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For \( n \) and \( k \) be positive integers, let \( S(n, k)_\mathbb{C} \) be the complex vector space of weight \( k \) cusp forms on \( \Gamma_1(n) \). In these propositions \( \mathbb{T}(n, k) \) will denote the cuspidal Hecke algebra of level \( n \) and weight \( k \), i.e. the \( \mathbb{Z} \)-subalgebra of \( \text{End}_\mathbb{C}(S(n, k)_\mathbb{C}) \) generated by the Hecke operators \( T_p \) for every prime \( p \) and the diamond operators \( \langle d \rangle \) for every \( d \in (\mathbb{Z}/n\mathbb{Z})^* \). Let \( \overline{\mathbb{Q}} \) denote an algebraic closure of the field of rational numbers \( \mathbb{Q} \).

1. Let \( E \) be an elliptic curve over a number field \( K \) with \( j \)-invariant different from 0 and 1728, and let \( \ell \) be a prime number. Suppose that \( E \) admits an \( \ell \)-isogeny locally at a set of primes with density one and \( E \) does not admit an \( \ell \)-isogeny over \( K \). Then \( \ell \leq \max \{\Delta, 6d+1\} \), where \( d \) is the degree of \( K \) over \( \mathbb{Q} \) and \( \Delta \) is the discriminant of \( K \) [Corollary 2.3.5].

2. Let \( X_{V_4}(5) \) be the modular curve \( G \backslash X(5) \) obtained by taking for \( G \subset GL_2(\mathbb{F}_5) \) the inverse image of the Klein 4-group \( V_4 \subset PGL_2(\mathbb{F}_5) \). The modular curve \( X_{V_4}(5) \) is isomorphic to \( \mathbb{P}^1 \) over \( \text{Spec}(\mathbb{Q}(\sqrt{5})) \) [Proposition 3.2.1].

3. Let \( n, m \) and \( k \) be positive integers with \( n \) a multiple of \( m \). Let \( \ell \) be a prime not dividing \( n \), and such that \( 2 \leq k \leq \ell + 1 \). Let \( f : \mathbb{T}(n, k) \rightarrow \overline{\mathbb{F}}_\ell \) be a morphism of rings and let \( \rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{F}}_\ell) \) be the unique, up to isomorphism, continuous semi-simple representation corresponding to it. Let \( g : \mathbb{T}(m, k) \rightarrow \overline{\mathbb{F}}_\ell \) be a morphism of rings and let \( \rho_g : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{F}}_\ell) \) be the corresponding Galois representation. Assume that \( \text{cond}(\rho_g) = m \) and that the weight of \( \rho_g \) is minimal. If \( \rho_f \) is ramified at \( \ell \) then \( \rho_f \) is isomorphic to \( \rho_g \) if and only if \( f \) is in the subspace of the old-space given by \( g \) at level \( n \) [Theorem 6.3.6].

4. Let \( n \) and \( k \) be two positive integers, let \( \ell \) be a prime such that \( \ell \) does not divide \( n \) and \( 2 \leq k \leq \ell + 1 \). Let \( f : \mathbb{T}(n, k) \rightarrow \overline{\mathbb{F}}_\ell \) be a morphism of rings and let \( \rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{F}}_\ell) \) be the representation attached to \( f \). Let \( \overline{\tau} : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow \overline{\mathbb{F}}_\ell^* \) be the character defined by \( \overline{\tau}(a) = f(\langle a \rangle) \) for all \( a \in (\mathbb{Z}/n\mathbb{Z})^* \). Assume that \( \rho_f \) is irreducible and that it does not arise from lower level. Let \( p \) be a prime dividing \( n \) and suppose that \( f(T_p) \neq 0 \). Let \( \chi : (\mathbb{Z}/p^i\mathbb{Z})^* \rightarrow \overline{\mathbb{F}}_\ell^* \), for \( i > 0 \), be a non-trivial character. Then \( N_p(\rho_f \otimes \chi) = N_p(\chi \overline{\tau}) + N_p(\chi) \) [Proposition 8.2.4].
5. Let $n$ and $k$ be positive integers and let $\ell$ be a prime not dividing $n$ such that $2 \leq k \leq \ell + 1$. Let $f : \mathbb{T}(n, k) \to \overline{\mathbb{F}}_{\ell}$ be a morphism of rings. Let us suppose that the representation $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_\ell)$ is irreducible and does not arise from lower level or weight. Let $S$ be the set:

$$S := \{ f(T_p) | p \text{ prime}, p \neq \ell \text{ and } p \leq B(n, k) \} \cup \{ f(\langle d \rangle) | d \in (\mathbb{Z}/n\mathbb{Z})^* \},$$

where $B(n, k) = k/12 \cdot n \prod_{p|n \text{ prime}} (1 + 1/p)$. Then the field of definition of $\rho_f$ is the smallest extension of $\mathbb{F}_\ell$ containing the elements of the set $S$ [Proposition 9.1.1].

6. Let $p \geq 5$ be a prime number. The equation $x^p + y^p + 129 z^p = 0$ has no solution with $x, y, z \in \mathbb{Z}$ and $xyz \neq 0$.

7. Let $E$ be an elliptic curve over a number field $K$, of degree $d$ over $\mathbb{Q}$, such that $j(E) \notin \{0, 1728\}$. Let $\ell$ be a prime and suppose $\sqrt{(-1)\ell} \in K$. Assume that $E/K$ admits an $\ell$-isogeny locally at a set of primes with density one but not globally. Suppose that $\mathbb{P} \rho_{E, \ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is not conjugated to a dihedral group. Then

- if $\ell < 16d+1$ then $\mathbb{P} \rho_{E, \ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to $\mathbb{A}_4$;
- if $\ell < 20d+1$ then $\mathbb{P} \rho_{E, \ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to either $S_4$ or $A_4$;
- if $\ell < 24d+1$ then $\mathbb{P} \rho_{E, \ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to either $S_4$ or $A_4$ or $A_5$.

8. Every odd irreducible 2-dimensional Artin representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has conductor greater or equal to 23.

9. Maths is in every kitchen, on every recipe card. The mathematics of cooking often goes unnoticed, but in reality, there is a large quantity of maths skills involved in cooking. Take a lemniscate, chop a piece off, pull it out and twist it around: in this way you obtain a shape of pasta which is more suited than others to dense sauces.

10. “Did you notice that this shop is called The Four Fours. This is a coincidence of unusual importance.” “A coincidence? Why?” “The name of this business recalls one of the wonders of calculus: using four fours, we can get any number whatsoever.”

$$10 = \frac{44 - 4}{4}; \quad 11 = \frac{44}{\sqrt{4} + \sqrt{4}}; \quad 113 = \Gamma(4) - \frac{4! + 4}{4}.$$