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Chapter 6

Microscopic Behavior

Our model directly describes the most important macroscopic behavior of the system: its rheology. In this chapter we will explore some of the remarkable microscopic behavior in the system, namely the relative velocities of the particles and the organisation of the forces.

6.1 Dissipation and Relative Velocity Distribution

As was discussed in both section 3.1.1 and section 4, energy dissipation is governed by the relative velocities of the bubbles, $\Delta v$. In this section we take a closer look at the probability distribution of $\Delta v$; we will be able to describe this distribution in surprising detail.

6.1.1 Second Moment

Specifically, we can look at the second moment of the distribution: $\langle \Delta v^2 \rangle$. We have already seen in section 2.3.4 that the distribution functions have widely varying width and therefore widely varying $\langle \Delta v^2 \rangle$. Naively, one would expect this second moment to scale with $\dot{\gamma}^2$: doubling the driving velocity would double the velocities of all bubbles and therefore all relative velocities. However, as can be seen in figure 6.1a, this is not true: the distribution functions of $\Delta v/\dot{\gamma}$ do not collapse. This is not surprising because we know that

$$\langle \Delta v^2 \rangle \sim \sigma_{xy} \dot{\gamma}$$

(6.1)

And indeed, we see that $p \left( \Delta v/ \langle \Delta v^2 \rangle \right) = p \left( \Delta v/ \sqrt{\sigma_{xy} \dot{\gamma}} \right)$ does result in distribution functions of comparable width, as can be seen in figure 6.1b. We also note, though, that the distributions functions themselves do not collapse.
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Figure 6.1: The probability distribution functions of $\Delta v$. a: the absences of collapse of $p(\Delta v/\dot{\gamma})$ illustrates the non-trivial nature of the velocity fluctuations. b: when scaled by $\sqrt{\sigma \dot{\gamma}}$, the second moment of $\Delta v$ collapses but the full distributions do not.

Below, we will first discuss the implications of Eq. 6.1 and then investigate the fact that the distribution functions do not collapse.

To get a good grasp on the implications of Eq. 6.1, we rewrite the original expression for power balance:

$$L_x L_y \dot{\gamma} \langle \sigma_{xy} \rangle_t = \frac{N b}{2} \langle Z \rangle_t \langle \Delta v^2 \rangle_{t,ij}$$

(6.2)

by rescaling $\Delta v$ with $\dot{\gamma}$, as discussed above, to take out the trivial dependence of velocities on the driving velocity; we denote $v^*_s := \Delta v/\dot{\gamma}$. We can also take together a number of the constants to form the contact density $C = NZ/2L_x L_y$. This results in

$$\langle v^2_s \rangle_{t,ij} C = \frac{\langle \sigma_{xy} \rangle_t}{b \dot{\gamma}}$$

(6.3)

Here we can interpret $b \dot{\gamma}$ as an affine shear stress: if all particles were to move affinely, they would all feel a force $b \Delta v_{ij} = b \dot{\gamma} r_{ij}$ on one side and the same force in the other direction on the other side, for a total stress of $(1/2V)2N (r b \dot{\gamma} r) = (N \pi r^2/V)(2/\pi)b \dot{\gamma} = (2/\pi)b \dot{\gamma}$.

We can now deduce the dependencies of $v_s$ on $\Delta \phi$ and $\dot{\gamma}$ by considering the right hand side of Eq. 6.3. These dependencies are complex since $\sigma_{xy}$ depends on both $\Delta \phi$ and $\dot{\gamma}$, and with different exponents in different regimes. We start by considering $\Delta \phi$: since $\sigma_{xy}$ never shrinks with increasing density, and grows with increasing density in two out of three regimes$^1$, we can generally say that increasing $\Delta \phi$ increases $v_s$. Physically, this means the following. In more densely packed foams, a higher stress is needed to sustain steady flow at a fixed shear rate. Therefore, more energy is put into the system and more

$^1$the Yield and Transition regimes
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Increasing Fluctuations

![Diagram](image)

Figure 6.2: Fluctuations in the relative velocity increase with increasing density or decreasing strain rate.

energy needs to be dissipated. The only way the system can dissipate more energy is by increasing the relative velocity between the particles.

Now we consider the dependence of $v_*$ on $\dot{\gamma}$. $\sigma_{xy}$ always increases with increasing $\dot{\gamma}$, though the degree with which it does so depends on the regime. However, the dependence is always sublinear. Because $\sigma_{xy}$ is sublinear in $\dot{\gamma}$, $\sigma_{xy}/\dot{\gamma}$ is a decreasing function of $\dot{\gamma}$. As a consequence, $v_*$ decreases as $\dot{\gamma}$ increases. Physically this is because increasing the strain rate, at constant density, will not increase the input power as much as the resulting extra (relative) movement will increase the dissipated power; therefore less relative velocity is needed/possible to keep a balance between the input and dissipated power.

Both dependencies are illustrated in figure 6.2. Note that the fact that fluctuations increase with increasing density is opposite to the trend in to static jamming, where various measures of randomness and non-affinity increase when decreasing the density towards the jamming density [5]. Here the randomness increases with increasing density. It should be noted, though that the static jamming results have been obtained in the quasi-static regime, while here we are at finite strain rate.

We can now return to the probability distribution function of $\langle \Delta v \rangle$: we know how the second moment scales, $\langle \Delta v^2 \rangle \sim \sigma_{xy}\dot{\gamma}$, and can take this into account in plotting the data. Figure 6.1 b shows that the distribution functions are now of equal width, but not of similar shape. This means that the second moment does not capture the full behavior of the pdfs. In other words: higher order moments will also display non-trivial scaling. To investigate this, we will look at the higher order even moments of the relative velocity distribution.
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Figure 6.3: The fourth moment of the relative velocity distribution function vs. powers of $\sigma/\dot{\gamma}$. a The naive Gaussian assumption of $\langle v_t^4 \rangle_{t,ij} C^2 \sim \left( \langle \sigma_{xy} \rangle_t / b\dot{\gamma} \right)^2$ does not hold as the data is not parallel to the black line that indicates linear scaling. b $\langle v_t^4 \rangle_{t,ij} C^2 \sim \left( \langle \sigma_{xy} \rangle_t / b\dot{\gamma} \right)^{2.35}$ describes a large range of the data well, as the data is parallel to the black line that indicates linear scaling. In both plots, colours and shapes represent density and strain rate as in table 3.3.

because they also scale as moments of the dissipation. For example, for the next even moment, the fourth moment, we have $\langle \Delta v^4 \rangle = \left( \langle \Delta v^2 \rangle \right)^2 \sim \langle P^2_{\text{diss}} \rangle$.

In appendix 9.2 we also discuss the simple scaling of the first moment.

6.1.2 Fourth Moment

The first higher order even moment of the relative velocity distribution that will contribute to the non-collapse of the rescaled probability distribution functions is the fourth moment. The most simple possible scaling would be if the fourth moment scaled as the square of the second moment:

$$\langle v_t^4 \rangle_{t,ij} C^2 \sim \left( \langle v_t^2 \rangle_{t,ij} C \right)^2 = \left( \frac{\langle \sigma_{xy} \rangle_t}{b\dot{\gamma}} \right)^2,$$

as is the case for central moments in a Gaussian distribution, for example. However, as we show in figure 6.3 a, this naive assumption does not hold as the data is not parallel to the black line that indicates linear scaling.

Therefore, we try a slightly different approach, namely the critical scaling-like description that has already been applied successfully to many quantities in static and non-static jamming. Thus, we hypothesise that we can describe the fourth moment of the relative velocity distribution in the following way:

$$\langle v_t^4 \rangle_{t,ij} C^2 = \left( \frac{\langle \sigma_{xy} \rangle_t}{b\dot{\gamma}} \right)^\alpha F(\Delta\phi, (b/k)\dot{\gamma}).$$
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Figure 6.4: a the fourth moment of the relative velocity distribution function with an exponent 2.35 power law of $\sigma_{xy}/\dot{\gamma}$ divided out. b, c: plotting vs. either of the two crossovers, $\dot{\gamma} \sim \Delta \phi^2$ and $\dot{\gamma} \sim \Delta \phi^{3.5}$, does not result in good collapse in the non-power law portion of the curve. In all three plots, colours and shapes represent density and strain rate as in table 3.3.
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Figure 6.5: The same scaling plot for the fourth moment of $v_*$ as figure 6.4 a, now for a system of one fourth the linear size, or one sixteenth the size. Note that the crossover from flat to curved takes place around $10^2$, just as in figure 6.4 a.

In figure 6.3 we see that $\langle v_*^4 \rangle_{t,ij} C^2 / (\langle \sigma_{xy} \rangle_t / b \dot{\gamma})^{2.35}$ describes a large part, but not all, of the data well. Therefore, we expect $\alpha = 2.35$. Next, we try to describe $F$ as precisely as possible. To plot $F$, we need to plot

$$\frac{\langle v_*^4 \rangle_{t,ij} C^2}{(\langle \sigma_{xy} \rangle_t / b \dot{\gamma})^{2.35}} = F(\Delta \phi, (b/k) \dot{\gamma})$$

(6.6)

on the vertical axis. Then, we can determine the combination of $\dot{\gamma}$ and $\Delta \phi$ of which $F$ is a function by determining for what combination the data collapses. Since we know that the power law description is worst for high density (red data points) and low strain rates (+’s and *’s) we start with trial combinations of the form $\Delta \phi^\chi / (b/k) \dot{\gamma}$, with $\chi$ an exponent that is essentially a fit-parameter. From figure 6.4 a we see that we get good collapse for $\chi = 1.4$. There is not enough data to make an accurate description of the functional form of $F$, however.

Note that the rescaling by $\Delta \phi^{1.4} / (b/k) \dot{\gamma}$ introduces a new combination of $\dot{\gamma}$ and $\Delta \phi$ in addition to $\Delta \phi^2 / \dot{\gamma}$ and $\Delta \phi^{3.5} / \dot{\gamma}$ that emerged as crossovers...
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Figure 6.6: Two different energy distributions. On the left, the energy is equally distributed and the participation ratio is 1; on the right the energy is concentrated in one particle and the participation ratio is $1/N$.

in our 3E scaling model. Intuitively these two relations would have been likely candidates for the argument of $F$, but as figure 6.4 b and c show, they perform markedly worse than $\Delta \phi^{1.4} / (b/k) \dot{\gamma}$. Currently, we do not have an interpretation of this new quantity. We have investigated whether this deviation from a simple power law is a finite size effect, by checking whether the part of the data that is described well by the power law grows smaller in smaller system sizes, but have found this not to be the case, as can be seen in figure 6.5, where we see that the position of the crossover does not change$^2$ if we reduce the size of the system by a factor of 16. However, it is surprising in itself that we are able to describe the fourth moment of the relative velocity distribution so accurately, even if the description is only empirical.

We can also offer an interpretation of what it means that the fourth moment does not show the naive Gaussian-like scaling that we introduced above and for that we have to consider a quantity called the participation ratio. The participation ratio of some quantity $E$, typically an energy, is defined as:

$$\text{Part}(E) = \frac{\langle E \rangle^2}{\langle E^2 \rangle}.$$  \hspace{1cm} (6.7)

With this definition, if the quantity $E$ has a value of $\bar{E}$ at each particle, as on the left of figure 6.6, $\langle E \rangle$ is just $\bar{E}$ and thus $\langle E \rangle^2 = \bar{E}^2$. Similarly, since the value of $E^2$ in each particle will just be $\bar{E}^2$, we also have $\langle E^2 \rangle = \bar{E}^2$. Substituting these quantities in the participation ratio yields $\text{Part}(E) = 1$, meaning that the system fully participates in the process associated with the quantity $E$. Conversely, if only one particle has a non-zero value of $E$, let's

$^2$though the collapse is worse, possibly as a consequence of reduced statistics
call it $E_*$, the average value of $E$ is $E_*/N$, where $N$ is the total number of particles. This means that $\langle E \rangle^2 = E_*^2/N^2$. Similarly $\langle E^2 \rangle = E_*^2/N$. Substituting these two quantities yields $\text{Part}(E) = 1/N$: only one of $N$ particles is participating in the process associated with $E$.

If we now take the dissipation as the relevant quantity, we have:

$$\text{Part}(P_{\text{diss}}) = \frac{\langle P_{\text{diss}} \rangle^2}{\langle P_{\text{diss}}^2 \rangle} \sim \frac{\langle \Delta v^2 \rangle^2}{\langle \Delta v^4 \rangle}. \quad (6.8)$$

Therefore, the fourth moment of the relative velocity distribution enters in the participation ratio. Since, as we have seen above, the fourth moment does not simply scale as the square of the second moment, this will mean that the participation ratio will differ between systems at different strain rates and/or densities. Moreover, since we have accurate expressions for the second and fourth moments, we can write down an expression for the participation ratio of the dissipation:

$$\text{Part}(P_{\text{diss}}) \sim \left( \frac{\dot{\gamma}}{\langle \sigma_{xy} \rangle} \right)^{0.35} F \left( \frac{(b/k)\dot{\gamma}}{\Delta \phi^{1.4}} \right). \quad (6.9)$$

We can now investigate the parameter values for which the participation ratio is large or small. Since the function $F$ is the same function as above and therefore a strictly increasing function and since the shear stress increases with density, increasing the density will decrease the participation ratio. Conversely, since the shear stress depends sub-linearly on the strain rate, increasing the strain rate will increase the participation ratio. Thus, for high strain rate and low density, closest to the critical point, the participation ratio will be highest and the dissipation will be most evenly spread throughout the system. On the other hand, if we have high density and low strain rate, approaching the quasi-static regime, the participation ratio will be very low. This means that all dissipation occurs in just a few bubbles. This means that we have identified the same regions with the extremes of randomness as above for the relative velocity fluctuations: high density, low strain rate is more localised; low density high strain rate is less localised, the parameter space picture of figure 6.2 therefore applies here as well.

To get a qualitative feeling for the actual difference between the distribution of the dissipation in these extreme cases we show two snapshots of our system coloured by dissipation in figure 6.7. Snapshot a is for $\Delta \phi = 0.98$ and $\dot{\gamma} = 10^{-5}$, the densest and slowest system we study. It therefore has the lowest participation ratio. Snapshot b is for $\Delta \phi = 0.8424$ and $\dot{\gamma} = 3 \cdot 10^{-3}$, the least dense and fastest system with the highest participation ratio. There is a clear difference between the two: in snapshot a we can clearly identify individual dissipation events, though they still involve multiple bubbles, while the rest of the bubbles do not participate in the dissipation; in snapshot b there is a large extended area of bubbles experiencing dissipation.
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Figure 6.7: Two snapshots of our systems coloured by dissipation. Red means high dissipation while white means no dissipation. Both colour scales have been individually rescaled by their maximum dissipation. a has $\phi = 0.98$, $\dot{\gamma} = 10^{-5}$ and $N = 1020$; b has $\phi = 0.8424 \approx \phi_c$, $\dot{\gamma} = 3 \cdot 10^{-3}$ and $N = 1200$.

6.1.3 Sixth and Higher Moments

Since we had such success in describing the fourth moment of the relative velocity distribution function, we can try a similar approach for the sixth moment. Based on our results for the fourth moment, we assume that $\langle v^6 \rangle \sim \langle v^4 \rangle^{3/2}$ or $\langle v^6 \rangle_{t,ij} C^3 = (\langle \sigma_{xy} \rangle / b \dot{\gamma})^{3.5} F^{3/2} (\Delta \phi^{1.4} / (b/k) \dot{\gamma})$. Figure 6.8 a shows that this simple assumption is not correct: the data collapses poorly. However, the shape is similar enough to that seen for the fourth moment in figure 6.4 that we can try a more general scaling expression of the form:

$$\langle v^6 \rangle_{t,ij} C^3 = \left( \frac{\langle \sigma_{xy} \rangle}{b \dot{\gamma}} \right)^{\chi} F^{3/2} \left( \frac{\Delta \phi^\xi}{(b/k) \dot{\gamma}} \right)$$  \hspace{1cm} (6.10)

As we show in figure 6.8 b, we get good collapse for $\chi = 3.85$ and $\xi = 1$. It is remarkable that it is possible to describe the sixth moment of a complicated distribution like $p(\Delta v)$ to such a level of detail.

Still, there is markedly more noise in the data for the sixth moment than for the fourth moment, which is not surprising given that we are looking at the distribution in more and more detail. Because of this and the fact that we have neither an explanation nor an interpretation of this result, we have not investigated the eighth or higher moments.

6.1.4 Conclusion

We have been able to describe the probability distribution function of the relative velocities, $\Delta v$, in surprising detail, finding expression for both the
Figure 6.8: Possible scaling expressions for the sixth moment. a assuming that \( \langle v_6^6 \rangle_{t,ij} \sim \langle v_4^4 \rangle_{t,ij}^{3/2} \). b the best fit is achieved for \( \langle v_6^6 \rangle_{t,ij} C^3 = (\langle \sigma_{xy} \rangle/b\dot{\gamma})^{3.85} F^{3/2} (\Delta \phi/(b/k)\dot{\gamma}) \). In both plots, colours and shapes represent density and strain rate as in table 3.3.

second, fourth and sixth moments. Analysis of each of these moments reveals a similar picture. For high strain rate and low density the system is in a relatively homogeneous state: few fluctuations in \( \Delta v \) and a high participation ratio of the dissipation. The more the strain rate is increased or the density is decreased, however, the more fluctuations appear and the more inhomogeneous the system becomes.

### 6.2 Forces and Stresses

We have seen above in section 2.3.1 that, in the range that we limit ourselves to, \( \sigma^{\text{el}} \gg \sigma^{\text{visc}} \), see again in figure 6.9. On the other hand, all viscous and elastic forces together must balance on each bubble. We also know that the forces determine the stress via the Born-Huang formula [35]

\[
\sigma_{xy} = \frac{1}{2V} \sum_{<ij>} r_{ij,x} f_{ij,y}, \tag{6.11}
\]

How can elastic and viscous forces balance, while elastic and viscous stresses differ by orders of magnitude? The answer must lie in the organization of the forces and the way they combine on each bubble.

In order to look at the distribution of the forces we look at four different quantities. The first two are the second moments of the elastic and viscous force distributions:

\[
\langle f_e^2 \rangle = \frac{1}{NZ} \sum_{<ij>} f_e^2 \tag{6.12}
\]

and similarly for \( \langle f_v^2 \rangle \), where the sum is over all contacts \( <ij> \), \( N \) is the total number of particles, \( Z \) is the average number of contacts per particle and
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Figure 6.9: Scatterplot of the viscous vs. the elastic shear stress. The black line represents equality, but is never reached. Colours and shapes represent density and strain rate as in table 3.3.

$NZ$ is the total amount of forces in the system\textsuperscript{3}. The data is also averaged over a strain of approximately 13 corresponding to 2000 data samples with our sampling rate of 0.0067 strain. The second two are the second moments of the distributions of net elastic and viscous forces on a particle:

$$< f_{e, \text{net}}^2 > = \frac{1}{N} \sum_i \left( \sum_j f_{ij}^e \right)^2 \quad (6.13)$$

and similarly for $< f_{v, \text{net}}^2 >$, where the first sum is over all particles $i$ and the second sum over all particles $j$ in contact with particle $i$. Averaging over time is the same as above.

First, to get an impression how the individual forces add to net forces, we plot these in a scatterplot: $< f_e^2 >$ vs. $< f_{e, \text{net}}^2 >$ and $< f_v^2 >$ vs. $< f_{v, \text{net}}^2 >$. These plots can be seen in figure 6.10. These figures clearly show different behavior for the viscous and elastic forces. Panel a shows that the elastic net forces, $< f_{e, \text{net}}^2 >$, are much smaller than the elastic forces, $< f_e^2 >$. This means that, typically, the elastic forces on a bubble largely balance and cancel each other. The magnitude of the imbalance, however, varies and clearly depends on both the density and the strain rate. Panel b, in contrast, shows that the viscous net forces, $< f_{v, \text{net}}^2 >$, are larger than the viscous forces, $< f_v^2 >$. This means that the viscous forces typically do

\textsuperscript{3}This is twice the number of contacts, because each contact $< ij >$ carries two forces: one from particle $i$ on particle $j$ and one from particle $j$ on particle $i$.
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Figure 6.10: The second moment of the average force vs. the second moment of the average net force per bubble. a for the elastic force, b for the viscous force. The black lines represent equality. In both plots, colours and shapes represent density and strain rate as in table 3.3.

\[ \sigma_{xy} = \left\langle f_e^2 \right\rangle - \left\langle f_{e,\text{net}}^2 \right\rangle \]

Figure 6.11: The various stress and force quantities that we will try to connect to each other.

not cancel per particle but instead add up. In addition there seems to be an approximately linear relation between the individual and net viscous forces. Finally, comparing the values of \( \langle f_e^2 \rangle \) and \( \langle f_v^2 \rangle \), we observe that \( \langle f_e^2 \rangle \), ranging from \( 10^{-11} \) to \( 10^{-8} \), is bigger than \( \langle f_v^2 \rangle \), which ranges from \( 10^{-15} \) to \( 10^{-11} \).

These observations answers the question how elastic and viscous forces can balance, while elastic and viscous stresses differ by orders of magnitude. Namely, elastic stresses are much larger than viscous stresses because the individual elastic forces are much larger than the individual viscous forces. However, since all elastic forces on a bubble largely cancel, the net elastic force on a bubble is relatively small, which allows it to be balanced by the net viscous force on the bubble, which is made up of the much smaller, but non-cancelling, individual viscous forces.
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Figure 6.12: The second moments of the distributions of the net elastic and viscous force per bubble in a scatterplot. The black line indicates equality. Colours and shapes represent density and strain rate as in table 3.3.

We will now fully explore the relationships between the forces and stresses discussed here. Figure 6.11 displays which quantities we will try to link together: the shear stress, its two components and the four force quantities discussed above. Some of the relations will be exact since they follow from, for example, force or power balance. Other relations will be approximate and/or empirical.

The first exact relation, which we have already tested before, is power balance. Although we have formulated power balance in terms of the velocity difference $\Delta v$ in our model, we can also formulate it in terms of the viscous force $f_v$ because the two are strictly proportional: $f_v = -b\Delta v$. This different formulation of power balance is:

$$L_x L_y \dot{\gamma} \sigma_{xy} = N^2 b Z \langle f_v^2 \rangle$$

(6.14)

This relationship is represented by the green line labeled with an $a$ in figure 6.14.

The second exact relation, also mentioned before, is force balance. Since there are only two kinds of forces in our system, elastic and viscous forces, and since we use overdamped dynamics, the net viscous and elastic force must balance each other on each bubble. The averages of their squares must therefore also be the same, or:

$$\langle f_{v,\text{net}}^2 \rangle = \langle f_{e,\text{net}}^2 \rangle$$

(6.15)

Figure 6.12 shows that this is indeed the case. This relationship is represented by the green line labeled with a $b$ in figure 6.14.

The third relation has been mentioned a number of times before and concerns the total and elastic stresses. Since we select our data on the condition
that the viscous stress is small compared to the elastic stress, we also know
that the total stress must be approximately equal to the elastic stress:

\[
\sigma_{xy}^e \approx \sigma_{xy}.
\]  
(6.16)

This relation differs from the previous two in that it is not an exact rela-
tion. This relationship is represented by the red line labeled with a c in figure

We have seen above that there is a roughly linear relationship between the
second moments of the distribution of viscous forces and the distribution of
net viscous forces. Here we will investigate this relationship in more detail.
The most simple and natural reason for such a linear relationship would be
a complete lack of correlation between the individual forces. In this case we
would have:

\[
<f_{v,net}^2> = Z <f_v^2>.
\]  
(6.17)

A different hypothesis that would also result in a linear dependence would be
complete coherence; in that case we would have:

\[
<f_{v,net}^2> = Z^2 <f_v^2>.
\]  
(6.18)

In general, the ratio of \(<f_{v,net}^2>\) and \(<f_v^2>\) can go from \(Z^2\), total correlation,
through \(Z\), total decorrelation to 0, total anti-correlation in such a way that
viscous forces alone balance on each bubble. In figure 6.13 a we plot \(<f_{v,net}^2>/Z <f_v^2>\). While the value of the ratio iremains between 1 and 2, clearly
this is not the entire story. Conversely, on figure 6.13 b we plot \(<f_{v,net}^2>/Z^2 <f_v^2>\). This ratio lies between 0.3 and 0.4, but there is much less scatter,
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\[ c \approx \sigma_{xy}^{e} \quad \langle f_{e}^{2} \rangle \quad \langle f_{e,net}^{2} \rangle \]

\[ \sigma_{xy} \quad a = \approx 1.5Z \quad b \]

\[ \sigma_{xy}^{v} \quad \langle f_{v}^{2} \rangle \quad \langle f_{v,net}^{2} \rangle \]

Figure 6.14: Tying together the various force and stress expressions. Green lines indicate exact relations, red lines indicate empirical and approximate relations. The relations are given by:

- **a** \( L_{x} L_{y} \langle \sigma_{xy} \rangle_{t} = (N/2b) \langle Z \rangle_{t} \langle f_{v}^{2} \rangle_{t,ij} \)
  - energy balance
- **b** \( < f_{v,net}^{2} > = < f_{e,net}^{2} > \)
  - force balance
- **c** \( \sigma_{xy}^{e} \approx \sigma_{xy} \)
  - elastic stress dominates
- **d** \( < f_{v,net}^{2} > \approx Z < f_{v}^{2} > \)
  - decorrelation of viscous forces.

especially for slower strain rates. Since the assumption of total decorrelation is never off by more than a factor of two we will write that:

\[ < f_{v,net}^{2} > \approx 1.5Z < f_{v}^{2} > . \quad (6.19) \]

This relationship is represented by the red line indicated with a d in figure 6.14.

A graphic overview of the four relations discussed above is given in figure 6.14. Two things can be noted. First, the four relations that we have described form a chain: each of the five quantities involved can be linked to any of the four other quantities. This allows us, for example, to link the elastic stress to the viscous forces.

Two quantities are currently not linked to the rest: \( \sigma_{xy}^{v} \) and \( < f_{e}^{2} > \). Two possible ingredients for a relation to \( \sigma_{xy}^{v} \) are \( \sigma_{xy} \) and \( < f_{e}^{2} > \). Two possible candidates for a relation to \( < f_{e}^{2} > \) are \( \sigma_{xy}^{e} \) and \( < f_{e,net}^{2} > \). For simplicity we will investigate the relationship between \( \sigma_{xy}^{v} \) and \( < f_{v}^{2} > \) and the relationship between \( \sigma_{xy}^{e} \) and \( < f_{e}^{2} > \).

The stress is calculated as a function of forces:

\[ \sigma_{xy} = \frac{1}{2V} \sum_{<ij>} r_{ij}^{x} f_{ij}^{y}, \quad (6.20) \]

where the sum runs over all contacts and \( r \) is the vector connecting the centres of the two contacting bubbles. If the orientations of all contacts and forces were uniformly distributed over the circle and there were no correlations between the orientation of a contact, the orientation of the force in that contact and
The ratio of actual to maximal elastic shear stress given by $\langle f^2 \rangle^{1/2} N Z < r > /2$. Colours and shapes represent density and strain rate as in table 3.3.

Figure 6.15: The ratio of actual to maximal elastic shear stress given by $\langle f^2 \rangle^{1/2} N Z < r > /2$. Colours and shapes represent density and strain rate as in table 3.3.

the size of that force then the total stress would be zero as all contributions would cancel on average. This means that the appearance of an overall stress implies some correlation or anisotropy.

The best way to approach the amount of anisotropy is from the other extreme: the case where the anisotropy maximises the stress. We first consider the elastic stress:

$$\sigma_{xy}^{el} = \frac{1}{2V} \sum_{<ij>} r_{ij}^x f_{ij}^{el} f_{ij}^y.$$  \hspace{1cm} (6.21)

Since $f^{el}$ is always in the same direction as $r$, this product will be maximised when all contacts are along 45 degree angles with respect to the $x$- and $y$-axes. On average $r_{ij}^x f_{ij}^{el}$ will then be $(1/\sqrt{2}) < r > (1/\sqrt{2}) < |f^{el}| >$. The full expression of the maximal elastic stress, given the average elastic force is then:

$$\sigma_{xy}^{el, max} = \frac{NZ}{4V} \langle r \rangle \sqrt{\langle f^2 \rangle^{el}}.$$ \hspace{1cm} (6.22)

The actually attained elastic shear stress, as a fraction of this maximal stress, is shown in figure 6.15, which shows that (this particular kind of) anisotropy is maximised for low density, the blue points, and high strain rate, the family of crosses on the right. In our range of $\Delta \phi$ and $\dot{\gamma}$ the anisotropy is never larger than 0.1 and never smaller than 0.03.

4 and the length of the connection vector, but since this is always of the order of the bubble radius anyway, we can ignore its variations.
Figure 6.16: The ratio of actual to maximal viscous shear stress given by $\langle f_r^2 \rangle^{1/2} NZ < r >$. Colours and shapes represent density and strain rate as in table 3.3.

A similar analysis can be performed for the viscous stress. Since the viscous force does not have to be along $r$, the configuration that maximises the $\sum_{<ij>} r_x^{ij} f_{y}^{ij}$ will be different. In fact, the optimal configuration will be one in which all contacts are horizontal, so that $r_x = r$, and all viscous force are vertical, so that $f_y^v = f^v$. The full expression for the maximal viscous stress, given the average viscous force is then:

$$\sigma_{x,y}^{v\text{max}} = \frac{NZ}{2V} \langle r \rangle \sqrt{\langle f_v^2 \rangle}.$$ (6.23)

The actually attained viscous shear stress, as a fraction of this maximal stress, is shown in figure 6.16, which shows that the (second kind of) anisotropy is also maximised for low density and high strain rate. But while these different forms of anisotropy are maximised in the same regime, they attain different values: the viscous anisotropy can reach values up to 0.4.

The two relationships that we have described, between $\sigma_{x,y}^{el}$ and $\langle f_{el}^2 \rangle$ on the one hand and $\sigma_{x,y}^{v}$ and $\langle f_v^2 \rangle$ on the other hand, are not identities. They cannot be used to write one of the quantities in terms of the other, they only maximise one, the shear stress, in terms of the other, the average force. More usefully, the degree to which this maximum is attained or not gives us information about correlations in the contact network. In this sense it is very similar to the relation between $\langle f_{v,net}^2 \rangle$ and $\langle f_v^2 \rangle$: one $\langle f_{v,net}^2 \rangle$ gives a maximum on the other $\langle f_v^2 \rangle$ and the degree to which this maximum is
Figure 6.17: Tying together the various force and stress expressions. Green lines indicate necessary and exact relations, red lines indicate empirical and approximate relations, purple lines indicate relations that cannot satisfactorily be reduced to an equality. The relations are given by:

\[ a \quad \sigma_{xy}^e - \langle f_e^2 \rangle = \langle f_{e,\text{net}}^2 \rangle \]

\[ b \quad \sigma_{xy}^v = \approx 1.5Z \quad \langle f_v^2 \rangle = \langle f_{v,\text{net}}^2 \rangle \]

attained gives some information about correlations (between the individual forces on a bubble). The relation between the two viscous force quantities is only different in the sense that a reasonably well-defined, and, perhaps more importantly, clearly interpretable, identity could be derived. Figure 6.17 adds the two stress-force relationships that we discussed here to the previous overview for completeness’ sake.

6.2.1 Conclusion

We have resolved the apparent contradiction between the balancing of viscous and elastic stresses on each bubble and the orders of magnitude size difference between the viscous and elastic stresses. Elastic forces are indeed, on average, larger than viscous forces, hence the difference between the stresses. However, since the elastic forces on each bubble largely cancel the resulting elastic force is small enough to be canceled by the viscous forces. In addition we tested and derived a number of other relations between the various components of the stress and forces, allowing us, for example, to express the elastic stress in the viscous forces.