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4 Ekedahl-Oort strata for CSpin-varieties

4.1 Orthogonal groups

Let $V$ be a $n + 2$-dimensional $\mathbb{Q}$-vector space with basis $\{e_1, e_2, \cdots, e_{n+2}\}$, equipped with a non-degenerate quadratic form

$$Q = -a_1x_1^2 - a_2x_2^2 + a_3x_3^2 + \cdots + a_{n+2}x_{n+2}^2, \quad a_i > 0 \text{ and square free}$$

under the above basis. Denote by $\langle -, - \rangle = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$ the associated bilinear form, and by

$$I_Q = \begin{pmatrix}
-a_1 & -a_2 \\
-a_2 & a_3 \\
\vdots & \ddots \\
& & & & a_{n+2}
\end{pmatrix}$$

the matrix corresponding to $Q$.

Let $SO(V)$ be the group scheme over $\mathbb{Q}$, whose $R$-valued points are

$$\{g \in GL(V)(R) \mid g^t I_Q g = I_Q, \det(g) = 1\}$$

for all $\mathbb{Q}$-algebra $R$. Consider the morphism $h : S \rightarrow SO(V)_{\mathbb{R}}$ given by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto \begin{pmatrix}
\frac{a^2 - b^2}{a^2 + b^2} & \frac{2ab}{a^2 + b^2} \sqrt{\frac{a^2}{a_1}} \\
-\frac{2ab}{a^2 + b^2} \sqrt{\frac{a^2}{a_2}} & \frac{a^2 - b^2}{a^2 + b^2}
\end{pmatrix} \begin{pmatrix}
\frac{a^2 - b^2}{a^2 + b^2} & \frac{2ab}{a^2 + b^2} \sqrt{\frac{a^2}{a_1}} \\
-\frac{2ab}{a^2 + b^2} \sqrt{\frac{a^2}{a_2}} & \frac{a^2 - b^2}{a^2 + b^2}
\end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

**Lemma 4.1.1.** The morphism $h$ gives a Hodge structure of type $(-1, 1) + (0, 0) + (1, -1)$ on $V$ with $\dim V^{-1,1} = 1$. Moreover, $h$ gives a Shimura datum.
Proof. Direct computation shows that \( h(z) \) acts on \( e_1 + i \sqrt{a_2} e_2 \) (resp. \( e_1 - i \sqrt{a_2} e_2 \)) in \( V_\mathbb{C} \) as multiplication by \( \frac{a_1}{a_2} \) (resp. \( \frac{a_2}{a_1} \)), and trivially on \( (e_3, \cdots, e_{n+2}) \). Hence \( h \) induces a Hodge structure of type \((-1,1) + (0,0) + (1,-1)\) on \( V \) with \( \dim V = 1 \).

Now we will check that it gives a Shimura datum.

(SV1) Under the basis \( \{i \frac{e_1}{\sqrt{2a_1}}, -i \frac{e_1}{\sqrt{2a_1}}, i \frac{e_2}{\sqrt{2a_2}}, -i \frac{e_2}{\sqrt{2a_2}}, \cdots, \frac{e_{n+2}}{\sqrt{b_{n+2}}} \} \), \( J \) becomes

\[
J = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & I_n
\end{pmatrix}.
\]

And

\[
\text{Lie}(SO(V)_\mathbb{C}) = \{ M \in M_{(n+2) \times (n+2)} | M^tJ = -JM \}
\]

\[
= \{ \begin{pmatrix} A_{2 \times 2} & B \\ C & D \end{pmatrix} | a_{11} = -a_{22}, a_{12} = a_{21} = 0, \\
b_{2i} = -c_{1i}, b_{1i} = -c_{i2}, D = -D^t \}.
\]

Denote by \( \delta_{ij} \) be the matrix whose elements are zero except the one at the \( i \)-th row and \( j \)-th colum which is 1. In our case, \( \text{Lie}(SO(2, n)_\mathbb{C}) \) has basis \( \{\delta_{11} - \delta_{22}, \delta_{2i} - \delta_{i1}, \delta_{ij} - \delta_{ji}, \delta_{st} - \delta_{ts} \} \) for \( i, j, s, t > 2 \). The conjugation action of \( h \), or equivalently, the conjugate action of

\[
\begin{pmatrix}
\frac{a_1}{a_2} & 0 & 0 \\
0 & \frac{a_2}{a_1} & 0 \\
0 & 0 & I_n
\end{pmatrix}
\]

has eigenvalue \( c_1/c_2 \) (resp. has eigenvalue \( c_2/c_1 \), resp. act trivially) on \( \delta_{1j} - \delta_{j2} \) (resp. \( \delta_{2i} - \delta_{i1} \), resp. \( \delta_{11} - \delta_{22} \) and \( \delta_{st} - \delta_{ts} \)). And hence (SV1) holds.

(SV2) \( \text{Inn}(h(i)) \) gives a Cartan involution on \( SO(V)^{\text{ad}}_\mathbb{R} \). By [7] Lemma 2.8, this is the same as to find a bilinear form \( \psi \) on \( V_\mathbb{R} \) which is \( \text{SO}(\mathbb{R}) \)-invariant, and such that \( \psi(u, h(i)v) \) is symmetric with \( \psi(v, h(i)v) \) positive definite. But one can just take \( \psi \) to be \( \langle -, - \rangle_\mathbb{R} \) defined at the beginning of this section.

(SV3) \( SO(V)^{\text{ad}} \) is simple, so it has no simple factor defined over \( \mathbb{Q} \) onto which \( h \) has trivial projection. \( \square \)
One can also determine the reflex field of this Shimura datum. First note that the torus $S$ is actually defined over $\mathbb{Z}[\frac{1}{2}]$, namely, one can take $S = \text{Spec}(\mathbb{Z}[\frac{1}{2}][x, y, z]/(x^2 + y^2)z - 1))$, with co-multiplication given by

\[
\begin{align*}
x &\mapsto x \otimes x - y \otimes y \\
y &\mapsto x \otimes y + y \otimes x \\
z &\mapsto z \otimes z.
\end{align*}
\]

The morphism $\mu : G_{m, \mathbb{C}} \to S_{\mathbb{C}}, z \mapsto (z, 1)$ is defined over $\mathbb{Z}[\frac{1}{2}]$. To be precise, the morphism $G_{m, \mathbb{Z}[\frac{1}{2}]} \to S_{\mathbb{Z}[\frac{1}{2}]}, t \mapsto \left(\frac{t+1}{2}, -\frac{it+1}{2}, \frac{t+1}{2}\right)$ is a model of it, which will also be denoted by $\mu$.

**Lemma 4.1.2.** If $n > 0$, then the reflex field of the above Shimura datum is $\mathbb{Q}$.

**Proof.** The cocharacter $h \circ \mu$ of $\text{SO}(V)$, given by

\[
\begin{pmatrix}
\frac{t^2+1}{2t} & -i\frac{t^2-1}{2t}\sqrt{\frac{a_2}{a_1}} \\
i\frac{t^2-1}{2t}\sqrt{\frac{a_1}{a_2}} & \frac{t^2+1}{2t} \\
1 & \ddots & 1
\end{pmatrix}
\]

is clearly defined over $\mathbb{Q}(i\sqrt{\frac{a_2}{a_1}})$, but the action of the non-trivial element in $\text{Gal}(\mathbb{Q}(i\sqrt{\frac{a_2}{a_1}})/\mathbb{Q})$ on $h \circ \mu$ is the same as the conjugation action of

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots \\
-1 & 1 & \cdots & 1
\end{pmatrix}
\]

$\in \text{SO}(V)$.

So the reflex field of the associated Shimura datum is $\mathbb{Q}$. \qed
Remark 4.1.3. One sees from the proof easily that if $n = 0$, the reflex field is $Q(i \sqrt{-a_2}) = Q(i \sqrt{a_1 a_2})$.

4.2 Clifford algebras and $C\text{Spin}$ groups

Definition 4.2.1. Let $V$ and $Q$ be as in 4.1, then the Clifford algebra $C(V)$ is the $\mathbb{Q}$-algebra with a map $i : V \rightarrow C(V)$ which is universal with respect to all the maps $\alpha : V \rightarrow A$, with $A$ a $R$-algebra, satisfying $\alpha(v)^2 = Q(v)$, $\forall v \in V$.

Remark 4.2.2. The Clifford algebra $C(V)$ can be constructed as follows. Let $T(V)$ be the tensor algebra of $V$, let $I$ be the two-side ideal of $T(V)$ generated by elements of the form $v \otimes v - Q(v), v \in V$. Then $C(V) \cong T(V)/I$. And $C(V)$ is finite dimensional with basis $\{e_{i_1} e_{i_2} \cdots e_{i_r} \}_{1 \leq i_2 < \cdots < i_r, 0 \leq r \leq n+2}$. Here $r = 0$ means the element 1 $\in \mathbb{Q}$.

Remark 4.2.3. $C(V)$ will be viewed as a representable ring functor:

$$((\mathbb{Q}\text{-schemes})) \rightarrow ((\text{rings})), \quad T \mapsto C(V) \otimes_{\mathbb{Q}} \Gamma(T).$$

This functor is represented by

$$C(V) = \text{Spec}(\text{Sym}_{\mathbb{Q}}(C(V)^\vee)) \cong \text{Spec}(\mathbb{Q}[\{X_{i_1, \ldots, i_r} \}_{1 \leq i_2 < \cdots < i_r, 0 \leq r \leq n+2}]).$$

The co-addition and co-multiplication

$$\mathbb{Q}[\{X_{i_1, \ldots, i_r} \}_{1 \leq i_2 < \cdots < i_r, 0 \leq r \leq n+2}] \rightarrow \mathbb{Q}[\{X_{i_1, \ldots, i_r} \}_{1 \leq i_2 < \cdots < i_r, 0 \leq r \leq n+2}]^\otimes 2$$

are given respectively by

$$X_{i_1, \ldots, i_r} \mapsto X_{i_1, \ldots, i_r} \otimes 1 + 1 \otimes X_{i_1, \ldots, i_r}, \quad \forall i_1 < i_2 < \cdots < i_r, 0 \leq r \leq n + 2$$

and

$$X_{i_1, \ldots, i_r} \mapsto f_{i_1, \ldots, i_r}(\{X_{i_1, \ldots, i_s} \otimes 1 \}_{1 \leq i_2 < \cdots < i_s, 0 \leq s \leq n+2} \cup \{1 \otimes X_{i_1, \ldots, i_t} \}_{1 \leq i_2 < \cdots < i_t, 0 \leq t \leq n+2}).$$

Here $f_{i_1, \ldots, i_r}$ is given by

$$\sum_{i_1, \ldots, i_s} x_{i_1, \ldots, i_s} e_{i_1, \ldots, i_s} \cdot \sum_{i_1, \ldots, i_t} y_{i_1, \ldots, i_t} e_{i_1, \ldots, i_t} = \sum_{i_1, \ldots, i_r} f_{i_1, \ldots, i_r} e_{i_1, \ldots, i_r},$$

which is a polynomial in $x_{i_1, \ldots, i_s}$ and $y_{i_1, \ldots, i_t}$.
Remark 4.2.4. We only defined Clifford algebras over \( \mathbb{Q} \), but in fact, they can be defined in a much more general setting. For our purpose, it will be enough to know that when there is a lattice \( V \subseteq V \), s.t. \( Q \) takes integral value on it, one can also define \( C(V) \) using the universal property. And it is a free \( \mathbb{Z} \)-module with basis \( \{ t_{i_1} t_{i_2} \cdots t_{i_r} \} \), where \( t_i \)s form a basis of \( V \).

We refer to [3] Chapter IX §9 for more details.

There is an involution \( \tau \) on \( C(V) \) given by \( e_{i_1} \cdots e_{i_r} \mapsto e_{i_r} \cdots e_{i_1} \). Here \( e_{i_1}, \cdots, e_{i_r} \) are in the chosen basis as in 4.1, and they are different from each other. We will write \( C^+(V) \) (resp \( C^-(V) \)) for the even (resp. odd) part of \( C(V) \). Note that \( C^+(V) \) is an algebra, while \( C^-(V) \) is just a \( C^+(V) \)-module. As we are more interested in \( C^+(V) \), a structure theorem will be given here.

**Theorem 4.2.5.** ([15], Theorem 7.7) Let \( Q, V \) be as at the beginning of 4.1. Then we have:

1) If \( n + 2 = 2m \), let \( d := (-1)^m a_1 \cdots a_{n+2} \). Then the even Clifford algebra \( C^+(V) \) is isomorphic to one of the following two algebras

(a) if \( \sqrt{d} \in \mathbb{Q} \), then \( C^+(V) = M_{2m-2}(D) \times M_{2m-2}(D) \) with a quaternion algebra \( D \) over \( \mathbb{Q} \).

(b) if \( \sqrt{d} / \in \mathbb{Q} \), then \( C^+(V) = M_{2m-2}(D) \) with a quaternion algebra \( D \) over \( \mathbb{Q}(\sqrt{d}) \).

2) if \( n + 2 = 2m + 1 \), then \( C^+(V) = M_{2m-1}(D) \) for a quaternion algebra \( D \) over \( \mathbb{Q} \).

**Remark 4.2.6.** There is a “reduction mod \( p \)” version of the above theorem. That is, let \( V \) be a \( n + 2 \) dimensional \( \mathbb{F}_p \)-vector space \( (p \geq 3) \), \( Q \) be a non-degenerate quadratic form on \( V \). Then precisely the same statement holds, if one use \( \mathbb{F}_p \) instead of \( \mathbb{Q} \). One can use precisely the same proof of [15], except that one should use Waring-Chevally instead of Meyer’s theorem on page 17 of [15].

**Remark 4.2.7.** ([26], Proposition 8.4) One can also see how the involutions on \( C^+(V) \) look like. The list is as follows.
1) If \( n \equiv 0 \mod 4 \), then the involution is unitary.

2) If \( n \equiv 2 \mod 4 \), then the involution fixes elements in center of \( C^+(V) \).

And the involution is orthogonal if \( n \equiv 6 \mod 8 \), symplectic if \( n \equiv 2 \mod 8 \).

3) If \( n \equiv 5, 7 \mod 8 \), then the involution is orthogonal.

4) If \( n \equiv 1, 3 \mod 8 \), then the involution is symplectic.

By the previous theorem, the functor whose \( R \)-points are invertible elements in \( C^+(V) \) is represented by a group scheme \( C^+(V) \times \) which is an open subscheme of \( C^+(V) \). The group scheme \( \text{CSpin}(V) \) is defined to be the scheme representing the functor

\[
\text{CSpin}(V)(R) = \{ g \in C^+(V)(R)^\times \mid gV R g^{-1} = V_R \},
\]

or in other words, the stabilizer of \( V \subseteq C(V) \) under the conjugation action of \( C^+(V)^\times \). One can show that \( \text{CSpin}(V) \subseteq G \), where

\[
G(R) = \{ g \in C^+(V)(R) \mid \tau(g) g \in R^\times \}.
\]

See Proposition 4 of [3] Chapter IX, 9.5 for more details.

**Remark 4.2.8.** For simplicity, let’s assume that \( k = \bar{k} \), and that \( q = \sum_{i=1}^{n+2} x_i^2 \).

Then \( \text{Lie}(G) \) and \( \text{Lie}(\text{CSpin}(V)) \) can be naturally viewed as subspaces of \( C^+(V) \). As by definition, \( \text{Lie}(G) \) is \( \{ r \in C^+(V) \mid (1 + r \varepsilon)(1 + \tau(r) \varepsilon) \in 1 + k \varepsilon \} \),

which is the same as the subspace of \( C^+(V) \) s.t. \( \tau(r) + r \in k \). One can see easily that it is the subspace of \( C^+(V) \) with basis 1 and \( \{ e_I \}_{\# I \equiv 2 \mod 4} \). And

\[
\text{Lie}(\text{CSpin}(V)) = \{ r \in \text{Lie}(G) \mid (1 + r \varepsilon) V \otimes_k k \varepsilon(1 + \tau(r) \varepsilon) \in V \otimes_k k \varepsilon \},
\]

which equals to \( \{ r \in C^+(V) \mid \tau(r) + r \in k, \text{ and } re_i + e_i \tau(r) \in V, \forall i \} \). Clearly, it is the subspace of \( C^+(V) \) with basis 1 and \( \{ e_i e_j \} \). This means that \( G \) and \( \text{CSpin}(V) \) are different once \( \dim(V) \geq 6 \).
### 4.3 The Kuga-Satake construction

Let \( S, V, Q, h \) be as in 4.1, then one can find an \( \tilde{h} : S \to \text{CSpin}(V) \), s.t.

\[
\begin{align*}
\text{CSpin}(V) & \xrightarrow{\tilde{h}} S \\
S & \xrightarrow{h} \text{SO}(V)
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{G}_{m, \mathbb{R}} & \xrightarrow{w} S \xrightarrow{\det} \mathbb{G}_{m, \mathbb{R}} \\
\mathbb{G}_{m, \mathbb{R}} & \xrightarrow{\bar{h}} \text{CSpin}(V) \xrightarrow{\text{norm}} \mathbb{G}_{m, \mathbb{R}}
\end{align*}
\]

are commutative (see [7] 4.2).

We will need an explicit version of the above construction which is given in [15] 5.6. Take \( J := e_1 e_2 \sqrt{a_1 a_2} \in \text{CSpin}(V) \), then a lifting \( \tilde{h} : S \to \text{CSpin}(V) \) is given by

\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a - b J.
\]

**Theorem 4.3.1.** ([7] Proposition 4.5) The morphism \( \tilde{h} \) induces a Hodge structure of type \((-1,0) + (0,-1)\) on \( C^+(V) \) via the embedding

\[
\text{CSpin}(V) \subseteq \text{GL}(C^+(V)), \quad g \mapsto g \cdot.
\]

We also want to study polarizations on \( C^+(V) \).

**Proposition 4.3.2.** The correspondence \( x \mapsto (e_1 e_2)^{-1} \tau(x)(e_1 e_2) \) is a positive involution on \( C^+(V) \), i.e. the bilinear form \( tr_{C^+(V)/\mathbb{Q}}((e_1 e_2)^{-1} \tau(x)(e_1 e_2)y) \) is symmetric and positive definite.

**Proof.** See [15] Proposition 5.9, or [47] Proposition 2. \( \square \)

Let \( \lambda = e_1 e_2, \langle x, y \rangle_\lambda = tr_{C^+(V)/\mathbb{Q}}(\lambda \tau(x)y) \). The previous proposition and the sentence before Remark 4.2.8 show that \( \text{CSpin}(V) \subseteq \text{GSp}(C^+(V), \langle, \rangle_\lambda) \) which implies the following statement.
Corollary 4.3.3. The morphism $\tilde{h}$ gives a Hodge type Shimura datum.

Denote by $B$ the opposite algebra of $C^+(V)$ with multiplication denoted by $*$, $C^+(V)$ is a left $B$-module induced by the right multiplication of $C^+(V)$. There is an involution on $B$

$$x \mapsto x^\lambda := \lambda^{-1} * \tau(x) * \lambda = \lambda \tau(x) \lambda^{-1}$$

Lemma 4.3.4. The involution $x \mapsto x^\lambda$ is a positive involution on $B$, and the $B$-module structure on $C^+(V)$ satisfies $\langle b^* u, v \rangle_\lambda = \langle u, b^\lambda * v \rangle_\lambda$. The group scheme $G$ defined before Remark 4.2.8 is the subgroup of $\mathrm{GSp}(C^+(V), \langle , \rangle_\lambda)$ that commutes with the action of $B$.

Proof. First, note that under the basis $\{e_i\}_{i \# I \text{ even}}$, we have

$$tr_{B/Q}(e_i) = tr_{C^+(V)/Q}(e_i) = \begin{cases} 0 & \text{if } I \neq \emptyset, \text{ i.e. } e_i \neq 1 \\ 2^{n+1} & \text{if } I = \emptyset, \text{ i.e. } e_i = 1 \end{cases}$$

This simply means that $tr_{B/Q}(c) = tr_{C^+(V)/Q}(c), \forall c \in B$.

For the first statement, one needs to show that $tr_{B/Q}(x^\lambda * x)$ is positive definite. We have

$$tr_{B/Q}(x^\lambda * x) = tr_{B/Q}(xx^\lambda) = tr_{B/Q}(x \lambda \tau(x) \lambda^{-1})$$

$$tr_{C^+(V)/Q}(\lambda^{-1} \tau(x) \lambda x) = tr_{C^+(V)/Q}(x \lambda^{-1} \tau(x) \lambda).$$

But $tr_{C^+(V)/Q}(\lambda^{-1} \tau(x) \lambda x)$ is positive definite by Proposition 4.3.2, so $tr_{B/Q}(x^\lambda * x)$ is also positive definite noting that $\lambda^{-1} = -\frac{\lambda}{a_1^2 a_2}$.

There are also equalities

$$\langle b * u, v \rangle_\lambda = \langle ub, v \rangle_\lambda = tr_{C^+(V)/Q}(\lambda \tau(b) \tau(u)v)$$

$$\langle u, b^\lambda * v \rangle_\lambda = \langle u, v \lambda \tau(b) \lambda^{-1} \rangle_\lambda = tr_{C^+(V)/Q}(\lambda \tau(u)v \lambda \tau(b) \lambda^{-1})$$

$$= tr_{C^+(V)/Q}(\lambda \tau(b) \tau(u)v).$$

So $\langle b * u, v \rangle_\lambda = \langle u, b^\lambda * v \rangle_\lambda$. 

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For the last statement, clearly $G \subseteq \operatorname{GSp}(C^+(V), \langle \cdot, \cdot \rangle) \cap \operatorname{End}_B(C^+(V))$. But by [15] Lemma 6.5, we have $C^+(V) = \operatorname{End}_B(C^+(V))$. For any $g \in C^+(V)^\times$ s.t. $tr(\lambda u) \tau(g)gv = r \cdot tr(\lambda u)gv$, $\forall \ u, v \in C^+(V)$, we can take $\tau(u) = \lambda^{-1}$ and $v = 1$, then the equality becomes $tr(\tau(g)g) = r \cdot tr(1)$, hence $\tau(g)g$ has to be a scalar.

**Remark 4.3.5.** The composition $S \to \operatorname{CSpin}(V)^\times \subseteq G^\times$ gives a Shimura datum of PEL-type.

**Lemma 4.3.6.** If $n > 0$, the reflex fields of the Shimura data given by $\operatorname{CSpin}(V)$ and $G$ are both equal to $\mathbb{Q}$.

**Proof.** By a theorem of Deligne (see, for example [34] Remark 12.3 c), we only need to show that the reflex field of $\operatorname{CSpin}(V)$ Shimura datum is $\mathbb{Q}$. The cocharacter of $\tilde{h} \circ \mu$ given by

$$
t \mapsto t + \frac{1}{2} + i \frac{t - 1}{2} J$$

is defined over $\mathbb{Q}(\sqrt{-a_1a_2})$, and the action on it by the non-trivial element in $\operatorname{Gal}(\mathbb{Q}(\sqrt{-a_1a_2})/\mathbb{Q})$ is the same as the conjugation action of $e_2e_3 \in \operatorname{CSpin}(V)$. Hence the reflex field of the Shimura datum is $\mathbb{Q}$. □

**Remark 4.3.7.** One can also compute the reflex field for $G$ directly, when $G$ is of type A or C. According to [5] page 1, the reflex field is generated by $\{tr_{C_0}(b) \mid b \in B\}$ over $\mathbb{Q}$. Here $C_0$ is the subspace of $C^+(V)_\mathbb{C}$ where $\mathbb{C}^\times$ acts trivially via $\mu$. Denote by $\Delta_e = \{e_I \mid I \subseteq \{3, 4, \cdots, n + 2\}, \#I \text{ even}\}$ and $\Delta_o = \{e_I \mid I \subseteq \{3, 4, \cdots, n + 2\}, \#I \text{ odd}\}$. Then the disjoint union $\Delta_e \bigsqcup e_1e_2 \Delta_e \bigsqcup e_1\Delta_o \bigsqcup e_2\Delta_o$ form a basis of $C^+(V)$, and $\{e_I - iJe_I \mid e_I \in \Delta_e \bigsqcup e_1\Delta_o \bigsqcup e_1\Delta_e \}$ form a $\mathbb{C}$-basis of $C_0$. Note that the $B$-action is induced by right multiplication of $C^+(V)$, so for an element $e_I$ in the basis $\{e_I \#I \text{ even}\}$ of $B$, it has non-trivial trace on $V_0$ if and only if $e_I = 1$.

**Remark 4.3.8.** Let $a_i$ be coefficients of $Q$ as in 4.1. Assume that all the $a_i$s are square free integers. Then for a prime $p$ s.t. $(p, 2 \prod_i a_i) = 1$, the polarization
\langle r \rangle \lambda is of degree prime to \( p \). As
\[
\langle e_i, e_j \rangle \lambda = \text{tr}_{C^+(V)/Q}(\lambda \tau(e_i)e_j),
\]
and it is non-zero if and only if \( e_j = e_i \lambda \) up to scalar. So under basis \( \{1, e_1e_2, \cdots, e_i, e_j, \lambda, \cdots\} \), the matrix of \( \langle r \rangle \lambda \) is of the form
\[
\begin{pmatrix}
0 & a \\
-a & 0 \\
0 & b \\
-b & 0 \\
\cdots \\
0 & c \\
-c & 0
\end{pmatrix},
\]
with all the none-zero entries products of \( 2^{n+1} \) and \( a_i \). So the degree, which is the determinant of the above matrix, is prime to \( p \).

### 4.4 Ekedahl-Oort strata for CSpin-varieties

Let \( p > 2 \) be a prime number and \( n \) be a positive integer. Let \( V \) be a \( \mathbb{Q} \)-vector space with basis \( e_1, e_2, \cdots, e_{n+2} \). Let \( Q \) be a quadratic form on \( V \), s.t. under basis \( e_1, e_2, \cdots, e_{n+2} \), \( Q = \sum_{i=1}^{n+2} a_i x_i^2 \) with \( a_i \in \mathbb{Q}_{>0} \cap \mathbb{Z}(p) \) and square free. In this case, if we denote by \( V_{\mathbb{Z}_p} \) the \( \mathbb{Z}_p \)-lattice of \( V \otimes \mathbb{Q}_p \) generated by \( e_1, e_2, \cdots, e_{n+2} \), then \( \text{SO}(V_{\mathbb{Z}_p}) \) is reductive, and so is \( \text{CSpin}(V_{\mathbb{Z}_p}) \). In particular, \( K_p = \text{CSpin}(V_{\mathbb{Z}_p})(\mathbb{Z}_p) \) is a hyperspecial subgroup of \( \text{CSpin}(V)(\mathbb{Q}_p) \). Let \( K^p \subseteq \text{CSpin}(V)(\mathbb{A}^p_f) \) be a compact open subgroup which is small enough. Then the \( \mathbb{Q} \)-variety \( \text{Sh}_{K_p,K^p}(\text{CSpin}(V), X) \) extends to a smooth quasi-projective \( \mathbb{Z}(p) \)-scheme \( \mathcal{Z}_{K_p,K^p}(\text{CSpin}(V), X) \) by Corollary 4.3.3 and Theorem 2.1.2. Let \( \mathcal{Z}_0 \) the special fiber of the scheme \( \mathcal{Z}_{K_p,K^p}(\text{CSpin}(V), X) \), then our results in the 3.1 work for \( \mathcal{Z}_0 \).

For simplicity of notations, we will take \( m \) s.t. \( n + 2 = 2m \) when \( n \) is even, and \( n + 2 = 2m + 1 \) when \( n \) is odd. And as we only work with reductions, the
\( \mathbb{F}_p \)-vector space \( V_{\mathbb{Z}_p} \otimes \mathbb{F}_p \) will be denoted by \( V \). Note that \( m \) is the dimension of a maximal torus in \( \text{SO}(V) \).

**Proposition 4.4.1.** There are at most \( 2^m \) Ekedahl-Oort strata for \( \mathcal{X}_0 \).

**Proof.** Let \( W \) be the Weyl group of \( \text{CSpin}(V) \), which is the same as the one for \( \text{SO}(V) \). By Theorem 3.1.5, all the possible Ekedahl-Oort strata are parameterized by the finite set \( ^dW \). We claim that \( \#(^dW) = 2^m \). To compute \( \#(^dW) \), we will work with \( \text{SO}(V) \).

1) Assume that \( n \) is odd. After passing to \( k = \overline{\mathbb{F}_p} \), one can multiply the basis \( \{e_1, \ldots, e_{n+2}\} \) by scalars s.t. \( Q = \sum_i (-1)^{i+1} x_i^2 \). We still denote this basis by \( \{e_1, \ldots, e_{n+2}\} \). Let \( f_1 = \frac{1}{2}(e_1+e_2) \), \( f_{n+2} = \frac{1}{2}(e_1-e_2) \), \( f_2 = \frac{1}{2}(e_3+e_4) \), \( f_{n+1} = \frac{1}{2}(e_3-e_4) \), \( \ldots \), \( f_m = \frac{1}{2}(e_n+e_{n+1}) \), \( f_{m+2} = \frac{1}{2}(e_n-e_{n+1}) \), \( f_{m+1} = e_{n+2} \). Then under basis \( \{f_1, f_2, \ldots, f_{n+2}\} \), \( Q \) becomes

\[
y_1 y_{n+2} + y_2 y_{n+1} + \cdots + y_m y_{m+2} + y_{m+1}^2.
\]

Consider the diagonal maximal torus and the the lower triangle Borel subgroup of \( \text{SO}(V) \). The cocharacter \( \mu \) (see the construction before Lemma 4.1.2) is given by

\[
t \mapsto \begin{pmatrix}
t & & \\
1 & \ddots & \\
& & 1 \\
& & t^{-1}
\end{pmatrix}.
\]

Then the descending filtration induced by \( \mu \) is

\[
V \supseteq \langle f_2, \ldots, f_{n+2} \rangle \supseteq \langle f_{n+2} \rangle \supseteq 0.
\]
And its stabilizer is the parabolic
\[
\begin{pmatrix}
* & 0 & \cdots & 0 & 0 \\
* & * & \cdots & * & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & \cdots & * & 0 \\
* & * & \cdots & * & * \\
\end{pmatrix}
\]

It is easy to compute that \( W \cong \{\pm 1\}^m \times S_m \). And \( W \) is generated by simple reflections \( \{s_i\}_{i=1,\ldots,m} \) where

\[
s_i = \begin{cases} 
(i, i + 1)(n - i + 2, n - i + 3), & \text{for } i = 1, \ldots, m - 1; \\
(m, m + 2), & \text{for } i = m.
\end{cases}
\]

The subgroup \( W_J \) is generated by \( \{s_i\}_{i=2,\ldots,m} \), moreover, \( W_J \cong \{\pm 1\}^{m-1} \times S_{m-1} \). So \( #(W_J \backslash W) = 2m \).

2) When \( n \) is even, one uses the above proof word by word, except that \( f_m = \frac{1}{2}(e_{n+1}+e_{n+2}) \), \( f_{m+1} = \frac{1}{2}(e_{n+1}-e_{n+2}) \) and \( s_m = (m-1, m+1)(m, m+2) \).

In this case, \( W \cong \{\pm 1\}^m_0 \times S_m \) and \( W_J \cong \{\pm 1\}^{m-1}_0 \times S_{m-1} \). Here \( \{\pm 1\}^i_0 \) means the set of elements in \( \{\pm 1\}^i \) with even number of \(-1\)s.

Now we will compute the dimension and describe the Zariski closure of each stratum (assuming that it is non-empty). We will first study the scheme of types for parabolic subgroups of a reductive group. This is necessary for the study of general Hodge type Shimura varieties, but not really necessary for CSpin-varieties. So people who only care about Ekedahl-Oort strata for CSpin-varieties can go to 4.4.5 and 4.4.7 directly.

### 4.4.2 The scheme of types

Here we give an effective version of A.5 of [53]. Let \( G \) be a reductive group (NOT necessarily quasi-split) over a perfect field \( k \), denote by \( \overline{k} \) the algebraic closure of \( k \). Take any pair \( (B, T) \) with \( B \) a Borel subgroup of \( G_{\overline{k}} \), \( T \subseteq B \) a maximal torus of \( G_{\overline{k}} \). For a \( \sigma \in Gal(\overline{k}/k) \), there is a unique \( g_\sigma \in G(\overline{k}) \) up
to right multiplication by $\sigma T(\overline{k})$, s.t. $g_\sigma(\sigma B, \sigma T) = (B, T)$. So one can define an action $*$ of $Gal(\overline{k}/k)$ on $B$ by $\sigma * b = g_\sigma(b)$, which takes $T$ to $T$. Note that the formula $g_{\sigma \tau} = g_\sigma \sigma(g_\tau)$ implies that it is an action. This new action takes positive roots (resp. simple roots) to positive root (resp. simple roots).

Denote by $S := S(B, T)$ the set of simple roots, and $P_S$ the power set of $S$. Then we have an action of $Gal(\overline{k}/k)$ on $S$ and hence on $P_S$, which gives a finite étale scheme $\mathcal{P}$ over $k$.

As $\mathcal{P}$ is independent of choices of $(B, T)$ in the sense that for a different $(B', T')$, denote by $g_\sigma'$ the unique element in $G(\overline{k})$ s.t. $g_\sigma'(\sigma B', \sigma T') = (B', T')$, $I = I(B', T')$ and $*$ be the action of $Gal(\overline{k}/k)$ on $B'$ given by $b \mapsto g_\sigma' b$, then the diagram

$$
\begin{array}{ccc}
I & \xrightarrow{g} & I' \\
\sigma_* & & \sigma_*' \\
I & \xrightarrow{g} & I'
\end{array}
$$

is commutative. This follows from that $g_\sigma = g^{-1} g'_\sigma \sigma(g)$.

As $\mathcal{P}$ is independent of choices of $(B, T)$, we can choose $T$ to be a maximal torus defined over $k$. Then $g_\sigma$ is a unique element in $W(T)$, denoted by $w_\sigma$.

Now we will show that $\mathcal{P}ar_G/G$ is isomorphic to $\mathcal{P}$. To do this, we only need to fix a pair $(B, T)$, and construct a $Gal(\overline{k}/k)$ equivariant bijection $\mathcal{P}ar_G(\overline{k})/G(\overline{k}) \rightarrow P_S$ with respect to the ordinary action on $\mathcal{P}ar_G(\overline{k})/G(\overline{k})$ and the $*$ action on $P_S$. In each $G(\overline{k})$-orbit $O$ of $Par_G(\overline{k})$, there is a unique parabolic $P$ containing $B$, and hence gives a subset of $J \subseteq I$. To see that it is Galois equivariant, note that $\sigma(O) = \{ \sigma(g) \sigma(P) \sigma(g^{-1}) \}$, and $\sigma(P)$ is the unique element in $\sigma(O)$ containing $\sigma(B)$. So $w_\sigma \sigma(P)$ is the unique element in $\sigma(O)$ containing $B$, and gives a subset $w_\sigma \sigma(J)$. This mean that the map is Galois equivariant, and $\mathcal{P}ar_G/G \cong \mathcal{P}$.

As main examples, we compute the scheme of types for orthogonal groups. And the perfect field $k$ will be of characteristic $> 2$. Let $V$ be a $k$-vector space with basis $\{ e_1, \cdots, e_n \}$. Let $q(x) = \sum_i a_i x_i^2$ be a non-degenerate quadratic form, where $x = \sum_i x_i e_i$, and $SO(V, q)$ be the special orthogonal group fixing $q$. 

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Example 4.4.3. Let $2m + 1 = n$. Consider the maximal torus

$$T^0 = \prod_{i \leq m} T_i \subseteq \text{SO}(V, q),$$

where

$$T_i = \{ g \in \text{SL}_2 \mid g^t \begin{pmatrix} a_i & \* \\ \* & a_{n+1-i} \end{pmatrix} g = \begin{pmatrix} a_i & \* \\ \* & a_{n+1-i} \end{pmatrix} \}.$$ 

Let $b_i \in \bar{k}$ be such that $b_i^2 = -\frac{a_i}{a_{n+1-i}}$ for $i \leq m$. Then under basis

$$\{ e_1 + b_1 e_n, e_2 + b_2 e_{n-1}, \ldots, e_m + b_m e_{m+2}, 2e_{m+1}, e_m - b_m e_{m+2}, \ldots, e_2 - b_2 e_{n-1}, e_1 - b_1 e_n \},$$

the quadratic form $q$ becomes $4x_{m+1}^2 + \sum_{i \leq m} 4a_i x_i x_{n+1-i}$. Note that under this basis, $T_0$ becomes

$$T = \begin{pmatrix} t_1 \\ \vdots \\ t_m \\ 1 \\ t_{m-1}^t \\ \vdots \\ t_1^{-1} \end{pmatrix},$$

which is a split maximal torus of $\text{SO}(V, q)_{\bar{k}}$.

The group of characters $X^*(T)$ of $T$ has a basis $\{ \chi_i \}_{i \leq m}$. Where $\chi_i$ is the character

$$\begin{pmatrix} t_1 \\ \vdots \\ t_m \\ 1 \\ t_{m-1}^t \\ \vdots \\ t_1^{-1} \end{pmatrix} \to t_i.$$
We will use the upper triangle Borel and $T$ to describe positive roots. In this case, the positive roots are $\chi_i$, $i \leq m$ and $\chi_i \pm \chi_j$, $i < j$; and the simple roots are $\chi_1 - \chi_2$, $\chi_2 - \chi_3$, $\ldots$, $\chi_{m-1} - \chi_m$, $\chi_m$. The action of the Galois group $\text{Gal}(\overline{k}/k)$ on $X^*(T)$ factor through the finite quotient $\text{Gal}(k'/k)$, where $k'$ is the field $k(b_i)_{i \leq m}$. To be more precise, we have $\sigma(\chi_i) = \frac{\sigma(b_i)}{b_i} \chi_i$.

Let $\mu_B = \sum_{i=1}^{m} (m + 1 - i) \chi_i^\vee = m\chi_1^\vee + \cdots + \chi_m^\vee$.

Then the set of positive (resp. negative) roots are the roots s.t. $\langle \mu_B, \alpha \rangle > 0$ (resp. $\langle \mu_B, \alpha \rangle < 0$). And hence the set of Borels containing $T$ is in bijection with the $W(T)$-orbit of $\mu_B$. So $w_\sigma$ is the unique element in $W(T)$ s.t. $w_\sigma(\sigma(\mu_B)) = \mu_B$.

But

$\sigma(\mu_B) = \sigma\left( \sum_{i=1}^{m} (m + 1 - i) \chi_i^\vee \right) = \sum_{i=1}^{m} (m + 1 - i) \frac{\sigma(b_i)}{b_i} \chi_i^\vee$.

So $w_\sigma = (\frac{\sigma(b_1)}{b_1}, \ldots, \frac{\sigma(b_m)}{b_m})$. Here we use the identification $W(T) \cong \{\pm 1\}^m \rtimes S_m$.

And the Galois action $*$ is always trivial on the set of simple roots. □

**Example 4.4.4.** If $n = 2m$, consider the torus $T^0 = \prod_{i \leq m} T_i \subseteq \text{SO}(V, q)$, where

$T_i = \{ g \in SL_2 \mid g^t \begin{pmatrix} a_i & \ast \\ a_{n+1-i} & \ast \end{pmatrix} g = \begin{pmatrix} a_i & \ast \\ a_{n+1-i} & \ast \end{pmatrix} \}$.

Let $b_i \in \mathbb{F}$ be such that $b_i^2 = -\frac{a_i}{a_{n+1-i}}$ for $i \leq m$. Then under basis

$\{e_1 + b_1 e_n, e_2 + b_2 e_{n-1}, \ldots, e_m + b_m e_{m+1}, e_m - b_m e_{m+1}, \ldots, e_2 - b_2 e_{n-1}, e_1 - b_1 e_n\}$,

$q$ becomes $\sum_{i \leq m} 4a_i x_i x_{n+1-i}$. Note that under this basis, $T_0$ becomes

$$
T = \begin{pmatrix}
    t_1 & t_2 & \cdots & t_2^{-1} \\
    t_2 & t_1 & \cdots & t_1^{-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_2^{-1} & t_1^{-1} & \cdots & t_1
\end{pmatrix}
$$

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which is a split maximal torus of $\text{SO}(V, q)_{\mathbb{K}}$.

The group of characters $X^*(T)$ of $T$ has a basis $\{\chi_i\}_{i \leq m}$. Where $\chi_i$ is the character

$$
\left( \begin{array}{cccc}
    t_1 \\
    t_2 \\
    \ddots \\
    t_2^{-1} \\
    t_1^{-1}
\end{array} \right) \rightarrow t_i.
$$

We will use the upper triangle Borel and $T$ to describe positive roots. In this case, the positive roots are $\chi_i \pm \chi_j$, $i < j$ and the simple roots are $\chi_1 - \chi_2$, $\chi_2 - \chi_3$, $\cdots$, $\chi_{m-1} - \chi_m$, $\chi_m - \chi_1$. The action of the Galois group $\text{Gal}(\mathbb{K}/k)$ on $X^*(T)$ factor through the finite quotient $\text{Gal}(k'/k)$, where $k'$ is the field $k(b_i)_{i \leq m}$. To be more precise, we have $\sigma(\chi_i) = \frac{\sigma(b_i)}{b_i} \chi_i$.

Next, we will compute $w_\sigma$. To do this, let

$$
\mu_B = \sum_{i=1}^{m} (m-i)\chi_i^\vee = (m-1)\chi_1^\vee + \cdots + \chi_{m-1}^\vee.
$$

Then the set of positive (resp. negative) roots are the roots s.t. $\langle \mu_B, \alpha \rangle > 0$ (resp. $\langle \mu_B, \alpha \rangle < 0$). And hence the set of Borels containing $T$ is in bijection with the $W(T)$-orbit of $\mu_B$. So $w_\sigma$ is the unique element in $W(T)$ s.t. $w_\sigma(\sigma(\mu_B)) = \mu_B$. But

$$
\sigma(\mu_B) = \sigma(\sum_{i=1}^{m} (m-i)\chi_i^\vee) = \sum_{i=1}^{m} (m-i) \frac{\sigma(b_i)}{b_i} \chi_i^\vee,
$$

so

$$
w_\sigma = \left( \frac{\sigma(b_1)}{b_1}, \cdots, \frac{\sigma(b_{m-1})}{b_{m-1}}, \frac{\sigma(b_1 \cdots b_{m-1})}{b_1 \cdots b_{m-1}} \right) = \left( \frac{\sigma(b_1)}{b_1}, \cdots, \frac{\sigma(b_m)}{b_m} \right) \cdot (1, \cdots, 1, \frac{\sigma(b_1 \cdots b_m)}{b_1 \cdots b_m}).
$$

Here we use the identification $W(T) \cong \{ \pm 1 \}_0^m \rtimes S_m$, where $\{ \pm 1 \}_0^m$ means the subset of elements in $\{ \pm 1 \}^m$ whose product of all factors are +1.
The Galois action $\ast$ on $\chi_i$ is given by

$$
\sigma \ast \chi_i = w_{\sigma} \cdot \sigma(\chi_i) = \begin{cases} 
\chi_i, & \text{if } i \neq m \\
\frac{\sigma(\sqrt{(-1)^m}d)}{\sqrt{(-1)^m}d} \chi_i, & \text{if } i = m
\end{cases}
$$

Here $d = \prod_{i=1}^n a_i$. Now we can describe the Galois action on $P_S$. Let $C = \{\chi_{m-1} - \chi_m, \chi_{m-1} + \chi_m\}$. Denote by $s(J) = \left(\frac{\sigma(\sqrt{(-1)^m}d)}{\sqrt{(-1)^m}d}\right)^{(J \cap C)}$ for a subset $J \subseteq I$. Then $J$ is Galois invariant if and only if $s(J) = 1$, and $\sigma(J) = (J - C) \cup (C - (J \cap C))$ if $s(J) = -1$. So if we denote by $k'$ the field extension $k(\sqrt{(-1)^m}d)$ of $k$, then

$$
P \cong \begin{cases} 
\prod_{2^m} \text{Spec}(k), & \text{if } (-1)^m d \text{ is a square;} \\
(\prod_{2^m} \text{Spec}(k)) \prod_{2^{m-2}} \text{Res}_{k'/k} \text{Spec}(k'), & \text{if } (-1)^m d \text{ is not a square.}
\end{cases}
$$

4.4.5 Ekedahl-Oort strata for odd dimensional CSpin-varieties

We will describe the dimension and Zariski closure of an Ekedahl-Oort stratum. The key points are our Proposition 3.1.6 and that the partial order $\preceq$ on $JW$ is finer than the Bruhat order (see [44] 1.5). Keep notations as what we did before Proposition 4.4.1. We will compute the case when $n$ is odd here, and when $n$ is even in 4.4.7.

**Proposition 4.4.6.** Let $V$ be of dimension $n + 2$ with $n$ odd. Then for any integer $0 \leq i \leq n$, there is at most one stratum $\mathcal{S}_i^i$ s.t. $\dim(\mathcal{S}_i^i) = i$. And these are all the Ekedahl-Oort strata on $\mathcal{S}_0$. Moreover, the Zariski closure of $\mathcal{S}_0^i$ is the union of all the $\mathcal{S}_0^i$, s.t. $i' \leq i$.

**Proof.** We use notations as in the proof of Proposition 4.4.1. Under basis $\{f_1, \cdots, f_{n+2}\}$, the largest element $\omega \in W$ has a reduced expression

$$
\omega = s_m(s_{m-1}s_{m-1})(s_{m-2}s_{m-1}s_{m-1}s_{m-2}) \cdots (s_1 \cdots s_{m-1}s_{m-1} \cdots s_1)
$$
Direct computation shows that
\[ \omega = (1, n+2)(2, n+1) \cdots (m, m+2), \]
which is a central element of order 2. One can also deduce this from [6] Remark 13.1.8, which says that for an algebraic group of type \( B_m \), the largest element in its Weyl group is the central element 
\[ -1 = (-1, -1, \cdots, -1) \in \{ \pm 1 \}^m \rtimes S_m. \]
So in this case, \( K := \varphi(J) = J \), and \( W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega \).
The last equality is because of that \( w_0 \) is central and that \( W_J \) is a group.
To get the shortest element \( x \) in \( W_K \omega W_{\varphi(J)} = W_J \omega \), we only need to find
the largest element \( w'_0 \) in \( W_J \).
So in this case, \( K := \omega \varphi(J) = J \) and \( W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega \).
The last equality is because of that \( w_0 \) is central and that \( W_J \) is a group.
To get the shortest element \( x \) in \( W_K \omega W_{\varphi(J)} = W_J \omega \), we only need to find
the largest element \( w'_0 \) in \( W_J \).
So in this case, \( K := \omega \varphi(J) = J \) and \( W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega \).
The last equality is because of that \( w_0 \) is central and that \( W_J \) is a group.
To get the shortest element \( x \) in \( W_K \omega W_{\varphi(J)} = W_J \omega \), we only need to find
the largest element \( w'_0 \) in \( W_J \).
So in this case, \( K := \omega \varphi(J) = J \) and \( W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega \).
The last equality is because of that \( w_0 \) is central and that \( W_J \) is a group.
To get the shortest element \( x \) in \( W_K \omega W_{\varphi(J)} = W_J \omega \), we only need to find
the largest element \( w'_0 \) in \( W_J \).
So in this case, \( K := \omega \varphi(J) = J \) and \( W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega \).
The last equality is because of that \( w_0 \) is central and that \( W_J \) is a group.
of $1, n+2$ with those of $2, n+1$. And hence $w_{n-1} = w_n s_1$. But $w_i$ is obtained from $w_{i+1}$ by interchanging the roles of $n - i, i + 3$ with those of $n - i + 1, i + 2$, so we get the claim inductively. We also see that the expression $w_{i-2} = s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_{n+3-i}$ is reduced, and hence the length of $w_{i-2}$ is $i - 2$ for $i > m + 1$. If $1 < i < m + 1$, write $w_{i-1}$ for the element s.t. $w^{-1}(1) = i$. Then by the same method, $w_{i-1} = s_1 \cdots s_{i-1}$ which is of length $i - 1$. This proves that for each $0 \leq i \leq n$, there is at most one stratum with dimension $i$, and that there are all the strata.

To see the Zariski closure of a stratum, we note that the partial order $\preceq$ is finer than the Bruhat order which is a total order. So the partial order $\preceq$ has to coincide with the Bruhat order. This proves the last statement.

4.4.7 Ekedahl-Oort strata for even dimensional CSpin-varieties

Now we turn to the cases when $n$ is even and positive. We have the following result.

**Proposition 4.4.8.** Let $V$ be of dimension $n + 2$ with $n$ even and positive. Then for any integer $0 \leq i \leq n$ and $i \neq n/2$, there is at most one stratum $\mathcal{S}_i$ s.t. $\dim(\mathcal{S}_i) = i$. There are at most 2 strata of dimension $n/2$. And these are all the Ekedahl-Oort strata on $\mathcal{S}_0$. Moreover, the Zariski closure of the stratum $\mathcal{S}_0^w$ is the union of $\mathcal{S}_0^w$ with all the strata whose dimensions are smaller than $\dim(\mathcal{S}_0^w)$.

**Proof.** We still use the notations as in the proof of Proposition 4.4.1. In this case, the reductive group is of type $D_m$, and the largest element $\omega \in W$ has a reduced expression

$$\omega = s_{m-1} s_m (s_{m-2} s_{m-1} s_m s_{m-2}) (s_{m-3} s_{m-2} s_{m-1} s_m s_{m-2} s_{m-3}) \cdots (s_1 \cdots s_{m-2} s_{m-1} s_m \cdots s_1)$$

by [1] page 15, Table 1. Direct computation shows that

$$\omega = (1, n+2)(2, n+1) \cdots (m, m+1)$$

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when \( m \) is even, and \( \omega = (1, n + 2)(2, n + 1) \cdots (m - 1, m + 2) \) when \( m \) is odd. One can also deduce this from [6] Remark 13.1.8, which says that for an algebraic group of type \( D_m \), the largest element in its Weyl group is the central element \(-1 = (-1, -1, \cdots, -1) \in \{ \pm 1 \}^m \rtimes S_m\) when \( m \) is even, and it is \(-1 = (-1, -1, \cdots, -1) \in \{ \pm 1 \}^m \rtimes S_m\) multiplied by the order 2 automorphism of the Coxeter diagram (i.e. interchanging the role of \( s_{m-1} \) and \( s_m \), which means mapping \( \chi_m \) to \(-\chi_m \), or multiplying by \((m, m + 1)\) in language of permutations). The subgroup \( W_J \) is generated by \( \{ s_i \}_{i=2, \cdots, m} \).

Let’s first assume that \( m \) is even. In this case, \( K := \omega \varphi(J) = J \), and \( W_K \omega \varphi(J) = W_{J\omega} \) as in the previous case. The largest element in \( W_J \) is \( w' = (2, n + 1)(3, n) \cdots (m - 1, m + 2) \), and so the shortest element in \( W_J \omega \) is \((1, n + 2)(m, m + 1)\), with a reduced expression \( s_1 \cdots s_{m-1}s_ms_{m-2} \cdots s_1 \). And \( JW = W' \coprod W'' \). Here

\[
W' = \{ w \in W \mid w^{-1}(1) \geq m + 1, w^{-1}(2) < \cdots < w^{-1}(m - 1) < w^{-1}(m + 1) \\
< w^{-1}(m) < w^{-1}(m + 2) < \cdots < w^{-1}(n + 1) \};
\]

\[
W'' = \{ w \in W \mid w^{-1}(1) \leq m, w^{-1}(2) < \cdots < w^{-1}(n + 1) \}.
\]

For \( i > m + 1 \) (resp. \( i < m \)), we denote by \( w_{i-2} \) (resp. \( w_{i-1} \)) the element s.t. \( w^{-1}(1) = i \). And we write \( w'_{m-1} \) (resp. \( w''_{m-1} \)) for the element in \( W' \) (resp. \( W'' \)) s.t. \( w^{-1}(1) = m + 1 \) (resp. \( w^{-1}(1) = m \)).

We see that \( w_n = (1, n + 2)(m, m + 1) \). Claim that

\[
w_i = \begin{cases} s_1 \cdots s_ms_{m-2} \cdots s_{n+1-i}, & \text{if } i > m - 1 \\
s_1 \cdots s_i, & \text{if } i < m - 1 \end{cases}
\]

and \( w'_{m-1} = s_1 \cdots s_{m-2}s_m \), \( w''_{m-1} = s_1 \cdots s_{m-1} \). To see this, by our construction,

\[
(\underbrace{w^{-1}_i(2), \cdots, w^{-1}_i(m-1), w^{-1}_i(m+1), w^{-1}_i(m), w^{-1}_i(m+2),}
\cdots, w^{-1}_i(n+1)}_{\text{\( }_{i=1, \cdots, n-i, n+2-i, \cdots, i+1, i+3, \cdots, n+2)\text{\( )}}
\]
for $i > m - 1$, and

$$(w_{m-1}^{'-1}(2), \ldots, w_{m-1}^{'-1}(m-1), w_{m-1}^{'-1}(m+1), w_{m-1}^{'-1}(m), w_{m-1}^{'-1}(m+2),$$

$$\ldots, w_{m-1}^{'-1}(n+1))$$

$=$(1, $\ldots$, $m-1$, $m+2$, $\ldots$, $n+2$).

Note that $w_i(m) = m+1$, $w_i(m+1) = m$, and similar method as in the proof of Proposition 4.4.6 shows that $w_{i-1} = w_i s_{n+1-i}$ for $i > m - 1$. Moreover, $w_{m-1}^\prime = w_m s_m$. And this proves the claim for $w_i, i > m - 1$ and $w_{m-1}^\prime$. The other half works similarly.

For $m$ odd, the longest element $\omega \in W$ is no longer central. But $\varphi(J) = J$ and $\omega J = J$ still hold. This is because $J$ is given by a cocharacter defined over $\mathbb{F}_p$, and $J$ is the subset $\{s_2, s_3, \ldots, s_m\}$ of $I$, while $\omega s_i = s_i$ for $1 < i < m - 1$, and $\omega s_{m-1} = s_m$, $\omega s_m = s_{m-1}$. So $W_K \omega W_{\varphi(J)} = W_J \omega W_J = W_J \omega$ as before.

Using the same method as in the case when $m$ is even, we conclude that the largest element in $J W$ is $w_n$ with a reduced expression $s_1 \cdots s_{m-1} s_m s_{m-2} \cdots s_1$.

And

$$w_i = \begin{cases} 
    s_1 \cdots s_m s_{m-2} \cdots s_{n+1-i}, & \text{if } i > m - 1 \\
    s_1 \cdots s_i, & \text{if } i < m - 1,
\end{cases}$$

$w_{m-1}^\prime = s_1 \cdots s_{m-2} s_m$, $w_{m-1}'' = s_1 \cdots s_{m-1}$.

Now it is clear that there is at most one stratum of dimension $i$ for $i \neq n/2$, and that there are almost 2 strata of dimension $n/2$. These are clearly all the possible strata. To describe the Zariski closure of a stratum, we still use the fact that the partial order $\preceq$ is finer than the Bruhat order. It simply implies that for $i \neq n/2$, the Zariski closure of the $i$-dimensional stratum is the union of all the strata with dimension $\leq i$. But the Zariski closure of a $n/2$-dimensional stratum can not contain other strata of dimension $\geq n/2$, and so the last statement holds.