

Figure 2.1: Hopf mapping and stereographic projection

We split $e^{i\phi}z^1$ into its real and imaginary parts: $e^{i\phi}z^1 = (x^1\cos\phi - x^2\sin\phi) + i(x^2\cos\phi + x^1\sin\phi)$. Also: $e^{i\phi}z^2 = (x^3\cos\phi - x^4\sin\phi) + i(x^4\cos\phi + x^3\sin\phi)$. Thus: $\pi_N^3(e^{i\phi}z^1, e^{i\phi}z^2) = \pi_N^3(x^1\cos\phi - x^2\sin\phi, x^2\cos\phi + x^1\sin\phi, x^3\cos\phi - x^4\sin\phi, x^4\cos\phi + x^3\sin\phi)$.
The result for $\phi = 0$ is $\pi_N^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4})$ and we write this as (X, Y, Z). A short calculation shows that the result for general ϕ can be expressed in terms of X, Y and Z as follows: $\pi_N^3(e^{i\phi}z^1, e^{i\phi}z^2) = \frac{1}{N}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi)$, where $N = \frac{1}{1-x^4} - \frac{x^4}{1-x^4}\cos\phi - Z\sin\phi$. We write this as $(x(\phi), y(\phi), z(\phi))$ or $(x, y, z)(\phi)$:

$$(x, y, z)(\phi) = \frac{1}{N}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi).$$

This is a closed parametrized curve, say C, through the point $(X, Y, Z) \in \mathbb{R}^3_C$ and every such point will be reached by some $p \in S^2$. This implies that all these curves fill $\mathbb{R}^3_C$, or: any point in $\mathbb{R}^3_C$ is on at least one such curve. In fact there can only be one curve through every point, as is clear from the form of the pre-image of p in S^3. The tangent vector to the curve at $(x, y, z)(\phi)$ is:

$$\partial_\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} (\phi) = \begin{pmatrix} \frac{-X\sin\phi - Y\cos\phi}{N} - \frac{X\cos\phi - Y\sin\phi}{N^2}\left(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi\right) \\ \frac{-Y\sin\phi + X\cos\phi}{N} - \frac{Y\cos\phi + X\sin\phi}{N^2}\left(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi\right) \\ \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} - \frac{Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi}{N^2}\left(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi\right) \end{pmatrix}.$$

It is easy to see that the first two components, when written in terms of x, y and z are $-y + xz$ and $x + yz$ respectively. The second term in the third component is also straightforward, it equals z^2. The first term in the third component is more difficult: it equals $\frac{N - \frac{1}{1-x^4}}{N}$. The third component can thus be written as $1 + z^2 - \frac{1}{N}\frac{1}{1-x^4}$. We proceed to show that it can also be written as $\frac{1}{N}\frac{1}{1-x^4} - x^2 - y^2$, and therefore $\frac{1}{N}\frac{1}{1-x^4} = \frac{1}{2}(1 + x^2 + y^2 + z^2)$:

Third component $= \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} - \frac{Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi}{N^2}\left(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi\right) = \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} + z^2 = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + 2Z\frac{x^4}{1-x^4}\cos\phi.\sin\phi + (\frac{x^4}{1-x^4})^2\cos^2\phi + Z^2\sin^2\phi] + z^2 = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi + Z^2 - \frac{x^4}{(1-x^4)^2}\cos\phi + (\frac{x^4}{1-x^4})^2] = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + \frac{1}{(1-x^4)^2} - X^2 - Y^2] = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + \frac{1}{(1-x^4)^2}] - x^2 - y^2 = \frac{1}{N}\frac{1}{1-x^4} - x^2 - y^2$.

Combining these results we have the tangent vector at (x, y, z) given by

$$\begin{pmatrix} -y + xz \\ x + yz \\ \frac{1}{2}(1 - x^2 - y^2 + z^2) \end{pmatrix}. \tag{2.2}$$

The integral curves of this vector field are the curves $(x, y, z)(\phi)$ we started with. A similar calculation shows that if instead of using stereographic projection from the north-pole we would have used projection from the south-pole, the tangent vector at (x, y, z) would have been: $\begin{pmatrix} -(y + xz) \\ x - yz \\ \frac{1}{2}(-1 + x^2 + y^2 - z^2) \end{pmatrix}$.

In the definition of H we used stereographic projection from the north-pole. It would have been possible to use projection from the south-pole. But since $\pi^2_S((\pi^2_N)^{-1}(z)) = \frac{1}{\bar{z}}$, it follows that $(\pi^2_S)^{-1}(\frac{\bar{z^2}}{\bar{z^1}}) = (\pi^2_N)^{-1}(\frac{z^1}{z^2})$ and therefore, except for a different relation between the z^i and the x^j, everything in the discussion of integral curves and tangent vector fields would stay the same.

We now investigate in closer detail the tangent vector field, or, what amounts to the same, the integral curves $(x, y, z)(\phi)$. It will be helpful to have a different view of S^3 and with it of $\mathbb{R}^3$, which will be developed first.

2.2.1 $\mathbb{R}^3$ as a collection of nested tori

There is an obvious reparametrization for $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 \mid ||z^1||^2 + ||z^2||^2 = 1\}$, namely if we write $r_1 e^{i\alpha_1}$ for z^1 and $r_2 e^{i\alpha_2}$ for z^2. Of course, $r_1^2 + r_2^2 = 1$, and this implies that there is a $\delta \in [0, \frac{\pi}{2}]$ such that $r_1 = \cos\delta$ and $r_2 = \sin\delta$.

The topology of the set $T \subset S^3$, for which $r_1 = r_2 = \frac{1}{2}\sqrt{2}$ $(\delta = \frac{\pi}{4})$ is that of a torus: $T = \{(\frac{1}{2}\sqrt{2}e^{i\alpha_1}, \frac{1}{2}\sqrt{2}e^{i\alpha_2}) \mid \alpha_1, \alpha_2 \in [0, 2\pi]\}$. According to general theorems, the topology of the stereographic projection of T on $\mathbb{R}^3_C$ is a torus too. This will also become clear by performing the calculation: $T_\pi \equiv \pi^3_N(T) = \pi^3_N(\frac{1}{2}\sqrt{2}e^{i\alpha_1}, \frac{1}{2}\sqrt{2}e^{i\alpha_2}) = (\frac{\cos\alpha_1}{\sqrt{2}-\sin\alpha_2}, \frac{\sin\alpha_1}{\sqrt{2}-\sin\alpha_2}, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. For any fixed α_2, varying α_1 through its range makes circles around and perpendicular to the z-axis. It suffices therefore to have a closer look at one particular section with α_1 fixed, for example $\alpha_1 = 0$. Now varying α_2 gives a closed loop in the $x > 0$ part of the x-z plane: $(\frac{1}{\sqrt{2}-\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. For $\alpha_2 \in [0, \pi]$, change to $\beta \equiv \alpha_2 - \frac{\pi}{2}$ $(\in [-\frac{\pi}{2}, \frac{\pi}{2}])$ and for $\alpha_2 \in [\pi, 2\pi]$ change to $\gamma \equiv \alpha_2 - \frac{3\pi}{2}$ $(\in [-\frac{\pi}{2}, \frac{\pi}{2}])$. For $\beta = 0$ and for $\gamma = 0$ we have two different points on the x-axis. Furthermore, the x-component of T_π is symmetric and the z-component of T_π anti-symmetric for $\beta \to -\beta$ and for $\gamma \to -\gamma$. This completes the picture of the loop in the x-z plane. With suitable choices for a symmetric function $r(\psi)$, it can be summarized as $(1 + r(\psi)\cos\psi, 0, r(\psi)\sin\psi)$ in which ψ is the polar angle as seen from (1,0,0). For T_π we now have $((1 + r(\psi)\cos\phi)\cos\alpha_1, (1 + r(\psi)\cos\phi)\sin\alpha_1, r(\psi)\sin\psi)$, which is the parametrization of a (when r is different from 1, deformed) torus.

For $r_1 = 0$ and $r_2 = 1$ $(\delta = \frac{\pi}{2})$ we have a circle: $\{(0, e^{i\alpha})\} \subset S^3$. Stereographic projection gives $\pi^3_N(0, e^{i\alpha}) = \pi^3_N(0, 0, \cos\alpha, \sin\alpha) = (0, 0, \frac{\cos\alpha}{1-\sin\alpha})$, and this is the z-axis in $\mathbb{R}^3$, or a circle with infinite radius in $\mathbb{R}^3_C$ when the point ∞ is included (and identified, as before, with -∞).

For $r_1 = 1$ and $r_2 = 0$ $(\delta = 0)$ we have a circle $\{(e^{i\beta}, 0)\} \subset S^3$. Stereographic projection now gives $\pi^3_N(e^{i\beta}, 0) = \pi^3_N(\cos\beta, \sin\beta, 0, 0) = (\cos\beta, \sin\beta, 0)$, the unit circle in the x-y plane. Note that this circle lies within the aforementioned torus and is identical with the degenerate torus, for which the function $r(\psi)$ is zero.

For any fixed r_1, r_2 with $r_1 > r_2$ $(\delta \in \langle 0, \frac{\pi}{4}\rangle)$, continuity demands that, after stereographic projection, we have a torus (in $\mathbb{R}^3$) contained within T_π

($\delta = \frac{\pi}{4}$) and containing the unit circle in the x-y plane ($\delta = 0$). A similar calculation as before ($\delta = \frac{\pi}{4}$) confirms this picture: for the closed loop in the x-z plane we now have $(\frac{\cos\delta}{1-\sin\delta.\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\frac{1}{\sin\delta}-\sin\alpha_2})$ instead of $(\frac{1}{\sqrt{2}-\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. The x-coordinate of this loop varies between $\frac{\cos\delta}{1+\sin\delta}$ and $\frac{\cos\delta}{1-\sin\delta}$. For $\delta \in \langle 0, \frac{\pi}{4}\rangle$ this interval encloses 1 and is enclosed by the previous interval for the x-coordinate, $[\frac{1}{\sqrt{2}+1}, \frac{1}{\sqrt{2}-1}]$. For every x in the new interval $[\frac{\cos\delta}{1+\sin\delta}, \frac{\cos\delta}{1-\sin\delta}]$ we want to check whether the new loop is closer to the x-axis than the corresponding loop for $\delta = \frac{\pi}{4}$. This is done as follows. A particular $x \in [\frac{\cos\delta}{1+\sin\delta}, \frac{\cos\delta}{1-\sin\delta}]$ corresponds to (possibly two) particular value(s) of α_2, say α_2^{new}, for the new loop: $x = \frac{\cos\delta}{1-\sin\delta.\sin\alpha_2^{new}}$. The same x for the ($\delta = \frac{\pi}{4}$)-loop corresponds to, say, α_2^{old}: $x = \frac{1}{\sqrt{2}-\sin\alpha_2^{old}}$. Thus: $\sin\alpha_2^{old} = \sqrt{2} - \frac{1-\sin\delta.\sin\alpha_2^{new}}{\cos\delta}$. We should now check $|z^{new}(\alpha_2^{new})| < |z^{old}(\alpha_2^{old})|$, where $z^{new/old}$ is on the new/old loop: $|\frac{\sin\delta.\cos\alpha_2^{new}}{1-\sin\delta.\sin\alpha_2^{new}}| < |\frac{\cos\alpha_2^{old}}{\sqrt{2}-\sin\alpha_2^{old}}| = |\frac{\cos\alpha_2^{old}.\cos\delta}{1-\sin\delta.\sin\alpha_2^{new}}| \Leftrightarrow |\sin\delta.\cos\alpha_2^{new}| < |\cos\delta.\cos\alpha_2^{old}|$. Squaring, using $\sin^2\alpha_2 + \cos^2\alpha_2 = 1$ and $\cos\delta > \frac{1}{2}\sqrt{2}$ leads ultimately to $|\sin\delta.\sin\alpha_2^{new}| < 1$, which is indeed satisfied.
This shows that it is possible to rewrite the new loop as $(1 + r^{new}(\psi)\cos\psi, 0, r^{new}(\psi)\sin\psi)$ in which the function r^{new}takes values between 0 and r^{old} for every ψ.

Note that for $\delta \in \langle 0, \frac{\pi}{4}\rangle$ we have $\frac{1}{\sin\delta} > \sqrt{2}$ and $|z^{new}(\alpha_2)| < |z^{old}(\alpha_2)|$ for every α_2. But this is not enough to conclude that the new loop is inside the old loop everywhere, since the same value for α_2 in both cases do correspond to different x values and also to different ψ values.

The remaining case where $r_2 > r_1$, or $\delta \in \langle \frac{\pi}{4}, \frac{\pi}{2}\rangle$, is now easy to guess. For any fixed δ the projection will be a (deformed) torus that contains torus T_π, and for $\delta \downarrow \frac{\pi}{4}$ the projection will approach T_π. For $\delta \uparrow \frac{\pi}{2}$ it will approach the z-axis (or circle in $\mathbb{R}^3_C$). With very minor changes (interchange of $<$ and $>$) the previous calculation can be repeated in order to confirm this guess.

In summary, we have seen that S^3 consists of a set of tori that after stereographic projection lead to a set of nested tori that fill all of $\mathbb{R}^3$. Starting with a degenerate torus for $\delta = 0$, the unit circle in the x-y plane, for increasing δ we get bigger and bigger tori, each containing the previous one. When δ approaches $\frac{\pi}{2}$ the tori approach the z-axis, that can be seen as a degenerate torus. This infinite or degenerate torus itself is reached for $\delta = \frac{\pi}{2}$.

2.2.2 Hopf fibration

From the foregoing it now follows that the curve C, the pre-image of the point p in S^2 for which $\pi_N^2(p) = \frac{z^1}{z^2}$ after stereographic projection, is fully contained within the torus that corresponds to $\delta = \tan^{-1}|\frac{z^2}{z^1}|$. The collection of all the curves that stem from $q \in S^2$ with $|\pi_N^2(q)| = |\frac{z^1}{z^2}|$ fill up this torus completely, since every point will at least be reached once. (Two such curves will either coincide completely or have no point in common, as we have seen before). It is obvious that these q form a circle in S^2 that is parallel to the equator. Thus the equator-parallel circles in S^2 correspond to the tori in $\mathbb{R}^3$ (or $\mathbb{R}^3_C$). It is readily verified that the degenerate circle (0,0,-1) corresponds to the z-axis, the equator to T_π and the degenerate circle (0,0,1) to the unit circle in the x-y plane. Furthermore, the (non-degenerate) equator-parallel circles in the southern hemisphere correspond to tori that contain T_π, those of the northern hemisphere to tori contained by T_π.

The curve C that contains the point (X, Y, Z) for $\phi = 0$ lies in the plane
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mu \begin{pmatrix} Y \\ -X \\ \frac{x^4}{1-x^4} \end{pmatrix}$, as can be seen by taking $\lambda = \frac{\cos\phi}{N}$ and $\mu = -\frac{\sin\phi}{N}$. C is therefore the intersection of this plane with the torus that corresponds to $\delta = \tan^{-1}|\frac{z^1}{z^2}|$. Since C also contains the point $\frac{1-x^4}{1+x^4}(-X, -Y, -Z)$ for $\phi = \pi$ (note: $\frac{1-x^4}{1+x^4} > 0$), C must be a Villarceau circle: depending on the orientation of the plane, the intersection of a plane through the origin with a torus with centre the origin consists of two circles that are both on opposite sides of the torus-symmetry axes, two concentric circles with the origin as centre, or two Villarceau circles [22–24]. The fact that (X, Y, Z) together with $\frac{1-x^4}{1+x^4}(-X, -Y, -Z)$ are on C excludes the first two possibilities, and C must be one of the Villarceau circles. The case $x^4 = 1$ corresponds to the curve through ∞ (the z-axis) and the case $x^4 = 0$ corresponds to the curve $(x, y, z)(\phi) = \frac{1}{1-Z\sin\phi}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi)$. For $|Z| = 1$ this again is the z-axis and for $|Z| < 1$, although (X, Y, Z) and also $(-X, -Y, -Z)$ are on the curve, the circle is not centred around the origin, since for $\phi = \frac{\pi}{2}$ we have $\frac{1}{1-Z}(-Y, X, 0)$ whereas for $\phi = \frac{3\pi}{2}$ we have $\frac{1}{1+Z}(Y, -X, 0)$ ($Z = 0$ gives the unit circle in the x-y plane).

Often one speaks of the Hopf fibration, and we can now understand where this comes from. S^3 can be thought of as base space S^2 with fibers S^1, since every point in S^3 is specified by one particular $p \in S^2$ and a $\phi \in [0, 2\pi\rangle$ and this interval can be identified with S^1. Locally we can write $S^3 = S^2 \times S^1$. Globally this is not correct.

2.2.3 linking

There is one more important property of the curves $(x, y, z)(\phi)$ that concerns the interconnection between any two of them. The analogues in S^3 (via $(\pi_N^3)^{-1}$) of the unit circle in the x-y plane and the z-axis are two circles that are linked once. The concept of linking is clear intuitively [25]. Stereographic projection does not change this linking property. Via the Hopf map and π_N^2 these two circles correspond to two specific points in $\mathbb{C}_C$ (or $\mathbb{R}_C^2$), say p and q. Two arbitrary points, not being p or q, can with the help of a suitably defined mapping be mapped to p and q respectively. This mapping induces a mapping in S^3 by which the curves (circles) that correspond to the two arbitrary points are being mapped into the analogues of the x-axis and unit circle in the x-y plane. It is intuitively clear that this mapping does not change the linking between the curves. Thus the two circles in S^3 (or $\mathbb{R}^3$) that are the pre-images of two arbitrary points in S^2 are also linked once. (These statements can be made mathematically precise with the help of algebraic topology [26].)

2.2.4 conclusion

We end this chapter by stating that a study of the Hopf map leads naturally to a picture of $\mathbb{R}^3$ as a collection of disjoined and singly linked circles, organised on nested tori, a truly remarkable and beautiful picture!