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CHAPTER 7

Supersymmetric cosmic strings in $\mathcal{N} = 2$ supergravity.

7.1 Introduction.

The present chapter is dedicated to the embedding of local supersymmetric cosmic string solutions in $\mathcal{N} = 2$ supergravity. The first known example of such string solutions was constructed in [73]. The authors of this work considered a model with the minimal matter content needed to obtain a half-BPS cosmic string solution in $\mathcal{N} = 2$ supergravity action: one hypermultiplet and one vector multiplet.

As in $\mathcal{N} = 1$ supergravity, (see section 2.5), the supersymmetric cosmic strings are solutions of the BPS equations, which are obtained imposing the field configuration to preserve half of the supersymmetries of the system. In [73], only field configurations compatible with a consistent truncation to $\mathcal{N} = 1$ were considered, i.e. the cosmic strings are also valid solutions for a $\mathcal{N} = 1$ supergravity model. Although these string configurations involve only the fields surviving the truncation, supersymmetry is not spontaneously broken to $\mathcal{N} = 1$ supergravity, and the full $\mathcal{N} = 2$ supersymmetry is preserved in the vacuum far away from the string. From a $\mathcal{N} = 2$ point of view, the use of an ansatz related to a consistent truncation to $\mathcal{N} = 1$ supergravity in [73] is not required but is useful to simplify the calculation, as the $\mathcal{N} = 2$ BPS equations are in general much more difficult to solve than those of $\mathcal{N} = 1$ theories. The cosmic string solution found in [73] can be seen as an embedding of a $\mathcal{N} = 1$ D-term string in $\mathcal{N} = 2$ supergravity. This is consistent with the discussion in [61] (reviewed in chapter 2), where it is argued that the only $\mathcal{N} = 1$ models admitting supersymmetric vortex solutions involve a D-term potential endowed with a constant Fayet-Iliopoulos term.

The main purpose of the work presented in this chapter is to enlarge the type of $\mathcal{N} = 2$ supergravity theories that can generate constant FI terms in $\mathcal{N} = 1$ supergravity. The special geometry used in [73] is a very particular case, it corresponds to the so-called *minimal special geometry*, which is based on a quadratic prepotential. Here we realize the construction of [73] with a special geometry based on a *cubic prepotential*.

For simplicity we will study a model with abelian gauging of isometries, and thus we will not be able to avoid the presence of hypermultiplets. Indeed, the hypermultiplet is required in order to provide the scalar acting as a Higgs field since, for Abelian gauging, supersymmetry forbids the scalars of vector multiplets to be charged under gauge transformations. As we discussed in the previous chapter, in the presence of hypermultiplets it is not possible to have constant FI-term in a $\mathcal{N} = 2$ supergravity model. Therefore, the FI-terms of the corresponding reduced $\mathcal{N} = 1$ theories are generated from field dependent moment maps in the mother $\mathcal{N} = 2$ theory, using mechanisms described in the previous chapter.

The field content of the model that we are going to study consists of one hypermultiplet and two vector multiplets, and the couplings are characterized by the following choice of quaternionic $\mathcal{M}_{\mathcal{Q}}$ and special Kähler manifolds $\mathcal{M}_{\mathcal{SK}}$:

$$\mathcal{M}_{\mathcal{Q}} = \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)} \qquad \mathcal{M}_{\mathcal{SK}} = ST[2,n] = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}. \tag{7.1.1}$$

In order to define the gauge couplings the we will use the so called *Calabi-Vesentini* symplectic section [139, 152], well-known from different compactifications of string theory [145, 146]. An interesting feature of this section is that it can be used to construct models that exhibit partial breaking of supersymmetry $\mathcal{N} = 2 \to \mathcal{N} = 1$ [153]. In this model we gauge a U(1) subgroup of the R-symmetry that rotates the complex structures of the quaternionic manifold. The corresponding compensator reduces to a constant after the truncation, which acts as an effective FI -term (see section 6.9). The truncated theory of this model contains an axiodilaton field $S = a - \mathrm{i}e^{\rho}$, which appears in the gauge kinetic function that defines the kinetic term of the vector field is $f = iS$. We shall see that the $\mathcal{N} = 2$ BPS equations imply that, in the background of the string, the axiodilaton must be an arbitrary constant. Once the winding number is fixed, the value of S in the string configuration parametrizes a one dimensional family of cosmic strings solutions degenerate in energy but with different radii.

7.2 The model.

7.2.1 Couplings in the hypermultiplets.

In the present model the couplings of the hypermultiplets are described by the quaternionic geometry we studied in the previous chapter, the quaternionic manifold of quaternionic dimension one $\frac{SO(4,1)}{SO(4)}$. This has a very simple quaternionic structure which can be derived from the vielbein and SU(2)-connection are respectively

$$f = \frac{1}{\sqrt{2}}\left(dh\mathbb{1} + \mathrm{i}\mathrm{e}^{-h}\mathrm{d}b^x\sigma^x\right), \qquad \omega^x = -\frac{1}{2}\mathrm{e}^{-h}\mathrm{d}b^x. \tag{7.2.1}$$

where $x = 1, 2, 3$ and h, b^x are real fields. The kinetic terms of the hyperscalars read

$$\mathcal{T} = -\partial_\mu h \partial^\mu h - \mathrm{e}^{-2h}(\partial_\mu b_1 \partial^\mu b_1 + \partial_\mu b_2 \partial^\mu b_2 + \partial_\mu b_3 \partial^\mu b_3). \tag{7.2.2}$$

We will consider the same Abelian gauging as in [73]. The U(1) symmetry that we gauge is $k = k_2 + (\eta + 2)k_1$, where k_1 and k_2 where defined in (6.9.8):

$$k = \begin{pmatrix} 4b_3 \\ 4b_1 b_3 \\ 4b_2 b_3 \\ 2\left[b_3^2 - \mathrm{e}^{2h} + 1 - b_1^2 - b_2^2\right] \end{pmatrix} + (\eta + 2)\begin{pmatrix} 0 \\ -2b_2 \\ 2b_1 \\ 0 \end{pmatrix}. \tag{7.2.3}$$

Here we have arranged the tangent vector in the order $(\frac{\partial}{\partial h}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3})$. The parameter in front of the second term of the left hand side, $(\eta + 2)$, has been written in this way for later convenience. Although the previous Killing vector seems complicated at first sight it is defined in a precise and simple way on any symmetric normal quaternionic manifold using a solvable parametrization of the quaternionic manifold. The moment map corresponding to k reads

$$\vec{\mathcal{P}} = \begin{pmatrix} -2b_2 - 2b_1 b_3 \mathrm{e}^{-h} \\ 2b_1 - 2b_2 b_3 \mathrm{e}^{-h} \\ -\mathrm{e}^{-h}\left[(b_3)^2 + 1 - (b_1)^2 - (b_2)^2\right] - \mathrm{e}^h \end{pmatrix} + (\eta + 2)\begin{pmatrix} \mathrm{e}^{-h}\, b_2 \\ -\mathrm{e}^{-h}\, b_1 \\ 1 \end{pmatrix}. \tag{7.2.4}$$

The killing vector k has a unique fixed point, $k = 0$, at the origin of the quaternionic manifold :

$$k = 0 \quad \Longrightarrow \quad b_3 = b_2 = b_1 = h = 0. \tag{7.2.5}$$

It is easy to check that the only non-vanishing component the moment map at the fixed point of the killing vector is $\mathcal{P}^3$:

$$\mathcal{P}^1|_{k=0} = \mathcal{P}^2|_{k=0} = 0, \qquad \mathcal{P}^3|_{k=0} = \eta. \tag{7.2.6}$$

We will see that η is proportional to the FI-term of the reduced $\mathcal{N}=1$ theory, and thus it will also be proportional to the tension of the cosmic string solution.

7.2.2 Couplings in the vector multiplets.

In order to characterize the couplings of the vector multiplets we consider the Kähler-Hodge manifold

$$\mathcal{M}_{\mathcal{SK}} = ST[2,2+n] = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2+n)}{\mathrm{SO}(2)\times\mathrm{SO}(2+n)}. \tag{7.2.7}$$

We will work in the Calabi-Visentini basis defined by the holomorphic section:

$$v = \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}, \quad \text{with} \quad Z^\Lambda = \begin{pmatrix} \frac{1}{2}(1+y^2) \\ \mathrm{i}\frac{1}{2}(1-y^2) \\ y^a \end{pmatrix}, \quad \text{and } F_\Lambda = S\pi_{\Lambda\Sigma}Z^\Lambda, \tag{7.2.8}$$

where $a=1,\cdots,n$ and $y^2 = y^a y^a$ and

$$\pi_{\Lambda\Sigma} = \begin{pmatrix} \mathbb{1}_2 & \\ & -\mathbb{1}_n \end{pmatrix}. \tag{7.2.9}$$

The fields $S = a - \mathrm{i}e^\rho$ and y^a parametrize the manifolds $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ and $\frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}$ respectively. The Calabi-Visentini basis does not admit a prepotential, but can be rotated to a symplectic section which can be obtained from the cubic prepotential

$$F(S,y) = -\frac{1}{2}Sy^a y^a. \tag{7.2.10}$$

The kinetic terms of the bosons from the vector multiplet can be calculated using formulae (6.3.14) and (6.3.15) given in the previous chapter. The Kähler potential for the Calabi-Visentini section is given by

$$\mathcal{K} = -\log\left[\mathrm{i}(S-\bar{S})\right] - \log\left[\frac{1}{2}\left(1 - 2\bar{y}^a y^a + |y^a y^a|^2\right)\right], \tag{7.2.11}$$

and the coupling matrix of the vector field takes the form

$$\mathcal{N}_{\Lambda\Sigma} = (S-\bar{S})\frac{Z_\Lambda \bar{Z}_\Sigma + \bar{Z}_\Lambda Z_\Sigma}{\bar{Z}^T \pi Z} + \bar{S}\pi_{\Lambda\Sigma}. \tag{7.2.12}$$

The metric is given as usual by the second derivative of the Kähler potential

$$g_{S\bar{S}} = \frac{1}{(2\,\mathrm{Im}\,S)^2}, \quad g_{b\bar{c}} = 2\frac{(\delta^{b\bar{c}} - 2y^b\bar{y}^{\bar{c}})}{1-2\bar{y}^a y^a + |y^a y^a|^2} + 4\frac{\left[\bar{y}^b - y^b(\bar{y}^{\bar{a}}\bar{y}^{\bar{a}})\right]\left[y^{\bar{c}} - \bar{y}^{\bar{c}}(y^a y^a)\right]}{(1-2\bar{y}^a y^a - |y^a y^a|^2)^2} \tag{7.2.13}$$

In order to calculate the scalar potential (6.5.3) we will need the following quantity, which in the Calabi Visentini symplectic section is given by:

$$U^{\Lambda\Sigma} - 3\mathrm{e}^{\mathcal{K}}\bar{Z}^{\Lambda}Z^{\Sigma} = -\frac{1}{\mathrm{i}(S-\bar{S})}\pi^{\Lambda\Sigma}, \tag{7.2.14}$$

for any n. Since $\mathrm{Im}\, S < 0$, it follows that the scalar potential is always positive and bounded from below in the Calabi-Visentini basis, provided that we gauge the vector A^{Λ}_{μ} with $\pi_{\Lambda\Lambda}$ negative. This corresponds to a gauge field associated with any coordinate y^a.

In the example we shall consider in this paper, we will restrict ourselves to the case $n = 1$, as it requires the minimum amount of fields: two complex scalar fields S and y and three vector fields $A^0_{\mu}, A^1_{\mu}, A^2_{\mu}$. This specific case is immediately generalized to any n. Then for $n = 1$ the metric for the scalar manifold $\mathcal{M}_{\mathcal{SK}}$ and the coupling matrix $\mathcal{N}$ simplify to:

$$g_{S\bar{S}} = \frac{1}{(2\,\mathrm{Im}\, S)^2}, \qquad g_{y\bar{y}} = \frac{2}{(1-y\bar{y})^2}, \qquad \mathcal{N} = \begin{pmatrix} S\mathbb{1}_2 & \\ & -\bar{S} \end{pmatrix}. \tag{7.2.15}$$

7.2.3 The $\mathcal{N} = 2$ supergravity lagrangian

With our choice of special geometry, on the submanifold $y = 0$ the graviphoton (6.7.7) depends only on A^0_{μ} and A^1_{μ}. As we would like to put the graviphoton to zero on the string configuration, so that we can perform the reduction to $\mathcal{N} = 1$, we shall gauge the killing vector k with the gauge field A^2_{μ}. Then the scalar potential (6.5.3) is given by

$$\mathcal{V} = 4\mathrm{e}^{-\rho}k^2\frac{y\bar{y}}{(1-y\bar{y})^2} + 2\mathrm{e}^{-\rho}\mathcal{P}^x\mathcal{P}^x, \tag{7.2.16}$$

where $k^2 = g_{XY}k^Xk^Y$. Which is non-negative as we anticipated in the previous section.

The bosonic sector of the $\mathcal{N} = 2$ supergravity action is:

$$\begin{aligned} e^{-1}\mathcal{L}_{bos} \;=\; & -\tfrac{1}{2}R - \tfrac{1}{2}g_{XY}D_{\mu}q^XD^{\mu}q^Y - \frac{1}{(2\,\mathrm{Im}\,S)^2}\partial_{\mu}S\partial^{\mu}\bar{S} - \frac{2}{(1-y\bar{y})^2}\partial_{\mu}y\partial^{\mu}\bar{y} \\ & +\tfrac{1}{4}(\mathrm{Im}\,\mathcal{N})_{\Lambda\Sigma}F^{\Lambda|\mu\nu}F^{\Sigma}_{\mu\nu} + \tfrac{e^{-1}}{8}(\mathrm{Re}\,\mathcal{N})_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} \\ & +\frac{2g^2}{\mathrm{Im}\,S}\left[2k^2\frac{y\bar{y}}{(1-y\bar{y})^2} + \mathcal{P}^x\mathcal{P}^x\right]^2, \end{aligned} \tag{7.2.17}$$

where:

$$(\mathrm{Im}\,\mathcal{N})_{\Lambda\Sigma} = \mathrm{Im}\,S\,\mathbb{1}_3 \qquad (\mathrm{Re}\,\mathcal{N})_{\Lambda\Sigma} = \begin{pmatrix} \mathrm{Re}\,S\mathbb{1}_2 & \\ & -\mathrm{Re}\,S \end{pmatrix}, \tag{7.2.18}$$

and the metric g_{XY} is given by (7.2.2), and the hyperscalars are organized as $q^X = (h, \vec{b})$. The covariant derivatives are defined by (6.5.2) and the killing vector (7.2.3) is gauged by A^2_μ. The square of the moment map, $\mathcal{P}^x\mathcal{P}^x$ is given in (7.3.7).

7.3 Analysis of the scalar potential

The scalar potential derived in the previous section has the following properties:

1. *The scalar potential $\mathcal{V}$ is bounded from below*

$$\mathcal{V} \geq 0, \tag{7.3.1}$$

this is in sharp contrast to the case of the minimal special geometry, used in [73], where the scalar potential was not bounded from below and could be positive, negative or vanish depending on the value of the scalar fields.

2. $y = 0$ *is a critical point of the scalar potential* $\mathcal{V}$:

$$\frac{\partial \mathcal{V}}{\partial y}|_{y=0} = 0. \tag{7.3.2}$$

3. The scalar potential $\mathcal{V}$ has a runaway behaviour in the dilaton field ρ :

$$\mathcal{V} \propto \mathrm{e}^{-\rho}. \tag{7.3.3}$$

7.3.1 Minkowski vacua

Since the scalar potential is a sum of squares it is easy to compute all its Minkowski vacua by looking at the zeroes of the different terms:

$$\mathcal{V} = 0 \Longrightarrow (k = 0 \text{ or } y = 0) \quad \text{and} \quad \mathcal{P}^x\mathcal{P}^x = 0. \tag{7.3.4}$$

To study the scalar potential, it is useful to introduce the following definitions:

$$\Phi = -b_3 + \mathrm{i}\,\mathrm{e}^h, \qquad \tilde{\Phi} = b_1 + \mathrm{i}\, b_2. \tag{7.3.5}$$

Using these coordinates the fixed point of the killing vector k is given by

$$\Phi = \mathrm{i} \quad \text{and} \quad \tilde{\Phi} = 0. \tag{7.3.6}$$

On the other hand, the square of the moment map reads

$$\mathcal{P}^x\mathcal{P}^x = \frac{4}{(\mathrm{Im}\ \Phi)^2}\left|\Phi - \tfrac{\mathrm{i}}{2}(\eta+2)\right|^2 |\tilde{\Phi}|^2 + \left(\frac{|\tilde{\Phi}|^2}{\mathrm{Im}\ \Phi} - \frac{|\Phi - \mathrm{i}|^2}{\mathrm{Im}\ \Phi} + \eta\right)^2, \tag{7.3.7}$$

then, we can see that that there is a Minkowski vacuum for each value of $\eta > -2$:

$$\mathcal{P}^x\mathcal{P}^x = 0 \Longrightarrow \begin{cases} \text{Case I}: \quad \Phi = \frac{\mathrm{i}}{2}(\eta+2), \qquad |\tilde{\Phi}|^2 = -\eta(1+\frac{1}{4}\eta), \quad (-2<\eta<0), \\ \\ \text{Case II}: \quad \frac{|\Phi-\mathrm{i}|^2}{\mathrm{Im}\,\Phi} = \eta, \qquad \tilde{\Phi} = 0, \quad (0 \leq \eta). \end{cases} \tag{7.3.8}$$

Minkowski vacua	
No Minkowski vacuum	$\eta \leq -2$
$y = 0, \quad \Phi = \frac{\mathrm{i}}{2}(\eta+2), \quad |\tilde{\Phi}|^2 = -\eta(1+\frac{1}{4}\eta)$	$-2 < \eta < 0$
$\Phi = \mathrm{i}, \qquad \tilde{\Phi} = 0$	$\eta = 0$
$y = 0, \quad \frac{|\Phi-\mathrm{i}|^2}{\mathrm{Im}\,\Phi} = \eta, \qquad \tilde{\Phi} = 0$	$\eta > 0$

Table 7.1 – *Type of vacua of the scalar potential for different values of the parameter η. Non-singular cosmic string solutions are only possible for $\eta > 0$.*

When $\eta \leq -2$ there are no Minkowski vacua. This implies in particular that for a gauging with $\eta \leq -2$, all the extrema of the potential are de Sitter vacua. However, the potential will not have an absolute minimum (for finite values of the fields) because of its runaway behaviour in the dilaton[1] (7.3.3).

We shall use table 7.1 to explain our choice for the cosmic string configuration. In order to have a cosmic string solution we need to have a circle in the vacuum manifold. If we want the string configuration to be compatible with a consistent reduction of supersymmetry, we shall have to truncate some of the scalar fields of the quaternionic manifold to end up with a Kähler-Hodge submanifold which is completely geodesic.

The appropriate choice of gauging to construct a cosmic string of the Nielsen-Olesen type is $\eta > 0$. Indeed, in that case, the vacuum is a circle

[1]In order to obtain a stable de Sitter vacuum we would have to resort to the techniques presented in [154], which involve using a rotated version of the Calabi-Visentini basis together with non-abelian gauge couplings.

defined by $\frac{|\Phi - \mathrm{i}|^2}{\mathrm{Im}\ \Phi} = \eta$. The Higgs field of the cosmic string is Φ. We shall keep $y = \tilde{\Phi} = 0$ not only in the vacuum but for all the string solutions in order to have a consistent truncation.

In the case $-2 < \eta < 0$, we also have a circle in the vacuum. However, $\Phi = \frac{\mathrm{i}}{2}(\eta + 2)$ does not define a consistent truncation of the quaternionic manifold. To see this note that a gauge transformation 6.5.1 with the killing vector k given by (7.2.3) does not respect this condition for every value of $\tilde{\Phi}$.

In the case where $\eta = 0$, the vacuum is just a point and therefore there is no room for a cosmic string solution of the Nielsen-Olesen type.

7.4 Consistent reduction of supersymmetry

The set of conditions that we impose on the bosonic fields defining the consistent reduction are:

$$\text{Consistent reduction ansatz :} \quad \begin{cases} y & = 0, \\ \tilde{\Phi} & = 0, \\ A^0_\mu = A^1_\mu & = 0. \end{cases} \tag{7.4.1}$$

The condition $y = \tilde{\Phi} = 0$ was explained in the previous section. The conditions $A^0_\mu = A^1_\mu = 0$ ensure that the graviphoton (see equation (6.7.7)) vanishes as it should be in a consistent truncation to $\mathcal{N} = 1$ supergravity. Indeed, the graviphoton appears in the supersymmetry transformations of the gravitini in $\mathcal{N} = 2$ supergravity (6.7.1) but is absent in those of the gravitino of $\mathcal{N} = 1$ supergravity (2.2.22).

In this field configuration the quaternionic and special Kähler manifold reduce as follow:

$$\mathcal{M}_{SK} \times \mathcal{M}_Q \overset{y=\tilde{\Phi}=0}{\Longrightarrow} \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_S \times \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_\Phi \simeq \left(\frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2) \times \mathrm{SO}(2)}\right)_{S,\Phi}, \tag{7.4.2}$$

with the corresponding Kähler potential given by

$$K = -\log\left[\mathrm{i}(S - \bar{S})\right] - 2\log\left[-\mathrm{i}(\Phi - \bar{\Phi})\right]. \tag{7.4.3}$$

Here S is an axion-dilaton field and Φ is the scalar field whose Higgs mechanism generates the cosmic string. Once we impose the condition $\tilde{\Phi} = 0$, the Killing vector of the quaternionic manifold that we have gauged only acts on Φ as:

$$\delta\Phi = 2g(\Phi^2 + 1), \tag{7.4.4}$$

which corresponds to the compact U(1) symmetry of $\left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_\Phi$.

In order to calculate the supersymmetry transformations in the reduced $\mathcal{N} = 1$ theory we will need the following relations:

$$\begin{aligned}
S^{ij} &= T^-_{\mu\nu} = N^S_{ij} = \mathcal{N}^{iA} = 0, \\
\mathrm{e}^{\mathcal{K}} &= \frac{1}{-\operatorname{Im} S}, \\
\mathcal{D}_S Z^\Lambda &= \frac{\mathrm{e}^{\mathcal{K}}}{4}\begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{D}_{y^a} Z^\Lambda = \delta^\Lambda_a, \\
V_{\mu i}{}^j &= \mathrm{i}(\omega^3_\mu + g A^2_\mu \mathcal{P}^3)_i{}^j, \\
N^y_{ij} &= -\mathrm{e}^{\mathcal{K}/2}\mathcal{P}^3_{ij},
\end{aligned} \tag{7.4.5}$$

7.4.1 Truncated $\mathcal{N} = 1$ Lagrangian and supersymmetry transformations.

After setting to zero the truncated fields (7.4.1) the bosonic sector of the $\mathcal{N} = 1$ reduced action reads:

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2}R + \frac{\operatorname{Im} S}{4}F^{\mu\nu}F_{\mu\nu} - \frac{e^{-1}}{8}\operatorname{Re} S\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\
& -\frac{1}{(2\operatorname{Im} S)^2}\partial_\mu S\partial^\mu \bar{S} - \frac{1}{2(\operatorname{Im}\Phi)^2}D_\mu\Phi D^\mu\bar{\Phi} \\
& -2\frac{g^2}{\operatorname{Im} S}\left[\frac{|\Phi - \mathrm{i}|^2}{\operatorname{Im}\Phi} - \eta\right]^2.
\end{aligned} \tag{7.4.6}$$

For convenience, we denote by $A_\mu \equiv A^2_\mu$ the vector gauging the U(1) isometry of the quaternionic manifold, and the corresponfing field strength by $F_{\mu\nu} = F^2_{\mu\nu}$. Using this notation the covariant derivative is given by

$$D_\mu\Phi = \partial_\mu\Phi - 2gA_\mu\left(\Phi^2 + 1\right). \tag{7.4.7}$$

The supersymmetry transformations of the fermions in the reduced $\mathcal{N} = 1$ theory can be found from (6.7.1) using (7.4.5). The transformations corresponding to the supersymmetry parameter ϵ^1 are

$$\begin{aligned}
\delta\psi^1_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_{\mu|mn}\gamma^{mn} + \tfrac{1}{2}\mathrm{i}A^{S\mathcal{K}}_\mu + \tfrac{1}{2}\mathrm{i}A^{\mathcal{Q}}_\mu)\,\epsilon^1, \\
\delta\lambda^y_2 &= -\mathrm{e}^{\frac{\mathcal{K}}{2}}(\operatorname{Im} S)F_{12}\gamma^{12}\epsilon^1 - \mathrm{i}\,\mathrm{e}^{\frac{\mathcal{K}}{2}}\mathcal{P}^3\,\epsilon^1, \\
\delta\lambda^S_1 &= \partial\!\!\!/ S\epsilon_1, \\
\delta\zeta^1 &= \frac{\sqrt{2}}{4\operatorname{Im}\Phi}D\!\!\!\!/\,\Phi\epsilon_1,
\end{aligned} \tag{7.4.8}$$

and for the second supersymmetry parameter ϵ^2 we have

$$\begin{aligned}
\psi^2_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_{\mu|mn}\gamma^{mn} + \tfrac{1}{2}\mathrm{i}A^{\mathcal{SK}}_\mu - \tfrac{1}{2}\mathrm{i}A^{\mathcal{Q}}_\mu)\,\epsilon^2, \\
\delta\lambda^y_1 &= \mathrm{e}^{\frac{\mathcal{K}}{2}}(\mathrm{Im}\,S)F_{12}\gamma^{12}\epsilon^2 - \mathrm{i}\,\mathrm{e}^{\frac{\mathcal{K}}{2}}\mathcal{P}^3\,\epsilon^2, \\
\delta\lambda^S_2 &= \partial\!\!\!/ S\epsilon_2, \\
\delta\zeta^2 &= -\frac{\sqrt{2}}{4\,\mathrm{Im}\;\Phi}D\!\!\!\!/\bar{\Phi}\epsilon_2.
\end{aligned} \tag{7.4.9}$$

In these equations $A^{\mathcal{Q}}_\mu$ is the quaternionic matter connection of the gravitini:

$$A^{\mathcal{Q}}_\mu = 2\omega^3_\mu + 2gA_\mu\mathcal{P}^3 = \frac{\left(\partial_\mu\Phi + \partial_\mu\bar{\Phi}\right)}{\mathrm{Im}\,\Phi} + 2gA_\mu\mathcal{P}^3, \tag{7.4.10}$$

and $A^{\mathcal{SK}}_\mu$ is the U(1) connection of the Special Kähler manifold

$$A^{\mathcal{SK}}_\mu = -\tfrac{1}{4}\frac{\partial_\mu S + \partial_\mu\bar{S}}{\mathrm{Im}\,S}. \tag{7.4.11}$$

The main difference with the supersymmetry transformations obtained in [73] on the string configuration is the presence of the axion-dilaton field S coming from the special geometry and parametrizing the manifold $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$. The gaugini λ^S_i and the U(1) connection $A^{\mathcal{SK}}_\mu$ of the axion-dilaton scalar manifold do not distinguish between the two supersymmetry transformations :

- In the supersymmetry transformations of the gravitini fields, the U(1) connection $A^{\mathcal{SK}}_\mu$ appears with the same charge for both transformations whereas the matter connection $A^{\mathcal{Q}}_\mu$ (coming from the SU(2) of the hypermultiplet) comes with opposite charge for the supersymmetries.
- The axion-dilaton field S enters in the same way in the supersymmetric transformations of the gaugini λ^S_i in contrast to the way Φ appears in the supersymmetric transformation of the hyperini.

This difference of behaviour will be more clear in the next section where we analyze the different BPS projectors obtained from the BPS equations.

7.5 Half-BPS cosmic string solution.

The coupling of the Higgs field to the gauge field is non standard (6.5.2), and thus it is difficult to see what would be the field configuration that corresponds to a cosmic string. To simplify the analysis we define, as in [73], the following field:

$$u = \frac{i-\Phi}{i+\Phi}. \tag{7.5.1}$$

This parametrization of the scalar manifold corresponds to the Poincaré disc model the hyperbolic space H^2. An appropriate Kähler potential to describe the geometry of the scalar manifold in terms of u and S is

$$K = -2\log[1-u\bar{u}] - \log\left[\mathrm{i}(S-\bar{S})\right], \qquad \text{with} \quad |u|^2 < 1, \text{ and } \operatorname{Im} S < 0. \tag{7.5.2}$$

This Kähler potential is related to the one given above in terms of Φ (7.4.3) by a Kähler tranformation

$$K^{\Phi} = K^u + h + \bar{h} \qquad \text{where} \quad h(u) = 2\log[1+u]. \tag{7.5.3}$$

The advantage of this parametrization is that the gauge transformations and the covariant derivatives of the field u have a simple form:

$$\delta u = 4gi u, \qquad D_\mu u = \partial_\mu u - 4gi u\, A_\mu. \tag{7.5.4}$$

Thus gauge transformations correspond to a change of phase of u, and the winding of this phase will be the one inducing the magnetic flux of the string. The corresponding moment map and the D-term potential are given by

$$\mathcal{P} = -\frac{D}{\operatorname{Im} S} = -2g\left(\frac{4u\bar{u}}{1-u\bar{u}} - \eta\right), \qquad V_D = -\tfrac{1}{2}\operatorname{Im}(S) D^2 = -\frac{2g^2}{\operatorname{Im} S}\left[\frac{4u\bar{u}}{1-u\bar{u}} - \eta\right]^2. \tag{7.5.5}$$

Note that η is proportional to the value of the moment map at the fixed point of the gauged isometry $u = 0$, i.e. the FI-term, $\mathcal{P}|_{u=0} = -2g\eta$. In terms of the field u the bosonic sector of the reduced lagrangian (7.4.6) reads

$$\begin{aligned} e^{-1}\mathcal{L} \;=\;& -\tfrac{1}{2}R - \frac{1}{(2\operatorname{Im} S)^2}\partial_\mu S \partial^\mu \bar{S} - \frac{2}{(1-u\bar{u})^2} D_\mu u D^\mu \bar{u} \\ & + \tfrac{1}{4}\operatorname{Im} S\, F^{\mu\nu}F_{\mu\nu} - \tfrac{e^{-1}}{8}\operatorname{Re} S\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ & - \frac{2g^2}{\operatorname{Im} S}\left[\frac{4u\bar{u}}{1-u\bar{u}} - \eta\right]^2. \end{aligned} \tag{7.5.6}$$

7.5.1 The BPS equations

The BPS equations for the cosmic string configuration are obtained by setting to zero the supersymmetry transformations (7.4.8) and (7.4.9). Note that the $\mathcal{N} = 2$ transformations in the reduced theory look like two sets of $\mathcal{N} = 1$ supersymmetry transformations, one for each parameter ϵ^i. Let us first consider the transformations of the gaugini and the chiralino corresponding to ϵ^1 written in terms of the field u

$$\begin{aligned} \delta\lambda_2^y &= -\mathrm{e}^{\frac{\mathcal{K}}{2}}(\operatorname{Im} S) F_{12}\gamma^{12}\epsilon^1 - \mathrm{i}\,\mathrm{e}^{\frac{\mathcal{K}}{2}}\mathcal{P}^3\,\epsilon^1, \\ \delta\lambda_1^S &= \partial\!\!\!/ S\,\epsilon_1, \\ \delta\zeta^1 &= -\tfrac{\mathrm{i}}{\sqrt{2}}(1-u\bar{u})^{-1} D\!\!\!\!/\, u\,\epsilon_1. \end{aligned} \tag{7.5.7}$$

The corresponding BPS equations can be found following the same steps we described in chapter 2 for the $\mathcal{N}=1$ supersymmetric cosmic string

$$(D_1 \pm \mathrm{i}D_2)u = 0, \qquad (\partial_1 \pm \mathrm{i}\partial_2)S = 0, \qquad F_{12} \mp D = 0, \tag{7.5.8}$$

Furthermore, the supersymmetry parameter ϵ^1 must satisfy the projector condition

$$\gamma^{12}\epsilon^1 = \mp\mathrm{i}\epsilon^1. \tag{7.5.9}$$

If we impose the cosmic string to preserve half of the supersymmetries of the full $\mathcal{N}=2$ theory, we also have to set to zero the second set of supersymmetry transformations. The supersymmetry transformations of the gaugini and the chiralino for the supersymmetry parameter ϵ^2 read

$$\begin{aligned}
\delta\lambda_1^y &= \mathrm{e}^{\frac{\mathcal{K}}{2}}(\mathrm{Im}\, S)F_{12}\gamma^{12}\epsilon^2 - \mathrm{i}\,\mathrm{e}^{\frac{\mathcal{K}}{2}}\mathcal{P}^3\,\epsilon^2,\\
\delta\lambda_2^S &= \not{\partial} S\epsilon_2,\\
\delta\zeta^2 &= \tfrac{\mathrm{i}}{\sqrt{2}}(1-u\bar{u})^{-1}\not{D}\bar{u}\,\epsilon_2.
\end{aligned} \tag{7.5.10}$$

This extra requirement leads to the following BPS equations

$$(D_1 \pm \mathrm{i}D_2)u = 0, \qquad (\partial_1 \mp \mathrm{i}\partial_2)S = 0, \qquad F_{12} \mp D = 0, \tag{7.5.11}$$

and to a projector condition for ϵ^2

$$\gamma^{12}\epsilon^2 = \pm\mathrm{i}\epsilon^2. \tag{7.5.12}$$

Thus, we see that in order to solve simultaneously the equations (7.5.8) and (7.5.11), the supersymmetry parameters ϵ^1 and ϵ^2 must have opposite chiralities on the cosmic string world sheet. Moreover, the two equations for the axio-dilaton S require it to be constant everywhere

$$S = \text{Constant}, \qquad A_\mu^{\mathcal{SK}} = -\tfrac{1}{2}\frac{\partial_\mu(\mathrm{Re}\, S)}{\mathrm{Im}\, S} = 0. \tag{7.5.13}$$

The BPS equations we have obtained are the same as those obtained in [73] modulo the factor of $\mathrm{e}^{-\rho}$ in the definition of the D-term.

Gravitini equations

We still have to take into account the supersymmetry transformations of the gravintini, which will lead to the equations that characterize the space-time metric in the background of the cosmic string

$$\begin{aligned}
\delta\psi_\mu^1 &= (\partial_\mu \mp \tfrac{\mathrm{i}}{2}\omega_{\mu|12} + \tfrac{\mathrm{i}}{2}A_\mu^{\mathcal{Q}})\epsilon^1 = 0,\\
\delta\psi_\mu^2 &= (\partial_\mu \pm \tfrac{\mathrm{i}}{2}\omega_{\mu|12} - \tfrac{\mathrm{i}}{2}A_\mu^{\mathcal{Q}})\epsilon^2 = 0.
\end{aligned} \tag{7.5.14}$$

Here we have already used the projector conditions (7.5.9) and (7.5.12), and $A_\mu^{\mathcal{Q}}$ is given by

$$A_\mu^{\mathcal{Q}} = \frac{\mathrm{i}}{1-u\bar{u}}(u\partial_\mu\bar{u} - \bar{u}\partial_\mu u) - 2gA_\mu\left(\frac{4u\bar{u}}{1-u\bar{u}} - \eta\right). \tag{7.5.15}$$

As we explained in section (2.5), in order to find the corresponding BPS equations we restrict ourselves to cylindrically symmetric cosmic string configurations. In that case we can take the space-time metric to be of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + C^2(r)\mathrm{d}\theta^2, \tag{7.5.16}$$

where we have used cylindrical coordinates $\{t, z, r, \theta\}$. With the choice of space-time vielbein

$$e^1 = dr \qquad \text{and} \qquad e^2 = C(r)d\theta, \tag{7.5.17}$$

the only non vanishing component of the spin connection is

$$\omega_\theta^{12} = -C'(r). \tag{7.5.18}$$

Then, using the following ansatz for the supersymmetry parameters

$$\epsilon^i(\theta) = \mathrm{e}^{\mp\frac{1}{2}\mathrm{i}\theta}\epsilon_0^i, \tag{7.5.19}$$

we obtain that the equation for the profile function $C(r)$ is

$$1 - C'(r) = \pm A_\theta^{\mathcal{Q}}. \tag{7.5.20}$$

This equation guarantees that the supersymmetry transformations of the two gravitini (7.5.14) are vanishing.

7.5.2 Cosmic string profile functions

We will use the following time independent ansatz to solve the BPS equations

$$u = f(r)e^{in\theta}, \qquad A_\theta = \frac{m}{4g}v(r), \qquad A_0 = A_r = A_z = 0. \tag{7.5.21}$$

It represents a straight cylindrically symmetric cosmic string of winding m along the z-axis. The BPS equations, (7.5.8) and (7.5.20), for the profile functions of the string, $f(r)$, $v(r)$ and $C(r)$, are:

$$\begin{aligned}
f'(r) &= \pm m\, C^{-1} f\,(1-v)\,,\\
v'(r) &= \mp\mathrm{e}^{-\rho}\,\frac{8g^2C}{m}\left(\frac{4f^2}{1-f^2} - \eta\right),\\
C'(r) &= 1 \mp m\,\frac{2f^2}{1-f^2}(1-v) \mp \tfrac{1}{2}m\eta\, v.
\end{aligned} \tag{7.5.22}$$

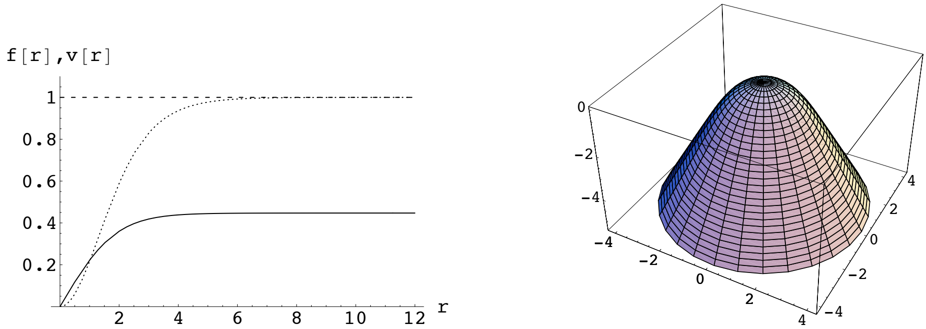

Figure 7.1 – *LEFT: Profile functions (7.5.21) $f(r)$ (solid line) and $v(r)$ (dotted line) which characterize the field configuration the $\mathcal{N}=2$ supersymmetric cosmic string. We have chosen $m=4g=\eta=1$, and the dilaton is set to zero $\rho=0$. RIGHT: Embedding of the metric on the plane orthogonal to the string in three dimensions. Far away from the string, which is located at the tip of surface, the metric approaches that of a cone with deficit angle $\Delta=\pi|m|\eta$.*

In order to have a regular solution at the origin $r=0$ we have to impose the boundary conditions $f(0)=v(0)=0$. We also require that far way from the string the field u is in the vacuum and its kinetic terms vanish, so that the string configuration has a finite action, thus

$$f(r)\to\sqrt{\frac{\eta}{\eta+4}}\qquad\text{and}\qquad v(r)\to 1\qquad\text{for}\quad r\to\infty.\tag{7.5.23}$$

A particular solution to these equations is shown in figure 7.1. The plots represents a unit winding cosmic string with coupling constant $g=1/4$ and $\eta=1$, and the dilaton has been set to zero $\rho=0$.

From the BPS equations we can find the asymptotic behavior of the profile functions, which is similar to the cases of [61, 73]. In the case $r\to 0$ we have:

$$f(r)\sim r^{\pm m},\qquad C\sim r,\qquad v(r)\sim\pm\frac{4g^2\eta}{m}r^2.\tag{7.5.24}$$

It can be seen that the equations only admit regular solutions provided we choose the upper sign for positive winding $m>0$, and the lower sign for negative winding $m<0$. In the opposite limit, $r\to\infty$ the metric is given by

$$\mathrm{d}s^2\approx-\mathrm{d}t^2+\mathrm{d}z^2+\mathrm{d}r^2+r^2\left[1-\tfrac{1}{2}|m|\eta\right]^2\mathrm{d}\theta^2.\tag{7.5.25}$$

At $r\to\infty$, the string creates a locally-flat conical metric with a deficit angle $\Delta=\pi|m|\eta$. The energy of the string per unit length can be computed as we discussed in section 2.5. One finds that the only non-vanishing contribution comes from the Gibbons-Hawking surface term:

$$\mu_{\text{string}} = -\int \mathrm{d}\theta\, C'\bigg|_{r=\infty} + \int \mathrm{d}\theta\, C'\bigg|_{r=0} = \pi|m|\eta > 0. \tag{7.5.26}$$

Note also that asymptotically, $r \to \infty$, since all the fields are in the vacuum the supersymmetry transformations (7.5.10), (7.5.10) and (7.5.14) become zero, and thus the full $\mathcal{N} = 2$ supersymmetry is restored.

7.5.3 Metastability of the string configuration.

An important issue is the study of the stability of these strings. Although supersymmetry ensures that these configurations are solutions to the equations of motion, a priori, there is nothing to prevent them from decaying into a different configuration. As we already mentioned in chapter 1, the stabilitiy of BPS cosmic string solutions has been investigated recently in the context of three and four dimensional $\mathcal{N} = 1$ supergravity in [99] [100] [101]. In these papers it was proven that these solutions are stable against all sorts of perturbations. However the present situation is more subtle since a complete analysis requires the study to be done in the full $\mathcal{N} = 2$ supergravity theory. In particular, it is not clear if the cosmic strings would survive a perturbation of the truncated fields. Moreover, these analyses do not prevent the presence of zero modes which, if they are excited, can also lead to the disappearance of the strings. The cosmic string solution we have presented in this chapter has one of such zero modes, the value of the axion-dilaton field.

The constant value of the axion-dilaton field is not fixed by the BPS equations nor by the scalar potential. The mass per unit length of the string is also independent of the value of the axion-dilaton field (7.5.26). The dilaton fixes the overall length scale of the configuration in the following sense. There are two natural lengths in the solution given by the inverse of the masses of the Higgs and the gauge field, and they are both functions of the dilaton field:

$$m_W^2 \propto -\frac{1}{\operatorname{Im} S}, \quad m_\Phi^2 \propto -\frac{1}{\operatorname{Im} S}, \tag{7.5.27}$$

so that the corresponding length scales are

$$l_W^2 \propto -\operatorname{Im} S, \quad l_\Phi^2 \propto -\operatorname{Im} S. \tag{7.5.28}$$

Suppose we have a solution to the BPS equations given by the profile functions $f(r)$, $W_\theta(r)$, $C(r)$ and ρ. Then it is easy to check that the functions $f(\lambda r)$, $W_\theta(\lambda r)$, $C(\lambda r)/\lambda$ and $\rho - 2\log(\lambda)$ also satisfy the BPS equations for any real $\lambda > 0$. From here it is obvious that the value of the dilaton determines the length scales in the transverse direction to the string, in particular the core radius.

This situation looks similar to the case of semilocal strings [155], where there is also a one parameter family of solutions with equal energy and different core radii. In that case finite energy perturbations can excite the zero mode connecting solutions within the same family, leading to the spread of the magnetic flux and eventually to the disappearance of the strings. An instability of this type was also found in a class of BPS cosmic strings solutions which appear in a $\mathcal{N} = 1$ supersymmetric model proposed by Blanco-Pillado *et al.* [59] to describe the last stage of a brane-antibrane inflation. We review this analysis, published in [72], in appendix B.

This is not going to occur in our model. In order to go from one solution to a different one, the dilaton has to change its value everywhere in the plane transverse to the string. The kinetic energy needed in order to excite the value of the dilaton globally diverges, and this implies that, once the system has chosen a given value for the dilaton, finite energy perturbations cannot drive the system to a solution with a different value of S. The radius of the string will remain unchanged.

7.6 Discussion

As a generalization of the work done in [73], in this chapter we have enlarged the family of $\mathcal{N} = 2$ supergravity actions which allow the embedding of $\mathcal{N} = 1$ supergravity actions containing a D-term potential and a constant FI term. We have extended the result of [73] to a class of special geometries more familiar in compactifications of string theory. We are using here a "very special Kähler geometry" characterized by a cubic prepotential, instead of the minimal special geometry used in [73]. To be specific we take the special manifold to be:

$$ST[2,n] \equiv \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)},$$

in the Calabi-Visentini basis (7.2.8), which is related to the cubic prepotential by a symplectic rotation [139, 152].

This choice of special geometry has two important consequences. An axion-dilaton field, $S = a - \mathrm{i}e^{\rho}$, is present in the reduced $\mathcal{N} = 1$ theory after truncation from $\mathcal{N} = 2$. Moreover, it is possible to define a gauging for which the scalar potential is bounded from below. However, it has a runaway dependence on the dilaton:

$$\mathcal{V} \propto \mathrm{e}^{-\rho}.$$

As an application, we have shown how to construct a half-BPS cosmic string solution from a $\mathcal{N} = 2$ supergravity action in $D = 4$. Following [73] we have used a string ansatz compatible with a consistent truncation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

In order to obtain the scalar potential we have gauged the same isometry used in [73]. We have found that the BPS equations imply that the axion-dilaton has to be simultaneously holomorphic and anti-holomorphic, which can only be satisfied if it is a constant:

$$S = \text{Constant}, \quad \text{Im}S < 0.$$

Despite the runaway behavior of the potential, we have proved that all the string solutions have the same energy per unit length, regardless of the value of the dilaton, and it is given by the Gibbons-Hawking surface term [96, 73]. The value of the dilaton fixes the masses of the Higgs and the gauge field and, hence, also the radius of the string. We have argued that the system cannot evolve between two solutions with different values of the dilaton, since this would require an infinite amount of energy. Thus, once the strings are formed their radii remain fixed.

Observations of the timing of milisecond pulsars give the constraint $\mu_{\text{string}} \lesssim 2 \times 10^{-7}$ [51]. However this constraint depends on the specific model used to calculate it, what leads to a considerable uncertainty. This implies for our model that the FI term has to satisfy:

$$0 < \pi|m|\eta \lesssim 2 \times 10^{-7},$$

where the lower bound is coming from the study of Minkowski vacua in section 7.3.