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Introduction

This thesis is concerned with the arithmetic of K3 surfaces over number fields. A K3 surface over a field k is a smooth, projective, and geometrically integral surface over k such that the canonical divisor class of X is trivial and the first cohomology of the structure sheaf of X vanishes. We will prove various results about p-adic density of rational points on certain types of K3 surfaces defined over $\mathbb{Q}$. In particular, we prove that, for each prime number p, there exist infinitely many K3 surfaces X over $\mathbb{Q}$ such that the rational points on X are p-adically dense. A fuller summary of the results in this thesis can be found at the end of this Introduction.

0.1 Diophantine geometry

Broadly speaking, this thesis is concerned with the topic of Diophantine equations, which are polynomial equations with coefficients in a number field k, for which one is only interested in solutions defined over the same number field k. This thesis applies geometric methods to the study of solution sets of Diophantine equations. In this context, one often speaks of Diophantine geometry.

Given a number field k and a system of Diophantine equations

$$f_1 = 0, \ldots, f_n = 0, \tag{1}$$

where $f_1, \ldots, f_n$ are polynomials over k, one may consider the algebraic variety X defined by the $f_1, \ldots, f_n$. There are several ways to describe the variety X. The classical viewpoint is, having chosen an algebraically closed field extension F of k, for example the field $\mathbb{C}$ of complex numbers, to identify X with the set of F-valued solutions to (1). A more modern viewpoint is to view X as a scheme, which is a topological space equipped with a sheaf of commutative rings, that admits an open covering by so-called spectra of commutative rings. Both viewpoints are equally acceptable for the purposes of this thesis, with the exception of some parts of chapter 4 which use scheme theory in an essential way. In both viewpoints, the variety X comes equipped with a topology, called the Zariski topology, in which the closed subsets are exactly the subsets Z of X that can be defined by imposing further polynomial equations

$$g_1 = 0, \ldots, g_m = 0 \tag{2}$$

on the points of Z.

We say that a variety is defined over a number field k if it arises from a set of polynomial equations whose coefficients lie in k. We also speak more simply of a variety over k. Note that the ground field k is often implicitly assumed to be part of the data of the variety.

Still writing X for the variety associated to the equations (1), the set of solutions over k to (1) is denoted by $X(k)$. The elements of $X(k)$ are called the rational points on X. For every field extension K of k, the set of solutions over K to (1) is denoted by $X(K)$.

Within the theory of Diophantine equations, the terminology afforded by the theory of algebraic varieties is considered to be so convenient that the central focus is often placed on the variety rather than its defining equations. Questions about Diophantine equations thus often take the following form: "Given a certain variety X defined over a number field k, what can one say about its set $X(k)$ of rational points?" In line with this, the results of this thesis are phrased in terms of varieties rather than their defining equations.

0.2 Topological aspects of rational points

0.2.1 Completions of a number field

It is possible to view $X(k)$ in a topological way, even leaving aside the Zariski topology for the moment. To this end, we will introduce the notion of completions of a number field k with respect to an (equivalence class of) absolute value(s) on k. Recall that an absolute value $|\cdot|$ on k is a function

$$|\cdot| : k \to \mathbb{R}$$

satisfying:

(i) for all $x \in k$ we have $|x| \geq 0$;
(ii) for all $x \in k$ we have $|x| = 0$ if and only if $x = 0$;

(iii) for all $x, y \in k$ we have $|xy| = |x|\,|y|$;
(iv) for all $x, y \in k$ we have $|x + y| \le |x| + |y|$.

Note that property (iv) is known as the triangle inequality for $|\cdot|$. An absolute value $|\cdot|$ is called non-archimedean if in addition to (iv) it satisfies the stronger property

(iv') for all $x, y \in k$ we have $|x + y| \le \max(|x|, |y|)$.

This last property is known as the ultrametric inequality for $|\cdot|$.

Two absolute values $|\cdot|_1$ and $|\cdot|_2$ of k are considered equivalent if there exists a non-zero real number e such that for all $x \in k$ we have $|x|_2 = |x|_1^e$. If v is a place of k, i.e. an equivalence class of absolute values on k, and $|\cdot|$ is an element of v, then the completion of k for v, often denoted by k_v, is obtained, analogously to the construction of $\mathbb{R}$ from $\mathbb{Q}$, by taking the Cauchy sequences in k for the metric $|\cdot|$ and identifying Cauchy sequences if their difference converges to 0. Note that this construction does not depend on the choice of $|\cdot|$. The set k_v is a field, with the addition and multiplication operations induced by the ones on k, and it is a metric space with the metric given by $|\cdot|$. If $|\cdot|$ is non-archimedean, then all elements of v are, and the field k_v is called a p-adic field.

By Ω_k we denote the set of all places of k. By Ostrowski's theorem, we have that the non-archimedean absolute values on k all arise from the valuations at the prime ideals of the ring of integers of k, whereas the archimedean absolute values on k are all obtained by composing the embeddings of k into the field $\mathbb{C}$ of complex numbers by the standard absolute value on $\mathbb{C}$. In particular, if we specialize to the case $k = \mathbb{Q}$, then all places of $\mathbb{Q}$ are given by either the standard absolue value on $\mathbb{Q}$, or the p-adic valuation for some prime number p

$$\begin{aligned} |\cdot| : \mathbb{Q} &\to \mathbb{R} \\ x &\mapsto p^{-v_p(x)} \\ 0 &\mapsto 0. \end{aligned}$$

Here v_p is the p-adic valuation on $\mathbb{Q}$: for every pair a, b of non-zero integers, we have that $v_p(a/b)$ is the number of prime factors p in a minus the number of prime factors p in b.

0.2.2 The Hasse principle

The sets $X(k_v)$, which we recall are the solution sets over k_v to (1), give a very useful tool for studying $X(k)$. Observe that $X(k)$ embeds in $X(k_v)$, by considering a solution over k to (1) as a solution over k_v to (1). It follows that if, for some place v of k, we have that $X(k_v)$ is empty, then $X(k)$ must be empty too. This gives a very useful sufficient criterion for the emptiness of $X(k)$. Its usefulness derives from the fact that there is an algorithm that checks in finite time whether or not there exists a place v of k such that $X(k_v)$ is empty. (By Hensel's lemma and the Lang–Weil estimates [17] one reduces this last problem to deciding the non-emptiness of $X(k_v)$ for only finitely many places v, which can be done in finite time by the main result of [24].)

Conversely, one might ask: if for all places v of k the set $X(k_v)$ is non-empty, may we then conclude that $X(k)$ is also non-empty? As we shall see, this implication does not hold for general X. If X is such that the implication does hold, we say that X satisfies the Hasse principle; if it does not hold, then it is said that X violates the Hasse principle. Note that X violates the Hasse principle if and only if $X(k)$ is empty, but $X(k_v)$ is non-empty for all places v of k.

We have the following classical theorem.

Theorem 0.1 (Hasse, Minkowski)**.** *Let C be a smooth plane conic curve over a number field k. Then the Hasse principle holds for C; that is, if $C(k_v)$ is non-empty for all places v of k, then $C(k)$ is non-empty.*

There exist varieties X over number fields k (even over $\mathbb{Q}$) that violate the Hasse principle. We will see examples of this later in this introduction.

0.2.3 Density of rational points

This thesis deals with the topological aspects of the solution set $X(k)$. For instance, one may ask whether $X(k)$ is dense in X for the Zariski topology. This is sometimes abbreviated slightly by asking whether $X(k)$ is Zariski-dense in X.

Other topological aspects of $X(k)$ can be made visible as follows. If v is a place of k, then the set $X(k_v)$ inherits a topology from the one on k_v. By viewing $X(k)$ as a subset of $X(k_v)$, we may then ask: is $X(k)$ dense in $X(k_v)$? Similarly, for any non-empty set S of places of $X(k)$, the set $X(k)$ embeds diagonally into the product $\prod_{v \in S} X(k_v)$, which we consider as having the product topology, and one may ask if $X(k)$ has dense image under this embedding.

Let X be a variety defined over a number field k. The following are some questions one may ask about the topological nature of $X(k)$.

(D1) Is $X(k)$ Zariski-dense in X?

(D2) For a non-empty finite subset $S \subset \Omega_k$, is the closure of $X(k)$ open in $\prod_{v \in S} X(k_v)$?

(D3) For a non-empty finite subset $S \subset \Omega_k$, is $X(k)$ is dense in $\prod_{v \in S} X(k_v)$?

(D4) Does there exist a finite subset $T \subset \Omega_k$ such that $X(k)$ dense in $\prod_{v \notin T} X(k_v)$?

(D5) Is $X(k)$ dense in $\prod_{v \in \Omega_k} X(k_v)$?

If X satisfies property (D2) with respect to some finite set S of places of v, one says that X has S-openness. If X satisfies property (D4) for some finite set T of places of k, one says that X satisfies weak weak approximation. If X satisfies property (D5), one says that X satisfies weak approximation. (For this terminology, see [21].) We note that (D5) implies that (D4) holds for every T; (D4) for some T implies that (D3) holds for every S disjoint from T; (D3) for some S implies that (D2) holds for the same S; lastly, (D2) for some S implies that (D1) holds.

0.3 Obstructions to rational points

Before, we mentioned that $X(k)$ may be empty whereas $X(k_v)$ is non-empty for all places v of k. An example of this is given by the famous Reichardt–Lind curve, which is the curve T defined over $\mathbb{Q}$ that is given by the equation

$$2y^2 = x^4 - 17. \tag{3}$$

We have that $T(\mathbb{R}) \neq \emptyset$ as well as $T(\mathbb{Q}_p) \neq \emptyset$ for all prime numbers p. On the other hand, it is an easy application of the law of quadratic reciprocity to show that there are no solutions over $\mathbb{Q}$ to (3) (see [32, X.6.5(a)]), or equivalently, that we have $T(\mathbb{Q}) = \emptyset$. Hence, T provides an instance of a violation of the Hasse principle.

In 1970, Yuri Manin defined a framework that explains the failure of the Hasse principle in certain cases [19]. For this, we need to introduce some additional concepts and set some notation. Assume that X is a smooth, projective, and geometrically integral variety defined over a number field k. Let $\mathrm{Br}(X)$ be the Brauer group of X, which is defined as the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. By functoriality, we have a map from $\mathrm{Br}(k)$, the Brauer group of the field k, to $\mathrm{Br}(X)$. The image of $\mathrm{Br}(k)$ in $\mathrm{Br}(X)$ is denoted by $\mathrm{Br}_0(X)$. Furthermore, one defines $\mathrm{Br}_1(X)$ as the subgroup of $\mathrm{Br}(X)$ consisting of the elements that become trivial over some finite extension of k. The elements of $\mathrm{Br}_1(X)$ are called algebraic Brauer classes. Elements of $\mathrm{Br}(X)$ that are not algebraic are called transcendental. By the fact that X is projective, we may write

$$X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v),$$

where $\mathbb{A}_k$ is the ring of adeles of k. We are now ready to describe Manin's theory. In [19], Manin defines a pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}.$$

This pairing is continuous in the first variable, for the discrete topology on $\mathbb{Q}/\mathbb{Z}$, and it has the property that $((x_v)_v, \alpha)$ maps to 0 if $(x_v)_v \in X(\mathbb{A}_k)$ is the image of a rational point. The set of points $(x_v)_v \in X(\mathbb{A}_k)$ that pair to 0 with every $\alpha \in \mathrm{Br}(X)$ is customarily denoted by $X(\mathbb{A}_k)^{\mathrm{Br}}$. By these facts, we thus have the crucial property

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}. \tag{4}$$

The properties of Manin's pairing mentioned above imply that $X(\mathbb{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbb{A}_k)$. Hence, if $\overline{X(k)}$ denotes the closure of the image of $X(k)$ in $X(\mathbb{A}_k)$, we have the following strengthening of (4):

$$\overline{X(k)} \subset X(\mathbb{A}_k)^{\mathrm{Br}}. \tag{5}$$

For a smooth, projective and geometrically integral variety over k, it may happen that $X(\mathbb{A}_k)$ is non-empty, but $X(\mathbb{A}_k)^{\mathrm{Br}}$ is empty, and, by (4), so is $X(k)$. Then X violates the Hasse principle, and Manin's pairing explains why this is true. In this case, one says that there is a Brauer–Manin obstruction to the Hasse principle on X. Similarly, if $X(\mathbb{A}_k)^{\mathrm{Br}}$ is a proper subset of $X(\mathbb{A}_k)$, then (5) shows that $X(k)$ is not dense in $X(\mathbb{A}_k)$, and one says that there is a Brauer–Manin obstruction to weak approximation on X.

In view of the inclusion (5), we may ask the following further question regarding the topological properties of $X(k)$.

(D4') Is $X(k)$ dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$?

If the answer to (D5) is positive for X, then so is the answer to (D4'). Now assume that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite, so that $X(\mathbb{A}_k)^{\mathrm{Br}}$ is open in $X(\mathbb{A}_k)$. Then if $X(k)$ is non-empty, and if the answer to (D4') is positive for X, then the answer to (D4) is also positive for some T.

0.4 Rational points on surfaces

By a surface X over a field k we will mean a smooth, projective, and geometrically integral variety X over k that has dimension 2. In the sequel, we again let k be a number field. The subject of this thesis mainly concerns the Diophantine geometry of surfaces over number fields, or, as is sometimes said, the arithmetic of surfaces. We will give an overview of what is known and conjectured about questions (D1)–(D5) for the case where X is a surface over a number field k.

0.5 Geometrically rational surfaces

One says that a variety X over k is rational if X is birational to $\mathbb{P}^n_k$ for some integer n. We say that X is geometrically rational if the base-change $X_{\overline{k}}$ of X to the algebraic closure of k is rational. Let X be a geometrically rational surface over a number field k.

It is well-known that, since X is geometrically rational, the quotient of $\mathrm{Br}(X)$ by $\mathrm{Br}_0(X)$ is finite, hence a positive answer to (D4') implies a positive answer to (D4) for some T. Moreover, we have that X is either a del Pezzo surface, which means that the anticanonical divisor $-K_X$ of X is ample, or X is a conic bundle, which means that there exists a surjective morphism $\pi\colon X \to C$, where C is a curve of genus 0 defined over k, such that the fibres of π are isomorphic to plane conics [16]. The degree d_X of X is defined as the self-intersection of K_X, where K_X is the canonical divisor of X.

The following conjecture is a special case of a conjecture by Colliot–Thélène [9, p. 319, Conjecture (d)]).

Conjecture 0.2. *Let X be a geometrically rational surface over a number field k. Then $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$.*

We will discuss some of the known facts about Conjecture 0.2.

Suppose first that X is a del Pezzo surface. Then we have $d_X \geq 1$ by ampleness of $-K_X$. If $d_X \geq 5$, then we have by [20, Theorem 29.4] that X satisfies both the Hasse principle and weak approximation. It follows that if $d_X \geq 5$ and $X(k) \neq \emptyset$, then all questions (D1)–(D5) have positive answers. If $d_X = 4$, then X may violate the Hasse principle (see [1]). However, if $X(k)$ is non-empty, then $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$ (see [27]). Hence if $d_X = 4$ and $X(k) \neq \emptyset$, then the questions (D1)–(D4') all have positive answers.

If $d_X = 3$, then the Hasse principle may fail (see [36]). Furthermore, if $d_X = 3$ and $X(k) \neq \emptyset$, then it is currently unknown in general whether any of the questions (D2)–(D5) have positive answers; however, it is known that if $X(k) \neq \emptyset$ then $X(k)$ is Zariski-dense in X, so that question (D1) does have a positive answer. Finally, if $d_X \leq 2$ and $X(k) \neq \emptyset$, we do not currently know the answers to any of the questions (D1)-(D5) in general; for the current state of the art in these cases, see [29] for $d_X = 2$ and [30] for $d_X = 1$.

If the geometrically rational surface X is not a del Pezzo surface, then it is shown in [16] that there exists a surjective morphism $\pi \colon X \to C$, where C is a curve of genus 0 defined over k, such that the fibres of π are isomorphic to plane conics.

By Theorem 0.1, the Hasse principle holds for C, hence a finite computation enables one to see whether $C(k)$ is non-empty. If $C(k) = \emptyset$, then we have $X(k) = \emptyset$. If $C(k) \neq \emptyset$, then C is isomorphic to $\mathbb{P}^1_k$. We assume that the latter is indeed the case, so that we have a surjective morphism $\pi \colon X \to \mathbb{P}^1_k$ whose fibres are isomorphic to plane conics. Then if the number of non-smooth fibres of π is at most 3, then X satisfies both the Hasse principle and weak approximation. Hence, if we are in this case and we have $X(k) \neq \emptyset$, the answers to the questions (D1)-(D5) are all positive. If the number of non-smooth fibres is 4 or 5, then we have that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$ (this follows from the results of [6], [7], [8], and [28] if the number of bad fibres is 4, and from [27] if the number of bad fibres is 5; see also the introduction to [4]). Finally, if the number of bad fibres is arbitrary, but every bad fibre is defined over $\mathbb{Q}$, then a very recent result [4, Theorem 1.1] says that $X(k)$ is always non-empty, and that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$.

0.6 K3 surfaces

Assume now that X is a K3 surface over a number field k, i.e., the class of the canonical divisor K_X in $\mathrm{Pic}(X)$ vanishes and we have $\mathrm{H}^1(X, \mathscr{O}_X) = 0$. In the case of K3 surfaces, the theory is far less complete than in the case for geometrically rational surfaces. We will describe some of the known results on the arithmetic of K3 surfaces, in particular the ones concerning density of rational points.

0.6.1 Existence of rational points

It is known that, in general, the Hasse principle fails for K3 surfaces over number fields. For example, Swinnerton-Dyer shows in [37] that the K3 surface over $\mathbb{Q}$ defined by

$$4x^4 + 9y^4 - 8z^4 - 8w^4 = 0$$

has points over $\mathbb{Q}_p$ for every prime number p, as well as over $\mathbb{R}$, but none over $\mathbb{Q}$.

0.6.2 Brauer group and density questions

By a remarkable result of Skorobogatov and Zarhin [34], one knows that $\mathrm{Br}_0(X)$ has finite index in $\mathrm{Br}(X)$, hence a positive answer to (D4') implies a positive answer to (D4) for some set of places T. In general, however, it is unknown whether any of the questions (D1)–(D5) has a positive answer. In fact, it is famously unknown whether $X(k) \neq \emptyset$ implies $X(k)$ to be even infinite!

0.6.3 Elliptic fibrations on K3 surfaces

It is known that K3 surfaces may admit fibrations into curves of genus 1. In this introduction, we will abuse terminology, and call such a fibration an elliptic fibration on X, even though the fibres are not elliptic curves since an identity for the group law is not specified.

Potential density

The presence of elliptic fibrations on a K3 surface is an important aid in proving density results. A seminal result by Bogomolov and Tschinkel [2, Theorem 1.1] says that if X possesses an elliptic fibration, then the rational points on X are potentially dense: there exists a finite field extension k'/k such that $X(k')$ is Zariski-dense in X.

If the rank of the abelian group $\mathrm{Pic}(X)$, which is free and finitely generated, is at least 5, then [13, Proposition 11.1] says that there exists a finite field extension k''/k such that the base-change of X to k'' possesses an elliptic fibration. From this and the result by Bogomolov and Tschinkel, it follows that if the rank of $\mathrm{Pic}(X_{\overline{k}})$ is at least 5, then the rational points on X are potentially dense.

Multiple elliptic fibrations

The result by Bogomolov and Tschinkel does not apply when one is solely interested in density over the ground field. We therefore turn to the case where X admits at least two elliptic fibrations. An example of a K3 surface over $\mathbb{Q}$ admitting at least two elliptic fibrations is the diagonal quartic surface

$$X_{a,b,c,d} \colon ax^4 + by^4 + cz^4 + dw^4 = 0,$$

where a, b, c, d are rational numbers such that $abcd \in \mathbb{Q}^{*2}$. It is a result by Logan, McKinnon, and Van Luijk [18, Theorem 1.1] that if $X_{a,b,c,d}$ contains a rational point that lies outside the coordinate planes and any of the 48 lines on $X_{a,b,c,d}$, then the rational points on $X_{a,b,c,d}$ lie dense in $X_{a,b,c,d}$ for the Zariski topology, as well as in $X_{a,b,c,d}(\mathbb{R})$ for the real-analytic topology.

The above is an example of a more general phenomenon. Assuming that X admits at least two elliptic fibrations, a result by Swinnerton-Dyer [38] provides sufficient conditions for $X(k)$ to be Zariski-dense in X. More precisely, under the assumption that every fibre belonging to one fibration is algebraically equivalent to none of the fibres belonging to the other fibration, Swinnerton-Dyer's result asserts the existence of an explicitly computable closed subset $Z \subsetneq X$ such that if X contains a rational point outside of Z, then $X(k)$ is Zariski dense in X.

0.6.4 Failure of weak approximation on K3 surfaces

We are still keeping the assumption that X is a K3 surface over a number field k. It is currently unknown whether or not it is true in general that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$. However, it is known that we may have $\overline{X(k)} \neq X(\mathbb{A}_k)$, even if $X(k)$ is non-empty. We give some examples of this. In [37, pp. 534–535], Swinnerton-Dyer shows that if X is the K3 surface over $\mathbb{Q}$ given by

$$7x^4 + 8y^4 - 9z^4 - 14w^4 = 0,$$

then $X(\mathbb{Q})$ does not lie dense in $X(\mathbb{Q}_3)$. In [41], Wittenberg shows that if X is the K3 surface over $\mathbb{Q}$ that is the minimal proper regular model of the elliptic surface

$$y^2 = x(x - 3(t-1)^3(t+3))(x - 3(t+1)^3(t-3))$$

over the projective line over $\mathbb{Q}$ with coordinate t, then $X(\mathbb{Q})$ is not dense in $X(\mathbb{Q}_2)$. In his PhD thesis [25], Preu shows that if X over $\mathbb{Q}$ is given by

$$x^4 + 3y^4 - 4z^4 - 9w^4 = 0,$$

then $X(\mathbb{Q})$ is not dense in $X(\mathbb{Q}_3)$. Finally, in [14], Hassett, Várilly-Alvarado, and Varilly construct a K3 surface with Picard rank equal to 1, for which weak approximation fails. In the last three cases, the failure of weak approximation is explained by a transcendental Brauer class.

0.7 An open question about K3 surfaces

In [34, p. 484], Skorobogatov and Zarhin ask the following question.

Question 0.3. Given a K3 surface X over a number field k, is $X(k)$ dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$?

Question 0.3 thus asks whether the answer to question (D4') is positive for every K3 surface X. In other words: does the Brauer–Manin obstruction explain the failure of the Hasse principle or weak approximation for *all* K3 surfaces X? It is this question that has guided the research of this thesis. We have restricted to certain classes of K3 surfaces, and for none of these we have been able to give a full answer to Question 0.3. On the other hand, we believe that the results do suggest that the answer to Question 0.3 should be positive for at least certain K3 surfaces.

0.8 Contents of this thesis

We briefly describe the contents of this thesis.

In chapter 1, we answer the following question: if p is a prime, and E is an elliptic curve over $\mathbb{Q}_p$ that has additive reduction, what are the possible isomorphism types of $E(\mathbb{Q}_p)$ as a topological group? Let $E_0(\mathbb{Q}_p) \subset E(\mathbb{Q}_p)$ be the subgroup of points of good reduction. We will give an easy criterion to determine the isomorphism type of $E_0(\mathbb{Q}_p)$ in terms of the coefficients of a Weierstrass equation for E. In particular, we show that $E_0(\mathbb{Q}_p)$ is topologically isomorphic to either $\mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}/p\mathbb{Z}$ as topological groups, where $\mathbb{Z}_p$ carries the p-adic topology, and $\mathbb{Z}/p\mathbb{Z}$ carries the discrete topology. If $p > 7$, then we find that $E_0(\mathbb{Q}_p)$ is always topologically isomorphic to $\mathbb{Z}_p$, and $E(\mathbb{Q}_p)$ is topologically isomorphic to the product of $\mathbb{Z}_p$ and a discrete finite group of order at most 4.

In chapter 2, we review, and slightly improve upon, a result by Sir Peter Swinnerton-Dyer [38]. This result concerns the 2-adic density of rational points on certain explicitly given diagonal quartic surfaces over $\mathbb{Q}$. We will mainly follow the proof of Swinnerton-Dyer, which employs the presence of multiple elliptic fibrations. The argument also builds on the results of chapter 1 to determine the structure of the groups of 2-adic points on the fibres that have additive reduction. The work by Swinnerton-Dyer represents the first known result concerning p-adic density of rational points on a K3 surface for any prime number p.

In chapter 3, we move beyond the work of Swinnerton-Dyer, and construct, for each prime number p, infinitely many pairwise non-isomorphic K3 surfaces over $\mathbb{Q}$ whose rational points are p-adically dense. All K3 surfaces constructed in this chapter will be Kummer surfaces. We will give criteria, in terms of an elliptic curve E and a set of primes S, for the density of the rational points on the Kummer surface X of $E \times E$ in the topological space $\prod_{p \in S} X(\mathbb{Q}_p)$. We construct a K3 surface X over $\mathbb{Q}$ whose rational points lie dense in the space $\prod_{p \in S} X(\mathbb{Q}_p)$, where S is a set of 331 primes. We construct a K3 surface over $\mathbb{Q}$ whose rational points are p-adically dense for all p with $p > 7$ and $p \equiv 3 \pmod 4$. Finally, we give a simple not-too-strong condition, in terms of an elliptic curve E over $\mathbb{Q}$, for the p-adic density of the rational points on the Kummer surface of $E \times E$ for infinitely many p.

In chapter 4, we collect more conditions on an elliptic curve E and a prime number p that imply that the rational points on the Kummer surface of $E \times E$ are p-adically dense. We use these additional criteria to perform a computer experiment. A significant result of this experiment is that, for all elliptic curves E over $\mathbb{Q}$ given by $y^2 = x^3 + ax + b$, with $a, b \in \mathbb{Z}$ such that $-5 \leq a \leq 5$ with $a \neq 0$, and $0 < b \leq 5$, if X is the Kummer surface of $E \times E$, then $X(\mathbb{Q})$ is dense in $X(\mathbb{Q}_p)$ for all prime numbers p such that $109 < p < 2000$ and p is of good reduction for E.

In chapter 5, we treat a result of a different nature. At the AIM workshop "Cohomological Methods in Abelian Varieties", held in Palo Alto from 26–30 March 2012, a group of eight people, namely Lisa Berger, Chris Hall, Jennifer Park, Karl Rubin, Shahef Sharif, Alice Silverberg, Doug Ulmer, and the author of this thesis, worked on the task of extending the result [39, Theorem 12.1] to curves of higher genus. In this thesis, we will prove the following result, which is only one among many results obtained by our group. We let K and K_d be as above, we let C be the curve $y^r = x^{r-1}(x+1)(x+t)$ over K for an odd prime r, and we let J be the Jacobian of C. Then the rank of the abelian group $J(K_d)$ is unbounded, more precise, it is at least $d-2$ for infinitely many values of d. Moreover, for the d for which it is shown that the rank of $J(K_d)$ is at least $d-2$, explicit generators of a rank $d-2$ subgroup of $J(K_d)$ are given.