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**Title:** G-zips and Ekedahl-Oort strata for Hodge type Shimura varieties  
**Issue Date:** 2013-10-25
3 Ekedahl-Oort Strata for Hodge type Shimura varieties

3.1 Basic properties of Ekedahl-Oort strata

In this section, we will define Ekedahl-Oort strata for Hodge Type Shimura varieties and study their basic properties. Before stating the next theorem, we will fix some notations. Since in this chapter we will simply work on $\kappa$, we will write $G$, $V$ and $\mu$ for their reduction mod $p$, and we will denote by $P_+$, $P_-$ and $L$ the parabolics and Levi subgroup induced by $\mu$ as at the beginning of 1.2.

**Definition 3.1.1.** The $G$-zip $(I, I_+, I_-, \iota)$ on $\mathcal{S}_0$ induces a morphism of smooth algebraic stacks $\zeta : \mathcal{S}_0 \to G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}}$. For a point $x$ in the topological space of $G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}}$, the Ekedahl-Oort stratum in $\mathcal{S}_0 \otimes \bar{\kappa}$ associated to $x$ is defined to be $\zeta^{-1}(x)$.

Now we will state our main result.

**Theorem 3.1.2.** The morphism $\zeta : \mathcal{S}_0 \to G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}}$ is smooth.

**Proof.** By Theorem 1.2.7, $G_\kappa \to G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}}$ is an $E_{G,\mu}$-torsor. To prove that $\zeta : \mathcal{S}_0 \to G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}}$ is smooth, it suffices to prove that in the cartesian diagram

$$
\begin{array}{ccc}
\mathcal{S}_0^\# & \to & \mathcal{S}_0 \\
\downarrow \zeta^\# & & \downarrow \zeta \\
G_\kappa & \to & G^{\text{-Zip}}_{\kappa \otimes \bar{\kappa}} 
\end{array}
$$

the morphism $\zeta^\#$ is smooth. Note that $\mathcal{S}_0^\#$ and $G_\kappa$ are both smooth over $\kappa$, so to show that $\zeta^\#$ is smooth, it suffices to show that the tangent map at each closed point is surjective (see [18] Chapter 3, Theorem 10.4).

Let $x^\# \in \mathcal{S}_0^\#$ be a closed point, its image in $\mathcal{S}_0$ is denoted by $x$ which is also a closed point. Let $R_G$ be as in 2.3.4 which is actually the modulo $p$ of
the universal deformation ring at $x$. Consider the cartesian diagram

$$
\begin{align*}
X & \longrightarrow \Spec(R_G) \\
\downarrow^{\alpha} & \downarrow \\
G_\kappa & \longrightarrow G\text{-Zip}^\mu.
\end{align*}
$$

The morphism $X \to \Spec(R_G)$ is a trivial $E_{G,\mu}$-torsor by our construction at the very end of 2.3.4: the $G$-zip over $R_G$ is isomorphic to $L_{ugt}$ (see Construction 1.2.4). The $R_G$-point $ugt$ of $G_\kappa$ gives a trivialization of the $E_{G,\mu}$-torsor $X$ over $R_G$. This trivialization induces an isomorphism from $\Spec(R_G) \times_\kappa E_{G,\mu}$ to $X$ that sends, for any $\kappa$-scheme $T$, a point $(u, l, u_+, u_-)$ to $lu_+ugt(l^{(\mu)}u_-)^{-1}$ (see Equation 1.2.6 and the line following it, and note that as $\kappa$-scheme, $E_{G,\mu} = L \times U_+ \times U_-^{(\mu)}$, and that $R_G$ is the complete local ring of $U_-$ at the origin). It follows that the tangent space at the origin of $\Spec(R_G) \times L \times U_+$ maps surjectively to that of $G_\kappa$.

3.1.3 Dimension and closure of a stratum

Thanks to Theorem 3.1.2, the combinatorial description for the topological space of $[E_{G,\mu}\backslash G_\kappa]$ developed in [43] can be used to describe Ekedahl-Oort strata for reduction of a Hodge type Shimura variety, and gives dimension formula and closure for each stratum. We will first present some notations and technical results following [53] and [43], and then state how to use them.

We will first collect some basic facts about Weyl groups following the appendix of [53]. Let $G$ be a reductive group over $k = \bar{k}$. Take a maximal torus $T$ and a Borel subgroup $B$, s.t. $T \subseteq B \subseteq G$. The centralizer of $T$ is $T$, the Weyl group $W(T) := \text{Norm}_G(T)(k)/T(k)$ is a finite group. It admits a Coxeter group structure, i.e. one can take the generating set of simple reflections $I(B, T)$ to be the set of simple reflections defined by $B$. A priori, these data depend on the pair $(B, T)$, but by remark 3.1.4 below, any other such pair $(B', T')$ is obtained by conjugating $(B, T)$ by some element $g \in G(k)$ which
is unique up to right multiplication by an element in $T(k)$. Thus conjugation by $g$ induces isomorphisms $W(T) \to W(T')$ and $I(B,T) \to I(B',T')$ that are independent of the choice of $g$. Especially, the isomorphisms associated to any three such pairs are compatible with each other. So $(W,I) := (W(T),I(B,T))$ for any choice of $(B,T)$ can be viewed as “the” Weyl group and “the” set of simple reflections, in the sense that the pair depends only on $G$ up to a unique isomorphism.

**Remark 3.1.4.** Fixing $T \subseteq B \subseteq G$, there are several interpretations for the Weyl group as a set

1) ([50], corollary 6.4.12) $W \to \{\text{Borel subgroups containing } T\}$, $w \mapsto wBw^{-1}$ is bijective.

2) ([38], 3.2) Let $B$ be the set of Borels of $G$. The map $W \to G \setminus (B \times B)$, $w \mapsto G$-orbit of $(B, wBw^{-1})$ is bijective. This is actually an restatement of the previous one, using the fact that the intersection of two Borel subgroups contains a maximal torus (not necessarily $T$).

Let $J \subseteq I$ be a subset, we denote by $W_J$ the subgroup of $W$ generated by $J$, and $W^J$ (resp. $J^W$) be the set of elements $w$ s.t. $w$ is the element of minimal length in some coset $w'W_J$ (resp. $W_Jw'$). Note that there is a unique element in $w'W_J$ (resp. $W_Jw'$) of minimal length, and each $w \in W$ can be uniquely written as $w = w^Jw_J = w'_Jw$ with $w_J, w'_J \in W_J, w^J \in W^J$, and $Jw \in J^W$. In particular, $W^J$ and $J^W$ are systems of representatives of $W/W_J$ and $W^JW^J$.

Furthermore, if $K$ is a second subset of $I$, then for each $w$, there is a unique element in $W_JwW_K$ which is of minimal length. We will denote by $J^W_K$ the set of element of minimal length. Then $J^W_K = J^W \cap W^K$, and $J^W_K$ is a set of representatives of $W_J \setminus W/W_K$.

For a parabolic subgroup $P \subseteq G$, let $B$ be a Borel subgroup contained in $P$ and $T \subseteq B$ be a maximal torus of $G$. We denote by $U_P$ the unipotent radical of $P$ (i.e. the maximal connected unipotent normal smooth subgroup of $P$). There is a unique reductive subgroup (called Levi subgroup) $L \subseteq P$ containing $T$, s.t. the natural map $L \to P/U_P$ is an isomorphism. One can
I as the set \( \{ gT(k) \mid g \in I \subseteq W \} \), then \( P \) gives a subset \( J \subseteq I \), which is defined to be the subset \( \{ g \in I \mid gT(k) \subseteq P(k) \} \subseteq I \), called the type of \( P \). Here, we first work with \( W(T) \) and \( I(B,T) \), and then use the canonical identification to see that \( J \) is independent of the choice of \( (B,T) \). All the parabolics form a smooth projective scheme on \( k \), and two parabolics are conjugate if and only if they have the same type.

There is another way to define the type of a parabolic subgroup \( P \). Using notations as in the previous paragraph, attached to the chosen pair \( (B,T) \subseteq P \), there are unique collections of positive roots and simple root. We can define \( I \) to be the set of all the simple roots, and the type \( J \) of \( P \) to be the set of simple roots whose inverse are roots of \( P \).

Assume that \( G \) is a reductive group over \( \mathbb{F}_p \). We will explain how to relate the set \( ^JW \) to the quotient stack \( [E_G,\mu \backslash G] \) following [43] and [44].

Choose a Borel subgroup \( B \subseteq G \) and a maximal torus \( T \subseteq G \) s.t. \( T \subseteq B \). Note that by [29] Theorem 2, such a \( B \) exists. And such a \( T \) also exists by [50] Theorem 13.3.6 and Remark 13.3.7. By what we have seen, such a pair \( T \subseteq B \) gives the Weyl group \( W \) together with a set of simple reflections \( I \). Let \( \varphi \) be the Frobenius on \( G \) given by the \( p \)-th power. It induces an isomorphism \( (W,I) \to (W,I) \) of Coxeter systems, which will still be denoted by \( \varphi \). The cocharacter \( \mu : G_m \to G_n \) gives a parabolic subgroup \( P \) (see the construction of \( P_- \) at the beginning of 1.2), and hence gives a subset \( J \subseteq I \) by taking the type of \( P \).

Let \( \omega_0 \) be the element of maximal length in \( W \), set \( K := \omega_0 \sigma(J) \). Here we write \( gJg^{-1} \) for \( gJg \). Let \( x \in K^W_{\sigma(J)} \) be the element of minimal length in \( W_K \omega_0 W_{\sigma(J)} \). Then \( x \) is the unique element of maximal length in \( K^W_{\sigma(J)} \) (see [53] 4.2). There is a partial order \( \leq \) on \( J^W \), defined by \( w' \preceq w \) if and only if there exists \( y \in W_J, yw'x\varphi(y^{-1})x^{-1} \leq w \) (see [53] Definition 4.8, Proposition 4.9). Here \( \preceq \) is the Bruhat order (see A.2 of [53] for the definition). The partial order \( \preceq \) makes \( J^W \) into a topological space.

Now we can state the the main result of Pink-Wedhorn-Ziegler that gives
a combinatorial description of the topological space of \([E_G,\mu \backslash G_\kappa]\) (and hence \(G\text{-Zip}_\kappa^\mu\)).

**Theorem 3.1.5.** For \(w \in \mathcal{J}_W\), take any \(T' \subseteq B' \subseteq G_\pi\) with \(T'(\text{resp. } B')\) a maximal torus (resp. Borel) of \(G_\pi\), s.t. \(T' \subseteq L_\pi\) and \(B' \subseteq P_{-\pi}\). Let \(g, \dot{w} \in \text{Norm}_{G_\pi}(T)\) be a representative of \(\varphi^{-1}(x)\) and \(w\) respectively, and \(G^w \subseteq G_\pi\) be the \(E_{G,\mu}\)-orbit of \(gB'\dot{w}B'\). Then

1) The orbit \(G^w\) does not depend on the choices of \(\dot{w}, T', B'\) or \(g\).

2) The orbit \(G^w\) is a locally closed smooth subvariety of \(G_\kappa\). Its dimension is \(\dim(P) + l(w)\). Moreover, \(G^w\) consists of only one \(E_{G,\mu}\)-orbit. So \(G^w\) is actually the orbit of \(g\dot{w}\).

3) Denote by \(\|E_{G,\mu}^G_\kappa \otimes \pi\|\) the topological space of \([E_{G,\mu}^G_\kappa] \otimes \pi\), and still write \(\mathcal{J}_W\) for the topological space induced by the partial order \(\preceq\). Then the association \(w \mapsto G^w\) induces an homeomorphism \(\mathcal{J}_W \rightarrow \|E_{G,\mu}^G_\kappa \otimes \pi\|\).

**Proof.** The first statement is Proposition 5.8 of [43]. The second statement is [43] Theorem 1.3, Proposition 7.3 and Theorem 7.5. And the third statement is [43] Theorem 1.4. \(\square\)

The next statement (including its proof) is a word by word adaptation of results in [53] (to be more precise, Proposition 4.7, Theorem 6.1 and Corollary 9.2). But note that it is actually Theorem 2.4.1 and Theorem 3.1.2 that make it work. So it is not simply implied by [53].

**Proposition 3.1.6.** Let \(J\) be the type of \(P_+\), then the Ekedahl-Oort strata are listed by the finite set \(\mathcal{J}_W\). For \(w \in \mathcal{J}_W\), the stratum \(\mathcal{I}_0^w\) is smooth and equi-dimensional of dimension \(l(w)\) if \(\mathcal{I}_0^w \neq \emptyset\). Moreover, the closure of \(\mathcal{I}_0^w\) is the union of \(\mathcal{I}_0^{w'}\) s.t. \(w' \preceq w\).

**Proof.** The first statement follows from our definition of Ekedahl-Oort strata and Theorem 3.1.5 3). For the second one, note that by Theorem 3.1.5 2), each \(G_w\) is equi-dimensional of codimension \(\dim(U_{-}) - l(w)\) in \(G_\kappa\), so each \(\mathcal{I}_0^w\) is equi-dimensional of codimension \(\dim(U_{-}) - l(w)\) in \(\mathcal{I}_{0,\pi}\), as \(\zeta\) is smooth.
by Theorem 3.1.2. So the dimension of $\mathcal{S}_w$ is $l(w)$, as $\dim(\mathcal{S}_0) = \dim(U_-)$. The smoothness of each stratum follows from a direct adaption of the proof of Proposition 10.3 of [53]. For the last statement, by Theorem 3.1.5 3), the closure of $\{w\}$ in $\overline{|E_{G,\mu}|G_\epsilon \otimes \pi|$ is $\{w' \mid w' \preceq w\}$. So $\mathcal{S}_0^w = \zeta^{-1}(\pi)$ by the universally-openness of $\zeta$.

Remark 3.1.7. By [44] Lemma 12.14, Theorem 12.17 and [40] 3.26, there is a unique open dense stratum corresponding to the unique maximal element in $JW$. This stratum will be called the ordinary stratum. And there is also a unique minimal element in $JW$, namely the element 1. Its corresponding stratum is called the superspecial stratum. We expect that it is non-empty (but we can not prove it now). And the non-emptiness of the superspecial stratum implies that every stratum is non-empty, as $\zeta$ is a open map by Theorem 3.1.2.

### 3.2 On extra structures on $F$-zips attached to $\mathcal{S}_0$

In this section, we will give some remarks and comments concerning extra structures on $F$-zips associated to reductions of Hodge type Shimura varieties. The structure of this part is as follows. We list some compatibilities between $s_{\text{dR}}$ and Dieudonné theory in 3.2.1 and 3.2.2. Then we describe the extra structures on $F$-zips attached to reduction of a Hodge type Shimura variety in 3.2.3. In 3.2.4 we prove that two points lie in the same Ekedahl-Oort stratum if and only if after passing to a common extension of the residue fields of those two points, there is an isomorphism of their $F$-zips respecting extra structures.

We will first give some technical remarks here. Let $G_{\mathbb{Z}_p}$, $V_{\mathbb{Z}_p}$ and $s$ be as in 2.1. Let $W$ and $C^\bullet$ be as in Corollary 2.2.13. If we denote by $(\mathcal{V}^o)^1 \subseteq \mathcal{V}^o$ the Hodge filtration on $\mathcal{V}^o$, then by Lemma 2.3.2 (and also use notations there), we know that

$$I := \text{Isom}_{\mathcal{S}}((V_{\mathbb{Z}_p} \otimes W, s) \otimes W O_{\mathcal{S}}, (\mathcal{V}^o, s_{\text{dR}}))$$

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is a right $G_{\mathbb{Z}_p} \otimes W$-torsor, and

$$I_+ := \text{Isom}_\mathscr{O}((V_{\mathbb{Z}_p} \otimes W, C^*, s) \otimes_W O_{\mathcal{O}}; (V^o, V^o \supseteq (V^o)^1, s_{\text{dR}}))$$

is a right $P_+$-torsor.

### 3.2.1 Compatibility of $s_{\text{dR}}$ with Frobenius and connection

Let $A/S$ be the abelian scheme as at the beginning of 2.2.5. We are going to state a certain compatibilities between $s_{\text{dR}} \in V^o \otimes$ and the Dieudonné crystal (in sense of [21] Definition 2.3.2) $D(A_{S_0}[p^\infty])$.

Let $S = \text{Spec } R \subset \mathcal{O}$ be an open affine subscheme. Let $\hat{R}$ be the $p$-adic completion of $R$ with any lift of Frobenius $\sigma$. Here we remark that there always exists a lifting of the Frobenius to $\hat{R}$. To see this, let $A_n = \hat{R}/(p^n)$ and $A = \hat{R}/(p)$. Let $I_{n+1} = \ker(A_{n+1} \rightarrow A_n)$, then $I_{n+1}$ is an $A$-module. Now we can use notations and methods as in [21] Lemma 1.1.2 and Lemma 1.2.2. The obstruction to lift $\sigma: A_n \rightarrow A_n$ to $A_{n+1}$ lies in $\text{Ext}^1_{A_n}(L_{A_n/(p^n)} \otimes_\sigma A_n, I_{n+1})$. The quasi-isomorphism $L_{A_n/(p^n)} \otimes_\sigma A_n \otimes A \cong \Omega^1_{A/F_p} \otimes_\sigma A$ induces an isomorphism

$$\text{Ext}^1_{A_n}(L_{A_n/(p^n)} \otimes_\sigma A_n, I_{n+1}) \cong \text{Ext}^1_A(\Omega^1_{A/F_p} \otimes_\sigma A, I_{n+1}).$$

But $\text{Ext}^1_A(\Omega^1_{A/F_p} \otimes_\sigma A, I_{n+1})$ is trivial as $\Omega^1_{A/F_p}$ is projective.

By evaluating $\mathcal{D}(A_{S_0}[p^\infty])$ at $(\hat{R}, \sigma)$, we get a tuple $(M, \nabla, \varphi)$ where $M$ is a locally free $\hat{R}$-module, $\nabla: M \rightarrow M \otimes_R \hat{R}$ is the integral and topologically quasi-nilpotent (see [21] Remark 2.2.4 c for the definition) connection coming from the descent data (see [21] Remark 2.2.4 d), and $\varphi: M \rightarrow M$ is a $\sigma$-linear map which is horizontal with respect to $\nabla$ (see [21] Definition 2.3.4, and the paragraph after it). More precisely, let $\hat{R}(1)$ be the $p$-adic completion of the PD-envelope of $\hat{R} \otimes \hat{R} \rightarrow \hat{R}$, the crystal structure gives an isomorphism $\varepsilon: \text{pr}^*_2(M) \cong \text{pr}^*_1(M)$. And $\nabla(x)$ is defined to be $\theta(x) - x \otimes 1$, with $\theta: M \rightarrow \text{pr}^*_1(M), x \mapsto \varepsilon(1 \otimes x)$.

For any $p$-adically complete and $p$-torsion free $W$-algebra $R'$ equipped with a lift of Frobenius $\sigma'$ and a homomorphism of $W$-algebras $\iota: \hat{R} \rightarrow R'$, we get
a triple \((M', \nabla', \varphi')\) as follows. Take \(M' = M \otimes_{\hat{R}'} R', \nabla' = \nabla \otimes 1, \) and \(\varphi'\) be the \(\sigma'\)-linear map whose linearization is

\[
\sigma'^* (M') = \sigma'^* t^* M \xrightarrow{\xi} t^* \sigma'^* M \to t^* M = M'.
\]

Here \(\varepsilon'\) is the base-change to \(\hat{R}'\) of \(\varepsilon\) via the homomorphism \(\hat{R}(1) \to \hat{R}'\) induced by \(\sigma' \cdot \iota \sigma\). If \(\hat{R}'\) is s.t. \(\hat{R}'/p\hat{R}'\) is a completion of a smooth \(\kappa\)-algebra, then the evaluation of \(\mathbb{D}(A_{\mathcal{X}[p^\infty]})\) at \(R'\) is the triple \((M', \nabla', \varphi')\).

Take a closed point \(x \in \text{Spec}(R/pR)\), denote by \(\hat{R}_x\) the completion of \(R\) with respect to the maximal ideal defining \(x\). Then \(\hat{R}_x\) is also \(p\)-adically complete, and there is a natural injective homomorphism \(\hat{R} \to \hat{R}_x\). Moreover, \(\hat{R}_x\) is also equipped with an endomorphism \(\sigma_0\) lifting the Frobenius on its reduction modulo \(p\). We will take \(\sigma_0\) to be the one described in [23] 1.5. Then \(\mathbb{D}(A_{\mathcal{T}[p^\infty]})(\hat{R}_x)\) gives the triple \((M' = M \otimes \hat{R}_x, \nabla' = \nabla \otimes 1, s_{\text{dR}}'\)). Here \(\varphi' : M' \to M'\) is the \(\sigma_0\)-linear map whose linearization is \(t^* \varphi^\text{lin} \circ (\sigma_0 \iota \cdot \iota \sigma)^* \varepsilon\).

There is a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nabla} & M \otimes \hat{\Omega}_{\hat{R}} \\
\downarrow & & \downarrow \\
M \otimes \hat{R}_x & \xrightarrow{\nabla \otimes 1} & M \otimes \hat{\Omega}_{\hat{R}_x}
\end{array}
\]

with injective vertical homomorphisms. But \(\nabla \otimes 1 (s_{\text{dR}} \otimes 1) = 0, \) so \(\nabla (s_{\text{dR}} \otimes 1) = 0\). Moreover, we know that \(\varepsilon = \nabla + \text{id} \otimes 1\) and \(\varphi'(s_{\text{dR}} \otimes 1) = s_{\text{dR}} \otimes 1\) (note that \(\varphi'\) is actually defined on \(M'^{\otimes}[1/p]\)). So \(s_{\text{dR}}\) is \(\varphi\)-invariant, as

\[
\varphi'(s_{\text{dR}} \otimes 1) = t^* \varphi^\text{lin} \circ (\sigma_0 \iota \cdot \iota \sigma)^* (\nabla + \text{id} \otimes 1)(s_{\text{dR}} \otimes 1) = s_{\text{dR}} \otimes 1.
\]

The same computation shows that for any \((R', \sigma')\) as before s.t. \(R'/pR'\) is a completion of a smooth \(\kappa\)-algebra, we have \(s_{\text{dR}} \otimes 1 \in V_{R'}^{\otimes}\) is \(\varphi'\)-invariant, and \(\nabla'(s_{\text{dR}} \otimes 1) = 0\).

### 3.2.2 A better description for Frobenius invariance of \(s_{\text{dR}}\)

Let \(T = \text{Spec}(A)\) be s.t. \(T \to \mathcal{X}\) is étale, and \(I(T) \neq \emptyset\). Here \(I\) is the \(G_{\mathbb{Z}_p} \otimes W\)-torsor over \(\mathcal{X}\) defined at the beginning of 3.2. Write \(\overline{A}\) and \(\hat{A}\) for the reduction
modulo $p$ and $p$-adic completion of $A$ respectively. We will assume that the image of $T$ in $\mathcal{S}$ is an open affine subscheme of $\mathcal{S}$, denoted by $S = \text{Spec}(R)$. Clearly, one can always find pairs like $(T, S)$, and one can choose finitely many $\{(T_\alpha, S_\alpha)\}_{\alpha \in \Delta}$ s.t. $\cup S_\alpha = \mathcal{S}$. Note that $\hat{A}$ is also equipped with a lift of Frobenius $\sigma'$, but we DO NOT assume that the diagram

$$
\begin{array}{ccc}
\hat{R} & \xrightarrow{\sigma} & \hat{R} \\
\downarrow & & \downarrow \\
\hat{A} & \xrightarrow{\sigma'} & \hat{A}
\end{array}
$$

commutes.

For any $t \in I(T)$, it induces an $\hat{A}$-point of $I$, which will still be denoted by $t$. By 3.2.1, we have

$$\mathbb{D}(A_{\mathcal{S}_0}[p^\infty])(\hat{A}, \sigma') = (M', \nabla', \varphi')$$

as before, and that $s_{\text{dR}} \otimes 1 \in M'^{\otimes}$ is $\varphi'$-invariant. The cocharacter $\mu$ and $t \in I(\hat{A})$ induces a splitting $M' = M'^0 \oplus M'^1$, and the same argument as in the proof of Proposition 2.2.9 shows that $\varphi'$ is defined on $(M'^{\otimes})^0$, and that $s_{\text{dR}} \otimes 1 \in (M'^{\otimes})^0$. In particular, the $\sigma'$-linear map $\varphi'_{\mu} + \varphi'_{s_{\text{dR}}}$ takes $s_{\text{dR}} \otimes 1$ to itself.

### 3.2.3 Description of the extra structures

Now we will describe what are the extra structures on $F$-zips attached to reduction of a Hodge type Shimura variety. We will simply write $\sigma$ for the $p$-Frobenius on $\mathcal{S}_0$, and $\varphi$ for the Frobenius on $\overline{V}$. We will write $(G, V, \mu, s)$ for the reduction modulo $p$ of $(G_{\mathbb{Z}_p}, V_{\mathbb{Z}_p}, \mu : G_{m, W} \to G_W, s)$. The residue field of $W$ is denoted by $\kappa$. We will write $C^*(V_\kappa)$ for the descending filtration induced by $\mu$ and $D_\bullet(V_\kappa)$ for the ascending filtration induced by $\mu^{(p)}$. There are several conditions that the tuple $(\overline{V}, C^\bullet, D_\bullet, \varphi_\bullet)$ has to satisfy.

1) As we have seen in Theorem 2.4.1,

$$I := \text{Isom}_{\mathcal{S}_0}( (V_\kappa, s) \otimes O_{\mathcal{S}_0}, (\overline{V}, s_{\text{dR}}))$$

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is a right $G_\kappa$-torsor over $\mathcal{S}_0$,

$$I_+ := \text{Isom}_{\mathcal{S}_0}((V_\kappa, s, C^* (V_\kappa)) \otimes O_{\mathcal{S}_0}, (\overline{V^\otimes}, \overline{s_{\text{dR}}}, C^*))$$

is a right $P_+$-torsor over $\mathcal{S}_0$, and

$$I_- := \text{Isom}_{\mathcal{S}_0}((V_\kappa, s, D_\bullet (V_\kappa)) \otimes O_{\mathcal{S}_0}, (\overline{V^\otimes}, \overline{s_{\text{dR}}}, D_\bullet))$$

is a right $P_{(p)}$-torsor over $\mathcal{S}_0$.

2) The last two sentences of 3.2.2 imply that $s_{\text{dR}} \in V^\otimes$ gives a Tate sub $F$-zip of weight zero of $V^\otimes$, as $\varphi'$ and $\varphi'_p$ gives $\varphi_0$ and $\varphi_1$ (see the proof of 2.4.1). In particular, étale locally, $\oplus \varphi_\bullet$ is induced by an element $g$ of $G(\mathcal{S}_0)$ via $t$ which is a section of $I$ (see Lemma 2.3.3. And one can use the construction of $\hat{g}_t$ to define $g$. Everything works if we change $O_{\mathcal{S}_x}$ to the $p$-adic completion of a $W$-algebra which is étale over $\mathcal{S}$).

The second condition in 1) is equivalent to that étale locally, the Hodge filtration on $V^\otimes$ is induced by a cocharacter of $G_{\mathcal{S}_0}$ conjugate to $\mu$, and the third condition in 1) is equivalent to that étale locally, the ascending filtration on $V^\otimes$ is induced by a cocharacter of $G_{\mathcal{S}_0}$ conjugate to $\mu^{(p)}$. Now we will define what are the extra structures on $F$-zips attached to reduction of a Hodge type Shimura variety, and show how to attach $G$-zips to them.

**Definition 3.2.4.** Let $(G, V, \mu, s)$ be as at the beginning of 3.2.3. Let $S$ be a locally Noetherian scheme over $\kappa$. By an $F$-zip with a Tate class $s_{\text{dR}}$ over $S$, we mean an $F$-zip $(V, C^\bullet_\bullet, D_\bullet_\bullet, \varphi_\bullet)$ over $S$ equipped with a section $s_{\text{dR}}$ of $V^\otimes$, s.t.

1) The scheme $I := \text{Isom}_S((V_\kappa, s) \otimes O_S, (V, s_{\text{dR}}))$ is a $G_{\kappa}$-torsor over $S$.

2) There is an étale covering $S'$ of $S$, s.t. $I(S') \neq \emptyset$. For any $t \in I(S')$, the filtration $C^\bullet_{S'}$ is induced by a cocharacter of $G_{S'}$ via $t$ which is $G_{\kappa}(S')$ conjugate to $\mu$.

3) Étale locally, the ascending filtration $D_\bullet$ is induced by a cocharacter of $G_S$.

4) The $O_S$-submodule $O_S \hookrightarrow V^\otimes$ corresponding to the section $s_{\text{dR}}$ is a Tate sub $F$-zip of weight zero.
Remark 3.2.5. The Tate sub $F$-zip $O_S \hookrightarrow V^\otimes$ in the above definition is a locally direct summand. As $O_S \hookrightarrow V^\otimes$ remains injective at the residue field of each closed point, so Theorem 22.5 of [31] implies that it is a locally direct summand. Moreover, the embedding $O_S \hookrightarrow V^\otimes$ is admissible in the sense of Definition 1.1.5.

We will first show that, étale locally, $D_\bullet$ is induced by a cocharacter conjugate to $\mu(p)$.

Lemma 3.2.6. Let $(V, C^\bullet, D^\bullet, \varphi^\bullet)$ be an $F$-zip with a Tate class $s_{\text{dR}}$ over $S$. Let $S'$ and $t$ be as in the definition. For simplicity and without loss of generality, let’s assume that $\mu$ induces a splitting $V_{S'} = V_0 \oplus V_1$ of $C^\bullet$ via $t$. Let $\xi : V \otimes S' \to (V \otimes S')^{(p)}$ be given by $v \otimes s \mapsto v \otimes 1 \otimes s$, $\forall v \in V$ and $\forall s \in O_S$, and $\mathcal{V}_{S'} = \mathcal{V}_0 \oplus \mathcal{V}_1$ be the splitting of $D_{\bullet, S'}$ induced by a cocharacter $\nu$ of $G_{S'}$ via $t$. We still write $\varphi_0$ for $V_0 \to V_{S'} / V_1 \to V_0$ and $\varphi_1$ for $V_1 \to V_{S'} / V_0 \to V_1$. Then the composition

$$V \otimes S' \xrightarrow{\xi} (V \otimes S')^{(p)} \xrightarrow{((\oplus \varphi^\bullet) \otimes (s_{\text{dR}}))} \mathcal{V}^{(p)}_{S'} = \mathcal{V}_0^{(p)} \oplus \mathcal{V}_1^{(p)} \xrightarrow{t^{-1}} V \otimes S'$$

is an element $g_t$ in $G(S')$. Moreover, the cocharacter $\nu$ equals to $g_t \mu(p) g_t^{-1}$ via $t$.

Proof. For the first statement, we only need to prove that $g_t (s \otimes 1) = s \otimes 1$. And to do so, we only need to check that $(\oplus \varphi^\bullet) \otimes : \mathcal{V}_{S'}^{\otimes} \to \mathcal{V}_{S'}^{\otimes}$ maps $s_{\text{dR}}$ to itself. Let $(\mathcal{V}_{S'}^{\otimes})^0$ (resp. $(\mathcal{V}_{S'}^{\otimes})_0$) be the subspace of weight zero in $\mathcal{V}_{S'}^{\otimes}$ with respect to the cocharacter $\mu$ via $t$ (resp. the cocharacter $\nu$ via $t$). Then $s_{\text{dR}} \in (\mathcal{V}_{S'}^{\otimes})^0$, and condition 3) in Definition 3.2.4 implies that $(\oplus \varphi^\bullet) \otimes (s_{\text{dR}})$ is an element of weight $\leq -1$ with respect to $\nu$. But $(\oplus \varphi^\bullet) \otimes (s_{\text{dR}})$ lies in $(\mathcal{V}_{S'}^{\otimes})_0$ and hence is of weight zero with respect to $\nu$. While $s_{\text{dR}}$ is $G(S')$-invariant via $t$, so $d_{-1} = 0$, and $(\oplus \varphi^\bullet) \otimes (s_{\text{dR}}) = s_{\text{dR}}$. The second statement is clear.

Remark 3.2.7. There is a $G$-zip attached to an $F$-zip $(V, C^\bullet, D_\bullet, \varphi^\bullet)$ with a Tate class $s_{\text{dR}}$. By Definition 3.2.4 and Lemma 3.2.6, we only need to
construct an isomorphism \( \iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)} \) of \( L^{(p)} \)-torsors. But this follows from the proof of Theorem 2.4.1 word by word. Conversely, we can get an \( F \)-zip with a Tate class \( s_{d\mathbb{R}} \) from a \( G \)-zip by taking contract products. More precisely, given a \( G \)-zip \( (I, P_+, P_-, \iota) \) over \( S \), we take \( V = I \times^G V_S \), \( C^1 = I_+ \times^P C_1(V_\kappa)_S \), \( D_0 = I_- \times^P D_0(V_\kappa)_S \). The Tate class is the image of \( I \times \{s\} \) in \( V^\otimes = I \times^G (V_S^\otimes) \), and \( \oplus \varphi_i : \oplus \text{gr}^i_C \to \oplus \text{gr}^i_D \) is the \( \sigma \)-linear map whose linearization is the morphism

\[
\iota \times (\phi_0 \oplus \phi_1) : I_+^{(p)}/U_+^{(p)} \times L^{(p)} \oplus \text{gr}^i_C((V_\kappa)_S^{(p)}) \to I_-/U_-^{(p)} \times L^{(p)} \oplus \text{gr}^i_D(V_\kappa)_S.
\]

Here \( \phi_0 \) and \( \phi_1 \) are as in 2.2.14. The condition that \( \iota \) is \( L^{(p)} \)-equivariant implies that \( \iota \times (\phi_0 \oplus \phi_1) \) is well defined. And that the submodule generated by \( s_{d\mathbb{R}} \) is a Tate sub \( F \)-zip of \( (V, C^\bullet, D^\bullet, \varphi^\bullet) \) is straightforward.

### 3.2.8 Defining Ekedahl-Oort strata using \( F \)-zips

In this section, we will follow the construction in [40] and [53] to show that the Ekedahl-Oort strata defined using \( G \)-zips are the same as those defined using \( F \)-zips with a Tate class. The main technical tool is still [43]. We will write \((G, V, \mu, s)\) for the reduction modulo \( p \) of \((G_{Z_p}, V_{Z_p}, \mu : G_{m,W} \to G_W, s)\). Fix the datum \((G, V, \mu, s)\), we consider the following functor \( Z_\mu \) which associates to a \( \kappa \)-scheme \( S \) the set of \( F \)-zip structures \((C^\bullet, D^\bullet, \varphi^\bullet)\) on \( V_S \) with Tate class \( s \otimes 1 \). Clearly, \( Z_\mu \) is representable. And it is an \( F \)-zip interpretation of \( C_Z \) constructed in [43] Section 12.

Now we will construct a morphism \( Z_\mu \to [E_{G,\mu}\setminus G_\kappa] \). By definition, to give such a morphism is the same as to give an \( E_{G,\mu} \)-torsor \( H \) over \( Z_\mu \), equipped with an \( E_{G,\mu} \)-equivariant morphism \( H \to G_\kappa \).

There is a distinguished element \((V_\kappa, C^\bullet(V_\kappa), D^\bullet(V_\kappa), \phi^\bullet)\) in \( Z_\mu(\kappa) \) attached to the datum \((G, \mu)\) constructed as in 2.2.14. One checks easily that the line generated by \( s \in V^\otimes \) is a Tate sub \( F \)-zip of weight 0. Using the proof of [43] Lemma 12.5, the group \( G_\kappa \times G_\kappa \) acts on \( Z_\mu \) transitively via

\[
(g, h) \cdot (C^\bullet, D^\bullet, \varphi^\bullet) = (gC^\bullet, hD^\bullet, h\varphi^\bullet g^{-1})
\]
where $h \varphi_i g^{-1}$ is the composition

$$g(C^i)/g(C^{i+1}) \xrightarrow{g^{-1}} C^i/C^{i+1} \rightarrow D_i/D_{i+1} \xrightarrow{h} h(D_i)/h(D_{i+1}).$$

Under the above action, the stabilizer of $(V_\kappa, C^\bullet(V_\kappa), D^\bullet(V_\kappa), \phi_\bullet)$ is $E_{G,\mu}$ (still use the proof of [43] Lemma 12.5), and hence the action induces an $E_{G,\mu}$-torsor $G_\kappa \times G_\kappa \rightarrow Z_\mu$ which is $G_\kappa$-equivariant with respect to the diagonal action on $G_\kappa \times G_\kappa$ and the restriction to diagonal on $Z_\mu$. The morphism $m : G_\kappa \times G_\kappa \rightarrow G_\kappa$, $(g, h) \mapsto g^{-1} h$ is a $G_\kappa$-torsor which is $E_{G,\mu}$-equivariant.

By the same reason as in [43] Theorem 12.7, we get an isomorphism of stacks 

$$\beta : [G_\kappa \backslash Z_\mu] \simeq [E_{G,\mu} \backslash G]$$

after passing to quotients.

The $G_\kappa$-torsor $I = \text{Isom}_{\mathcal{F}_0}((V_\kappa, s) \otimes O_{\mathcal{F}_0}, (V^\circ, s_d^\circ))$ induces a morphism $\zeta' : \mathcal{S}_0 \rightarrow [G_\kappa \backslash Z_\mu]$, while our Ekedahl-Oort strata are defined by the morphism $\zeta : \mathcal{F}_0 \rightarrow [E_{G,\mu} \backslash G_\kappa]$ defined in subsection 3.1. By what we have seen, one can identify $[G_\kappa \backslash Z_\mu]$ with $[E_{G,\mu} \backslash G]$ via $\beta$. So it is natural to ask whether they induce the same theory of Ekedahl-Oort strata. One can prove that two points lie in the same Ekedahl-Oort stratum defined using $\zeta$ if and only if after passing to a common extension of the residue fields of those two points, there is an isomorphism of their $F$-zips respecting the Tate $F$-zip. The following more conceptual statement holds.

**Proposition 3.2.9.** We have an equality $\beta \circ \zeta' = \zeta$.

**Proof.** By [43] 12.6, there is a cartesian diagram

$$
\begin{array}{ccc}
G_\kappa \times G_\kappa & \xrightarrow{m} & G_\kappa \\
\downarrow^{n} & & \downarrow \\
Z_\mu & \rightarrow & [E_{G,\mu} \backslash G_\kappa]
\end{array}
$$

with vertical arrows $G_\kappa$-equivariant $E_{G,\mu}$-torsors and horizontal arrows $E_{G,\mu}$-equivariant $G_\kappa$-torsors. One only needs to check that the pull back to $G_\kappa \times G_\kappa$ of $\mathcal{F}^\#$ to $G_\kappa$ and $I \rightarrow Z_\mu$ are $G_\kappa \times E_{G,\mu}$-equivariantly isomorphic over $G_\kappa \times G_\kappa$.
Let $\mathcal{S}_0$ be the pull back

\[ \begin{array}{ccc}
\mathcal{S}_0 & \rightarrow & \mathcal{S}_0^# \\
\downarrow & & \downarrow \\
G_\kappa \times G_\kappa & \overset{m}{\rightarrow} & G_\kappa.
\end{array} \]

For any $T/\kappa$,

\[ \mathcal{S}_0(T) = \{ (g_1, g_2, a, b) \mid g_i \in G_\kappa(T), (a, d) \in \mathcal{S}_0(T) \text{ s.t. } g_1^{-1} g_2 = a^{-1} b \}. \]

For any $(g, p_1, p_2) \in G_\kappa \times E_{G, \mu}(T)$, the action is given by

\[(g, p_1, p_2) \cdot (g_1, g_2, a, b) = (g_1 p_1^{-1}, g g_2 p_2^{-1}, a p_1^{-1}, b p_2^{-1}). \]

Let $\tilde{I}$ be the pull back

\[ \begin{array}{ccc}
\tilde{I} & \rightarrow & I \\
\downarrow & & \downarrow \\
G_\kappa \times G_\kappa & \overset{n}{\rightarrow} & Z_\mu.
\end{array} \]

For any $T/\kappa$,

\[ \tilde{I}(T) = \{ (g_1, g_2, t) \mid g_i \in G_\kappa(T), t \in I(T) \text{ s.t. } (g_1 C^\bullet(V_\kappa)_T, g_2 D^\bullet(V_\kappa)_T, g_2 \phi^\bullet g_1^{-1}) \]
\[ = t^{-1}(C^\bullet(V_\kappa)_T, D^\bullet(V_\kappa)_T, \phi^\bullet) \}. \]

For any $(g, p_1, p_2) \in G_\kappa \times E_{G, \mu}(T)$, the action is given by

\[(g, p_1, p_2) \cdot (g_1, g_2, t) = (g_1 p_1^{-1}, g g_2 p_2^{-1}, g \cdot t). \]

There is a $G_\kappa \times G_\kappa$-morphism $\mathcal{S}_0 \rightarrow \tilde{I}$ mapping $(g_1, g_2, a, b)$ to $(g_1, g_2, a g_1^{-1})$. This is clearly an isomorphism. One also checks easily that it is $G_\kappa \times E_{G, \mu^*}$ equivariant.