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Tetrads, spinors and twistors

Here we introduce mathematical concepts that are needed throughout this work. Some of them are very familiar. However, the form in which they are presented is not standard.

1.1 tetrads

We consider Minkowski space (flat space-time) M^4 and orthonormal basis vectors $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) \equiv g_{ab} = \eta_{ab} \ , \quad (\eta_{ab}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Here $\mathbf{g}$ is the metric tensor and indices $a, b, ...$ run from 0 to 3, whereas $k, l, ...$will run from 1 to 4. The collection [0,1,2,3] is used to stress the difference of the first component from the others as a time component.
It is possible to change to a basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$ such that:

$$(\mathbf{g}(\mathbf{E}_k, \mathbf{E}_l)) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{1.1}$$

Since all $\mathbf{E}_k$ are null vectors, this basis is called a null tetrad. Two of them need be complex in order to satisfy the definition (1.1). An example is provided by:

$$\begin{aligned}
\mathbf{E}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_0 + \mathbf{e}_3)\\
\mathbf{E}_2 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_0 - \mathbf{e}_3)\\
\mathbf{E}_3 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2)\\
\mathbf{E}_4 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2).
\end{aligned}$$

If we define $(\mathbf{e}_{AX'}) = \begin{pmatrix} \mathbf{e}_{11'} & \mathbf{e}_{12'} \\ \mathbf{e}_{21'} & \mathbf{e}_{22'} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_4 \\ \mathbf{E}_3 & \mathbf{E}_2 \end{pmatrix}$, we find from the form of $(\mathbf{g}(\mathbf{E}_k, \mathbf{E}_l))$ that:

$$\mathbf{g}(\mathbf{e}_{AX'}, \mathbf{e}_{BY'}) = \epsilon_{AB}\epsilon_{X'Y'},$$

in which

$$(\epsilon_{AB}) = (\epsilon_{X'Y'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.2}$$

Indices A, B take the values 1 and 2, indices A', B' also take the values 1 and 2, but to avoid confusion this is sometimes written as $1'$ and $2'$. Both $\{\mathbf{e}_{AX'}\}$ and $\{\mathbf{e}_a\}$ form a basis and therefore coefficients exist that relate the two:

$$\mathbf{e}_{AX'} = \sigma^a{}_{AX'}\mathbf{e}_a.$$

These coefficients are called Infeld Van Der Waerden symbols and from the definition and the form of the metric tensor in both bases it follows that they satisfy:

$$\sigma^a{}_{AX'}\sigma^b{}_{BY'}g_{ab} = \epsilon_{AB}\epsilon_{X'Y'}. \tag{1.3}$$

Einstein summation convention is used throughout this thesis. In the example given above they are given by:

$$\boxed{\begin{aligned}(\sigma^0{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ (\sigma^1{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ (\sigma^2{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ (\sigma^3{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}} \tag{1.4}$$

1.2 spinors

For most physicists the first encounter with spinors is in a course on non-relativistic quantum mechanics when considering particles with spin. Every physicist is therefore familiar with (Schrödinger) two-spinors and possibly (Dirac) four-spinors. Lesser known is the fact that all of physics can be rewritten, sometimes to great advantage, in spinor form. In a sense a spinor can be considered as the "square root of a vector" (see for example (1.6)), and is thus more basic than a vector. Although a solid introduction to spinors should mention the group theoretical background [14], for our purposes it suffices to introduce spinors in a more elementary and direct way.

The considerations in the previous paragraph show that, in a sense, Minkowski space can be considered as a subspace of the tensor product of two complex vector spaces of dimension two:

$M^4 \subset M^4_{\mathbb{C}} \cong S \otimes S'$,

in which the tensor product is regarded as complexified Minkowski space.

$S \equiv$ spin-space $\equiv \{$1-index spinors$\} \equiv \{\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} | \psi^A \in \mathbb{C}\}$,

basis vectors: $\mathfrak{e}_1, \mathfrak{e}_2$

$S' \equiv$ conjugate spin-space$\equiv \{$1-index conjugate spinors$\} \equiv$
$\{\begin{pmatrix}\psi^{1'} & \psi^{2'}\end{pmatrix} | \psi^{X'} \in \mathbb{C}\}$,

basis vectors: $\mathfrak{e}_{1'}, \mathfrak{e}_{2'}$.

The column and row vector notation is arbitrary, but handy in a first encounter on spinors. Now we can make the following identification, relating the basis in $M^4_{\mathbb{C}}$ (*and* M^4) with those in S and S':

$$\mathbf{e}_{AX'} = \mathfrak{e}_A \otimes \mathfrak{e}_{X'}. \tag{1.5}$$

Corresponding to S and S' we define the dual spaces S^* and S'^* in which the vector components have subindices, and we can identify S with S^* via:

$\psi_A \equiv \psi^B \epsilon_{BA}$ or $\psi^A \equiv \epsilon^{AB}\psi_B$ in which $(\epsilon^{AB}) = (\epsilon_{AB})$.

The same relations hold for primed spinor indices. This shows that (ϵ_{AB}) can be considered as the metric spinor and plays the same role in S as the metric tensor in M^4, including the raising and lowering of spinor indices. The antisymmetry of the metric spinor has consequences unfamiliar in ordinary vector language, such as (for arbitrary spinors $(\psi^A), (\phi^A)$ and 2-component spinor $(\psi_{AB}) \in S^* \otimes S^*$):

$\psi^1 = \psi_2$ and $\psi^2 = -\psi_1$
$\psi^A\psi_A = 0$
$\psi^A\phi_A = -\psi_A\phi^A$
${\epsilon^A}_B = -\delta^A_B = -{\epsilon_B}^A \quad (\delta^A_B = 1$ if $A = B$, otherwise $0)$
$\psi_{AB} = -\psi_{BA} \iff \psi_{AB} = \frac{1}{2}\epsilon_{AB}{\psi_R}^R$ (and obvious generalization for many-index spinors)
$\psi_{AB} = \psi_{BA} \iff \psi_{AB} = \alpha_{(A}\beta_{B)}$ for some (α_A) and (β_B) (and obvious generalization)

Round brackets ( ) around indices means symmetrization and square brackets [ ] anti-symmetrization. In the last example (α_A) and (β_A) are called the principal spinors of (ψ_{AB}).

An arbitrary tensor T can be written $T = T^{ab...c}\mathbf{e}_a \otimes \mathbf{e}_b \otimes ... \otimes \mathbf{e}_c$, but also as $T^{AX'...BY'}\mathbf{e}_{AX'} \otimes ... \otimes \mathbf{e}_{BY'}$ and therefore:

$T^{a...c} = {\sigma^a}_{AX'}...{\sigma^c}_{BY'}T^{AX'...BY'}$ or $T_{a...c} = {\sigma_a}^{AX'}...{\sigma_c}^{BY'}T_{AX'...BY'}$

$T_{AX'...BY'}$ is called the spinor equivalent of the tensor $T_{a...c}$ and is given by:

$T_{AX'...BY'} = {\sigma^a}_{AX'}...{\sigma^c}_{BY'}T_{a...c}$.

Note that ${\sigma^a}_{AX'}$ is a mixed quantity, since the tensor index a is raised and lowered with the metric tensor, whereas the spinor indices A, X' are raised and lowered with the metric spinor. The general rule for going from tensor indices to spinor indices or vice versa with the help of the Infeld Van Der Waerden symbols can be read off from the foregoing three examples. We see that the spinor equivalent of the metric tensor is given by (1.3). With the choice (1.4) for the Infeld Van Der Waerden symbols, they satisfy $\sigma^{a*}{}_{AX'} \equiv (\sigma^a{}_{AX'})^* = \sigma^a{}_{XA'}$, and it can be shown that this relation holds for every possible choice in agreement with the requirement (1.3). As a consequence, the spinor equivalent of a real tensor T satisfies $T^*_{AX'\dots BY'} = T_{XA'\dots YB'}$. The Hermitian conjugate $\bar{T}$ of a spinor T is defined by $\bar{T}_{AX'\dots BY'} = T^*_{XA'\dots YB'}$ and this spinor is called Hermitian if $\bar{T} = T$. Thus the spinor equivalent of a real tensor is Hermitian. It follows that the spinor that corresponds to a real null vector, $V^a = \sigma^a{}_{AX'}V^{AX'}$, can be written

$$V^{AX'} = \xi^A \bar{\xi}^{X'}, \tag{1.6}$$

so, in this sense, it is ξ^A that corresponds to V^a. Of course, there is freedom in the choice of ξ^A. The spinor equivalent of a real anti-symmetric tensor, $T_{ab} = -T_{ba}$, can be written

$$T_{AX'BY'} = \tau_{AB}\epsilon_{X'Y'} + \bar{\tau}_{X'Y'}\epsilon_{AB}, \tag{1.7}$$

with $\tau_{AB} = \frac{1}{2}T_{AR'B}{}^{R'} = \tau_{BA}$. In this sense, it is the symmetric τ_{AB} that corresponds to the anti-symmetric T_{ab}. The principal spinors of τ define two principal null directions via (1.6) that correspond to the eigenvectors of (T_{ab}). For a totally anti-symmetric tensor with three indices, T_{abc}, we have the spinor equivalent

$$T_{AX'BY'CZ'} = t_{BX'}\epsilon_{AC}\epsilon_{Y'Z'} - t_{AY'}\epsilon_{BC}\epsilon_{X'Z'}$$

for some $t_{BX'}$, as can be seen by using previous result. This result in turn can be used to derive the spinor corresponding to a totally anti-symmetric four-index tensor, T_{abcd}, as

$$T_{AW'BX'CY'DZ'} = a(\epsilon_{BD}\epsilon_{W'Z'}\epsilon_{AC}\epsilon_{X'Y'} - \epsilon_{AD}\epsilon_{W'Y'}\epsilon_{BC}\epsilon_{X'Z'})$$

for some constant a. For the completely anti-symmetric Levi-Civita pseudotensor, ϵ_{abcd}, this leads to $a = \pm i$. We use the convention $\epsilon_{0123} = 1 = -\epsilon^{0123}$. The choice (1.4) for the Infeld Van Der Waerden symbols gives $+i$. Other choices might give $-i$! With $a = +i$ we have a spinor equivalent for the dual, $*T^{ab}$, of an anti-symmetric tensor T_{ab}, defined by $*T^{ab} = \frac{1}{2}\epsilon^{abcd}T_{cd} \Leftrightarrow *T_{ab} = \frac{1}{2}\epsilon_{abcd}T^{cd}$

$$*T_{AX'BY'} = i(-\tau_{AB}\epsilon_{X'Y'} + \bar{\tau}_{X'Y'}\epsilon_{AB}) \tag{1.8}$$

where $T_{AX'BY'}$ is given by (1.7). Therefore, the first part in (1.7), $\tau_{AB}\epsilon_{X'Y'}$ is called the anti-self-dual part of $T_{AX'BY'}$, and the last part, $\bar{\tau}_{X'Y'}\epsilon_{AB}$, its self-dual part.

Since in differential geometry the basis vectors are associated with differential operators, it is natural to extend the equivalence between tensors and their spinor form to differential operators:

$\nabla^{AX'} = \sigma_a{}^{AX'}\partial^a.$

Using (1.4) (the speed of light will be put equal to 1 in this work) we find

$$(\nabla^{AX'}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial^t + \partial^z & \partial^x + i\partial^y \\ \partial^x - i\partial^y & \partial^t - \partial^z \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_t - \partial_z & -\partial_x - i\partial_y \\ -\partial_x + i\partial_y & \partial_t + \partial_z \end{pmatrix}.$$

This enables us to rewrite differential equations in spinor form. As an example, Maxwell's equations in vacuum can be written as

$\nabla^{AX'}\phi_{AB} = 0$

and linearized Einstein's equations in vacuum are

$\nabla^{AX'}\phi_{ABCD} = 0.$

In the sequel this will be shown in more detail and we will profit from the resemblance of these two equations, which constitutes one of the advantages of the spinor formalism.

1.2.1 geodesic shear-free null congruences

From equation (1.5) we infer that in spin-space there exist a normalized dyad, $((\vartheta^A), (\iota^A))$ with $\epsilon_{AB}\vartheta^A\iota^B = 1$, from which we can construct a null tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ in M^4 satisfying (1.1), as follows:

$l^a = \sigma^a_{AX'}\vartheta^A\bar{\vartheta}^{X'}, n^a = \sigma^a_{AX'}\iota^A\bar{\iota}^{X'}, m^a = \sigma^a_{AX'}\vartheta^A\bar{\iota}^{X'}, \bar{m}^a = \sigma^a_{AX'}\iota^A\bar{\vartheta}^{X'}.$

In these formula there is still an unnecessary reference to the basis $\{\mathbf{e}_0, ..., \mathbf{e}_3\}$. This can be avoided by writing:

$\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta}, \;\; \mathbf{n} \leftrightarrow \iota\bar{\iota}, \;\; \mathbf{m} \leftrightarrow \vartheta\bar{\iota}, \;\; \bar{\mathbf{m}} \leftrightarrow \iota\bar{\vartheta}.$

It is clear that the spinors that form the normalized dyad could be spinor fields on M^4. The corresponding null tetrad now consists of four vector fields that satisfy (1.1) everywhere. This tetrad is a basis that possibly changes from point to point.

We now consider a congruence of null geodesics in M^4, a space filling family of curves with the property that all curves are geodesics and the tangent vectors to the curves are null. The collection of tangent vectors, $\{\mathbf{l}(x)|x \in M^4\}$, is a vector field that satisfies $l^a l_a = 0$ everywhere, and therefore there exists a spinor field, ϑ, such that $\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta}$. We complement ϑ with a spinor field ι such that (ϑ, ι) forms a normalized dyad. The condition that the integral curves of $\mathbf{l}$ are geodesics is $l^b\nabla_b l^a = f(x^\mu)l^a$, for some arbitrary function f (see also section **7.2.4**). In terms of ϑ this becomes $\vartheta^A\vartheta^B\bar{\vartheta}^{X'}\nabla_{BX'}\vartheta_A = 0$.

The behaviour of neighbouring null geodesics can be expressed in terms of the so called optical scalars, rotation, expansion and shear (see section **7.5**). Following the time development of the geodesics that make up a circle at one instant of time, it is clear what rotation and expansion do to this circle. Shear distorts the circle into an ellipse, without changing its area. The mathematical definition of (complex) shear is $m^a m^b \nabla_b l_a$, or, equivalently $\vartheta^A\vartheta^B\bar{\iota}^{X'}\nabla_{BX'}\vartheta_A$.

If the congruence of null geodesics is shear-free, a property of great importance in this thesis as we will see, the two equations $\vartheta^A\vartheta^B\bar{\vartheta}^{X'}\nabla_{BX'}\vartheta_A = 0$ and $\vartheta^A\vartheta^B\bar{\iota}^{X'}\nabla_{BX'}\vartheta_A$ have to be fulfilled simultaneously. Since $\bar{\vartheta}$ and $\bar{\iota}$ form a basis we arrive at

$$\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic and shear-free } \iff \vartheta^A\vartheta^B\nabla_{BX'}\vartheta_A = 0 \tag{1.9}$$

1.2.2 Kerr's theorem

We can try to solve (1.9) by decomposing ϑ in the *constant* basis $(\mathfrak{e}_1, \mathfrak{e}_2)$ as $\vartheta = \lambda(-Y\mathfrak{e}_1 + \mathfrak{e}_2)$. For $\mathfrak{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathfrak{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ we have $\vartheta = \lambda\begin{pmatrix}-Y\\1\end{pmatrix}$. With

$u \equiv \frac{1}{\sqrt{2}}(t+z),\ v \equiv \frac{1}{\sqrt{2}}(t-z),\ w \equiv \frac{1}{\sqrt{2}}(x+iy)$ and $\bar{w} \equiv \frac{1}{\sqrt{2}}(x-iy)$

we find $(\nabla_{AX'}) = \begin{pmatrix} \partial_u & \partial_w \\ \partial_{\bar{w}} & \partial_v \end{pmatrix}$, and equation (1.9) therefore becomes $Y\partial_u Y - \partial_{\bar{w}} Y = 0$ and $Y\partial_w Y - \partial_v Y = 0$. Note that λ plays no role, since $\vartheta^A(-Y\mathfrak{e}_1 + \mathfrak{e}_2)_A = 0$. The solution to these two equations can be given implicitly by an arbitrary analytic function F in the following way:

$$F(Y, w + Yv, u + Y\bar{w}) \equiv 0,\ \ Y \equiv -\frac{\vartheta^1}{\vartheta^2} \iff \mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic and shear-free} \tag{1.10}$$

This is known as Kerr's theorem and is of importance in chapter 4 and 8. In words: the general (analytic) geodesic shear-free null congruence in flat space-time is related to the zero set of an arbitrary analytic function. Although this theorem has been shown to work in Minkowski space, its action can be extended to the curved Kerr-Schild spaces of chapter 8.

1.3 twistors

Twistors were invented almost half a century ago by Roger Penrose [15] and, though not widely used by physicists, have proven very useful in solving some nonlinear equations. There has been a renewed interest in twistors within the physical community since E. Witten combined string theory with twistor theory [16], but it is only very recently that twistor theory is used by high energy physicists for calculational purposes in scattering processes [17]. Although twistors do appear in books for undergraduate students [18], it is considered as an advanced subject. The full body of twistor theory, with all its mathematical intricacies, its beautiful geometrical structure and its relations with group theory, certainly needs some time to digest. However, in this thesis we would like to add only one very tiny thing to that body (see chapter 4), and we do not need the full theory for that. We therefore present here only the barest essentials of twistor theory, just enough for the aforementioned purpose.

We now combine two spinors to form a twistor: twistor space consists of pairs of spinors $(\Omega^A, \bar{\Pi}_{X'}) \equiv (\mathcal{Z}^\alpha) \equiv \mathcal{Z}$ $(\alpha = 1,2,3,4)$ with corresponding conjugate $(\bar{\mathcal{Z}}_\alpha) \equiv (\Pi_A, \bar{\Omega}^{X'})$ and inner product:

$$\mathcal{Z}^\alpha \bar{\mathcal{Z}}_\alpha = \Omega^A \Pi_A + \bar{\Omega}^{X'} \bar{\Pi}_{X'} = \mathcal{Z}^1 \bar{\mathcal{Z}}^3 + \mathcal{Z}^2 \bar{\mathcal{Z}}^4 + \bar{\mathcal{Z}}^1 \mathcal{Z}^3 + \bar{\mathcal{Z}}^2 \mathcal{Z}^4. \tag{1.11}$$

A twistor for which the inner product is 0 is called null, otherwise it is non-null. Twistor space is related to Minkowski space by the incidence relation:

$$\Omega^A = iX^{AX'}\bar{\Pi}_{X'} \tag{1.12}$$

in which $X^{AX'}$ is the spinor equivalent of the space-time point x^μ:

$$(X^{AX'}) = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \tag{1.13}$$

It is not difficult to proof that if and only if the (fixed) twistor $(\Omega^A, \bar{\Pi}_{X'})$ in eq. (1.12) is null does there exist a real solution for x^μ: in that case the solution space is a null geodesic. (The incidence relation shows that it is really projective twistor space that should concern us, since any non zero complex multiple of a twistor gives rise to the same solution space. For reasons of succinctness, here, as elsewhere in this thesis, we will not be too concerned about technical subtleties (like analyticity, compactification, points at infinity . . .) that are needed for a complete treatment, whenever these subtleties do not influence the result that concerns us.) For future reference we give here the spinorial form of the null geodesic:

$$X^{AY'} = X_0^{AY'} + \lambda\Pi^A\bar{\Pi}^{Y'} \tag{1.14}$$

in which $(X_0^{AY'})$ corresponds to an arbitrary point on the geodesic, and λ is a real parameter. A null twistor thus corresponds to a null geodesic in M^4. For a non-null twistor the solution space is a subspace of $M^4_{\mathbb{C}}$ that does not intersect the real part M^4. There is therefore no direct correspondence between a non-null twistor and a subspace of M^4 defined via (1.12). The most natural we can do is to consider the collection of null twistors that are orthogonal to the non-null twistor $\mathcal{A}^\alpha$: $\{\mathcal{Z}^\alpha | \bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha = 0\}$. This zero set of an analytic function defines a collection of disjoint null geodesics that fill Minkowski space and is known as a Robinson congruence. Now, for $\mathcal{A}^\alpha = (0, \frac{1}{\sqrt{2}}, 0, 1)$ and constant t, the structure of the projection of the corresponding Robinson congruence onto this time slice is a Hopf fibration. As a function of time, this fibration moves with the speed of light along the z-axis.
The details of these statements will be presented in chapter 2 and chapter 4.

It is one of the goals of this work to show that the "optical analogy" between null twistors and null geodesics (light rays), can be extended or generalized to non-null twistors.

Finally, we rewrite Kerr's theorem (1.10) in terms of twistors. Kerr's theorem states that for any spinor field ϑ that defines a geodesic shear-free null congruence via (1.6), there exist a function of three variables, F, such that ϑ is given implicitly by $F(-\frac{\vartheta^1}{\vartheta^2}, w - \frac{\vartheta^1}{\vartheta^2}v, u - \frac{\vartheta^1}{\vartheta^2}\bar{w}) = 0$. We can rewrite this as a function of four variables, g, that is homogeneous to some degree, say n:

$F(-\frac{\vartheta^1}{\vartheta^2}, w - \frac{\vartheta^1}{\vartheta^2}v, u - \frac{\vartheta^1}{\vartheta^2}\bar{w}) =$

$g(-i(u - \frac{\vartheta^1}{\vartheta^2}\bar{w}), -i(w - \frac{\vartheta^1}{\vartheta^2}v), 1, -\frac{\vartheta^1}{\vartheta^2})$ (this *defines* g) $=$

$-(\frac{1}{\vartheta^2})^n g(i(u\vartheta^2 - \bar{w}\vartheta^1), i(w\vartheta^2 - v\vartheta^1), -\vartheta^2, \vartheta^1) =$

$-(\frac{1}{\vartheta^2})^n g(-i(u\vartheta_1 + \bar{w}\vartheta_2), -i(w\vartheta_1 + v\vartheta_2), \vartheta_1, \vartheta_2).$

So there exists a function, f (homogeneous with the same degree) such that:

$f(i(u\bar{\vartheta}_{1'} + w\bar{\vartheta}_{2'}), i(\bar{w}\bar{\vartheta}_{1'} + v\bar{\vartheta}_{2'}), \bar{\vartheta}_{1'}, \bar{\vartheta}_{2'}) = 0.$

But the argument $(i(u\bar{\vartheta}_{1'} + w\bar{\vartheta}_{2'}), i(\bar{w}\bar{\vartheta}_{1'} + v\bar{\vartheta}_{2'}), \bar{\vartheta}_{1'}, \bar{\vartheta}_{2'})$ defines a null twistor:

$$\mathcal{Z} = (iX^{AY'}\bar{\vartheta}_{Y'}, \bar{\vartheta}_{X'}). \tag{1.15}$$

Therefore the Kerr theorem can be stated as follows. The zero set of a homogeneous analytic twistor function $f(\mathcal{Z}^\alpha)$ defines a geodesic shear-free null congruence in M^4, or, with $\mathcal{Z}$ as in (1.15):

$$\{\mathcal{Z}^\alpha \mid f(\mathcal{Z}^\alpha) = 0 \ \wedge \ \mathcal{Z}^\alpha\bar{\mathcal{Z}}_\alpha = 0\} \iff \mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic shear-free.} \tag{1.16}$$