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CHAPTER 6

$\mathcal{N} = 2$ supergravity and effective Fayet-Iliopoulos terms.

6.1 Introduction

This chapter is intended to provide the basis to discuss our work about supersymmetric cosmic strings in $\mathcal{N} = 2$ supergravity, thus we will review here some aspects of extended $\mathcal{N} = 2$ locally supersymetric theories.

In $\mathcal{N} = 2$ supergravity the couplings of the vector multiplets and of the hypermultiplets are governed by *special geometry* and *quaternionic geometry* respectively. In sections 6.3 and 6.4 we will present the basic features of the kinetic terms of the bosons in both sectors, and then in section 6.5 we will discuss the form of the scalar potential. Contrary to what happens in $\mathcal{N} = 1$ supergravity, in $\mathcal{N} = 2$ there is no analogue of the superpotential, and thus the only contribution to the scalar potential resembles the $\mathcal{N} = 1$ $D-$term potential, which is non-zero provided we promote to local some of the global symmetries of the action.

According to our discussion in chapter 2, Fayet-Iliopoulos terms play a crucial röle in the construction of supersymmetric cosmic string solutions. In particular, they induce the spontaneous gauge symmetry breaking which leads to the formation of the string, and the string tension is proportional to the value of the FI-term. In $\mathcal{N} = 1$ supergravity, if the gauged isometry has a fixed point, then we can obtain a model with a non-zero FI-term using the freedom to add a constant to the moment map (2.2.18). However, as we will show in section 6.5.1, in $\mathcal{N} = 2$ supergravity a killing vector has associated a triplet of moment maps, $\mathcal{P}^x$ with $x = 1, 2, 3$, which are completely determined by the killing vector, and thus we are not free to shift them. This statement can also be understood in terms of the gauge couplings of the gravitino. In the last chapter we identified the $\mathcal{N} = 1$ FI-term with the charge of the gravitino under the gauge symmetry, then the freedom to shift the moment map (2.2.18) ensures that we are free to choose this charge. However in $\mathcal{N} = 2$ supergravity the coupling of the gravitino to the corresponding gauge boson is completely determined by the choice of the killing vectors, i.e. the gauge couplings of the scalar fields.

Identifying a constant contribution of the $\mathcal{N} = 2$ moment maps $\mathcal{P}^x$ which could act as an FI-term is not obvious. Since FI-terms are better understood in $\mathcal{N} = 1$ supergravity, in order to solve this problem we will discuss consistent truncations of $\mathcal{N} = 2$ supergravity models with partial reduction of supersymmetry down to $\mathcal{N} = 1$ [106, 107, 135]. This technique, reviewed in section 6.8, is similar to the one we presented in chapter 3, and consists in truncating part of the field content of the theory in such a way that the reduced model is only invariant under $\mathcal{N} = 1$ local supersymmetry transformations. As for $\mathcal{N} = 1 \to \mathcal{N} = 1$ truncations, this method also ensures that any solution of the equations of motion of the reduced theory is also a solution of the full equations of motion. Within the reduced $\mathcal{N} = 1$ theory the identification the FI-term is straightforward using the arguments of the previous chapter, indeed its magnitude is given by the value of the moment map of the reduced theory at the fixed point of the killing vector.

In section 6.9 we will present the method to construct $\mathcal{N} = 2$ supergravity actions leading $\mathcal{N} = 1$ supergravity models with a constant FI-term after a consistent reduction of supersymmetry. We will end the chapter with a couple of explicit examples, and in particular we discuss the embedding in $\mathcal{N} = 2$ supergravity of the gauged axio-dilaton system which was studied in the previous chapter.

6.2 Overview

The $\mathcal{N} = 2$ supersymmetry algebra involves two supersymmetry generators represented by Majorana spinors, which can be decomposed into two chiral spinors. We shall work with the corresponding four chiral spinors that we denote (ϵ^i, ϵ_i). The index $i = 1, 2$ labels the original Majorana spinors and the position of that index represents the chirality. ϵ^i is a left-handed spinors while ϵ_i is right-handed:

$$\epsilon^i = \tfrac{1}{2}(1+\gamma_5)\epsilon^i, \quad \epsilon_i = \tfrac{1}{2}(1-\gamma_5)\epsilon_i. \tag{6.2.1}$$

We follow the notation and conventions of [94, 84]. Charge conjugation relates the two chiral projections of a given Majorana spinor. We shall use the same convention for other chiral spinors.

In four space-time dimensions with a Minkowski signature the automorphism group of the $\mathcal{N} = 2$ supersymmetry algebra, the R-symmetry group, is $H_R = \mathrm{U}(2)$. Under the SU(2) part of the R-symmetry, the supersymmetry generators ϵ^i transform as a doublet. The U(1) part of the R-symmetry acts on the generators by a change of phase.

In $\mathcal{N} = 2$ supergravity, we shall consider three type of supermultiplets, the *graviton multiplet*, the *vector multiplet*, and the *hypermultiplet*. Inside each multiplet, the fields are arranged into representations of the R-symmetry group:

- The graviton multiplet: it contains the vielbein of the spacetime metric, two gravitini ψ^i_μ and one graviphoton A^0_μ. The label $i = 1, 2$ is associated to the SU(2) R-symmetry transformations.
- The vector multiplet: it contains one complex scalar z^α, two gaugini λ^α_i and one gauge field A^α_μ. Here $\alpha = 1, \ldots, n_V$ labels n_V different vector multiplets.
- The hypermultiplet: it contains four real scalars q^X and two hyperini, ζ^A, where the labels are $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$ for n_H hypermultiplets.

For convenience, the $\mathcal{N} = 2$ supersymmetry transformations will be reviewed later on, after discussing the bosonic part of the action

$$S_{bos} = \int d^4x\sqrt{-g}[-\tfrac{1}{2}R + \mathcal{L}_{hyper} + \mathcal{L}_{vector} - \mathcal{V}], \tag{6.2.2}$$

where $\mathcal{L}_{hyper}$ are the kinetic terms of the hyperscalars, $\mathcal{L}_{vector}$ are the kinetic terms of the bosonic fields in the vector multiplets and $\mathcal{V}$ is the scalar potential.

In $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets, the scalar manifold that characterizes the kinetic terms of the scalars is a direct product

$$\mathcal{M} = \mathcal{M}_{\mathcal{SK}} \times \mathcal{M}_{\mathcal{Q}},, \tag{6.2.3}$$

where $\mathcal{M}_{\mathcal{SK}}$ and $\mathcal{M}_{\mathcal{Q}}$ are respectively the scalar manifold of vector and hypermultiplets. The restrictions coming from supersymmetry impose that $\mathcal{M}_{\mathcal{SK}}$ is a so-called *special manifold*, whereas $\mathcal{M}_{\mathcal{Q}}$ is a *quaternionic manifold*. We will pay particular attention to the geometry of the scalar manifold and the gauging of isometries.

Gravity multiplet	vielbein	e_μ^m	$i = 1, 2$ $\mu, m = 0, \cdots, 3$
	gravitini	$\psi_\mu^i, \psi_{i\mu}$	
	graviphoton	"A_μ^0"	
Vector multiplet	gauge fields	A_μ^α	$\alpha = 1, \ldots, n_V$
	gaugini	$\lambda_i^\alpha, \lambda_\alpha^i$	
	scalars	z^α	
Hypermultiplets	hyperscalars	q^X	$X = 1, \ldots, 4n_H$ $A = 1, \ldots, 2n_H$
	hyperini	ζ^A, ζ_A	

Table 6.1 *– Field content of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. The physical graviphoton is not necessarily A_μ^0 but whatever field appears in the supersymmetric transformation of the gravitini through its field strength $T_{\mu\nu}$. The latter is a linear combination of field strengths of all the gauge fields A_μ^Λ present in the theory ($\Lambda = 0, \ldots, n_V$) with coefficients that depend on the scalar fields z^α of vector multiplets. The couplings of all the gauge fields A_μ^Λ and the scalar fields z^α is controlled by* special geometry*.*

6.3 Vector multiplets and special geometry

We consider $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets [136, 137, 138]. For a modern review, see [139, 140, 141].

Since the gravity multiplet contains a vector field, the *graviphoton*, this theory admits $n_V + 1$ vector fields A_μ^Λ where $\Lambda = 0, \ldots, n_V$. The n_V vector multiplets contain as well n_V complex scalar fields z^α, $\alpha = 1, \ldots, n_V$, parametrizing a Special Kähler manifold. The kinetic terms of the scalar and vector fields are :

$$\begin{aligned} \mathcal{L}_{vector} &= \frac{1}{4} \operatorname{Im}(\mathcal{N})_{\Lambda\Sigma} F^{\Lambda|\mu\nu} F^\Sigma_{\mu\nu} - \frac{1}{8} e^{-1} \operatorname{Re}(\mathcal{N})_{\Lambda\Sigma} \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} - g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}, \\ &= \tfrac{1}{2} \operatorname{Im}\left(\mathcal{N}_{\Lambda\Sigma} F^{+\Lambda}_{\mu\nu} F^{+\Sigma|\mu\nu} \right) - g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}, \end{aligned} \tag{6.3.1}$$

where $F^{\pm\Lambda}_{\mu\nu}$ is the self-dual combination[1]:

$$F^{\pm\Lambda}_{\mu\nu} = \tfrac{1}{2}\left(F^{\Lambda}_{\mu\nu} \mp \tfrac{1}{2}\mathrm{i}e\varepsilon_{\mu\nu\rho\sigma}F^{\Lambda|\rho\sigma}\right), \qquad F^{\Lambda}_{\mu\nu} = \partial_\mu A^{\Lambda}_{\nu} - \partial_\nu A^{\Lambda}_{\mu}. \tag{6.3.2}$$

The couplings of vector multiplets to $\mathcal{N}=2$ supergravity are characterized by *special geometry.* The latter relies heavily on the existence of *duality transformations* for vector fields in supersymmetric theories. Duality transformations generalize the electric-magnetic duality of Maxwell's equations without sources. Defining the magnetic duals of the field strengths $F^{+\Lambda}_{\mu\nu}$ as

$$G^{\mu\nu}_{+\Lambda} = 2\mathrm{i}\frac{\partial\mathcal{L}}{\partial F^{+\Lambda}_{\mu\nu}} = \mathcal{N}_{\Lambda\Sigma}F^{+\Sigma|\mu\nu}, \tag{6.3.3}$$

the equations of motion and Bianchi identities read

$$\begin{aligned} &\partial^\mu \operatorname{Im} F^{+\Lambda}_{\mu\nu} = 0, \quad \text{Bianchi Identity,} \\ &\partial_\mu \operatorname{Im} G^{\mu\nu}_{+\Sigma} = 0, \quad \text{Equations of motion.} \end{aligned} \tag{6.3.4}$$

Then, the duality transformations are linear transformations $\mathcal{S} \in \mathrm{G}\,\ell(2n_V+2,\mathbb{R})$ of the field strengths and their magnetic duals

$$\begin{pmatrix} \tilde{F}^+ \\ \tilde{G}_+ \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^+ \\ G_+ \end{pmatrix}, \tag{6.3.5}$$

that preserve the Bianchi identities and equations of motion for vector fields. The relation between the field strengths and their magnetic duals (6.3.3) determines the transformation rule of the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ under (6.3.5):

$$\mathcal{N} \Longrightarrow (C+D\mathcal{N})(A+B\mathcal{N})^{-1}, \tag{6.3.6}$$

where A, B, C, D are $(n_V+1)\times(n_V+1)$ real matrices that characterize the duality transformation

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{G}\,\ell(2n_V+2,\mathbb{R}). \tag{6.3.7}$$

If equations (6.3.3) are derived from a Lagrangian $\mathcal{L}$, the matrix $\mathcal{N}_{\Lambda\Sigma}$ should be symmetric. Asking the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ to be preserved under a fractional transformation (6.3.6) restricts the duality transformations to be given by symplectic matrices $\mathcal{S}$:

$$\mathcal{S} \in \mathrm{Sp}(2n_V+2,\mathbb{R}). \tag{6.3.8}$$

As the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ transforms under duality transformations, the scalar fields z^α should also transform in a specific way. The action of the duality

[1] In our convention the Levi-Civita tensor satisfies $\varepsilon_{0123}=1$.

transformations on the scalar fields is much more transparent once we introduce a *symplectic section* which depends on the scalar fields of vector multiplets and transforms linearly under symplectic rotations. Special geometry can be completely defined in terms of this symplectic section.

The symplectic section is given by:

$$v = \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}, \quad \Lambda = 0, \cdots n_V \tag{6.3.9}$$

and is endowed with a symplectic scalar product

$$\langle v|\bar{v}\rangle = -v^T \begin{pmatrix} \mathbf{0}_{n_V} & -\mathbb{1}_{n_V} \\ \mathbb{1}_{n_V} & \mathbf{0}_{n_V} \end{pmatrix} \bar{v}. \tag{6.3.10}$$

Here Z^Λ and F_Λ are functions of the coordinates z^α $(\alpha = 1, \ldots, n_V)$ of the scalar fields of the vector multiplets. Recall that the index Λ runs from 0 to n_V where n_V is the number of vector multiplets whereas $\alpha = 1, \ldots, n_V$ because the graviphoton that appears in the graviton multiplet is not related to any scalar fields. This is compensated by the freedom to re-scale the symplectic section: the symplectic section is a projective section.

Symplectic rotations act linearly on the symplectic section as:

$$\begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix} \Longrightarrow \mathcal{S} \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}. \tag{6.3.11}$$

We see that the upper part Z^Λ and the lower part F_Λ of the symplectic section transform respectively as the field strengths $F^{\pm\Lambda}_{\mu\nu}$ and their magnetic duals $G^{\pm\Lambda}_{\mu\nu}$. This can be understood from the following remark: Z^Λ and F_Λ are the fermi-fermi components of the superspace generalization of electric and magnetic field strenghts $\hat{F}^\Lambda_{\mu\nu}$ and $\hat{G}_{\Lambda|\mu\nu}$:

$$\begin{aligned} \hat{F}^\Lambda_{\mu\nu} &= F^\Lambda_{\mu\nu} + \bar{Z}^\Lambda \bar{\psi}^i_\mu \psi^j_\nu \varepsilon_{ji} + Z^\Lambda \varepsilon^{ji} \bar{\psi}_{i\mu} \psi_{j\nu}, \\ \hat{G}_{\Lambda|\mu\nu} &= G_{\Lambda|\mu\nu} + \bar{F}_\Lambda \bar{\psi}^i_\mu \psi^j_\nu \varepsilon_{ji} + F_\Lambda \varepsilon^{ji} \bar{\psi}_{i\mu} \psi_{j\nu}. \end{aligned} \tag{6.3.12}$$

A *special manifold* is a Kähler manifold in which the Kähler potential is not a fundamental quantity but is given by the following symplectic invariant expression:

$$\mathcal{K} = -\log\left(-\mathrm{i}\langle v|\bar{v}\rangle\right) = -\log\left[-\mathrm{i}\left(Z^\Lambda \bar{F}_\Lambda - F_\Lambda \bar{Z}^\Lambda\right)\right]. \tag{6.3.13}$$

In special geometry, the kinetic terms of both scalar and vector fields are computed from the symplectic section as follow

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z,\bar{z}) = \mathrm{i}\langle \mathcal{D}_\alpha v | \mathcal{D}_{\bar{\beta}} \bar{v}\rangle, \qquad \mathcal{N}_\Lambda \equiv \begin{pmatrix} F_\Lambda & \bar{\mathcal{D}}_{\bar{\alpha}} \bar{F}_\Lambda \end{pmatrix} \begin{pmatrix} Z^\Sigma & \bar{\mathcal{D}}_{\bar{\alpha}} \bar{Z}^\Sigma \end{pmatrix}^{-1}. \tag{6.3.14}$$

Here the covariant derivatives are defined by

$$\mathcal{D}_\alpha v = \partial_\alpha v + (\partial_\alpha \mathcal{K})v, \qquad \mathcal{D}_{\bar\alpha} \bar v = \partial_{\bar\alpha} \bar v + (\partial_{\bar\alpha} \mathcal{K})\bar v. \tag{6.3.15}$$

In contrast to the metric $\mathcal{N}_{\Lambda\Sigma}$ of the vector fields, the metric of the scalar fields is a symplectic invariant quantity. When the theory is gauged, the electric-magnetic duality is explicitly broken by the introduction of electric charges. In particular, the scalar potential generated by the gauging is not symplectic invariant.

Special geometry and prepotentials

A special geometry is said *to admit a prepotential* when the lower component of the symplectic section (the variable F_Λ) can be expressed as derivative of a scalar function $F(Z)$ depending only on the upper part of the symplectic section (Z^Λ):

$$F_\Lambda = \frac{\partial}{\partial Z^\Lambda} F(Z). \tag{6.3.16}$$

$F(Z)$ is restricted to be an homogeneous function of second degree in the Z^Λ fields and is called the *prepotential*. Interestingly, any symplectic section can be rotated to a section admitting a prepotential [140]. In the presence of a prepotential the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ can be written in the form[2]

$$\mathcal{N}_{\Lambda\Sigma} = \bar F_{\Lambda\Sigma} + 2\mathrm{i}\frac{\mathrm{Im}(F_{\Lambda\Delta})\,\mathrm{Im}(F_{\Sigma\Gamma})Z^\Delta Z^\Gamma}{\mathrm{Im}(F_{\Delta\Gamma})Z^\Delta Z^\Gamma}. \tag{6.3.17}$$

For phenomenogical reasons the most interesting sections are those related to a prepotential only after a symplectic rotation. For instance, in $\mathcal{N} = 2$ compactifications of string theory it is not possible to describe chiral fermions, and partial supersymmetry breaking to $\mathcal{N} = 1$ is only possible with symplectic sections that do not admit a prepotential [142]. Nevertheless, prepotentials are still interesting since they provide a handy way to classify special manifolds, and they simplify the study of symplectic invariant properties of the action.

Minimal special geometries correspond to quadratic prepotential defined with a metric $\eta_{\Lambda\Sigma}$ of signature $(1, n)$:

$$F(Z) = -\mathrm{i}Z^\Lambda \eta_{\Lambda\Sigma} Z^\Sigma, \quad \eta_{\Lambda\Sigma} = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & -\mathbb{1}_n \end{pmatrix}. \tag{6.3.18}$$

The corresponding scalar manifold of the vector multiplets is $\frac{\mathrm{SU}(1,n)}{\mathrm{U}(1)\,\mathrm{SU}(n)}$. Although unusual, minimal special geometry is not incompatible with string theory. To the best of our knowledge, there is so far only one case in which it

[2] $F_{\Lambda\cdots\Sigma} = \partial_\Lambda \cdots \partial_\Sigma F$.

occurs in string theory [143]. This are the $\mathcal{N}=2$ vacua coming from the $\mathcal{N}=3$ flux compactification on $T^6/\mathbb{Z}_2$ studied by Frey and Polchinski [144].

Very special Kähler geometries are characterized by cubic prepotentials

$$F(Z) = \mathrm{i} d_{\Lambda\Sigma\Gamma} \frac{Z^\Lambda Z^\Sigma Z^\Gamma}{Z^0}, \tag{6.3.19}$$

where $d_{\Lambda\Sigma\Gamma}$ is a real symmetric tensor. Very special Kähler geometries are familiar in string theory where they occur in many different compactifications as for example in toroidal compactifications of the heterotic string with possible Wilson lines [139], in compactifications of type II string theories on Calabi-Yau threefolds, in compactification of type II string theories on orientifolds like $K_3 \times T^2/\mathbb{Z}_2$ in the presence of $D3$ and $D7$ branes [145, 146].

6.4 Hypermultiplets and quaternionic-Kähler geometry

In four dimensional spacetime with the usual Minkowski signature, a hypermultiplet is composed of four real scalar fields and two Majorana spinors. As a Majorana spinor can be decomposed into two chiral spinors of opposite chirality, one can describe n_H hypermultiplets in terms of $4n_H$ real scalar fields q^X ($X = 1, \cdots, 4n_H$) and $2n_H$ chiral spinors ζ^A ($A = 1, \ldots, 2n_H$) of positive chirality and $2n_H$ chiral spinors ζ_A of negative chirality. The spinors (ζ^A, ζ_A) of hypermultiplets are called the *hyperini* and the $4n_H$ real scalar fields q^X are the *hyperscalars*. The kinetic terms of the scalar fields q^X are characterized by a sigma model with target space the manifold $\mathcal{M}_Q$, which is constrained by $\mathcal{N}=2$ supersymmetry to be a quaternionic manifold [147] [3]

$$\mathcal{L}_{\text{hyper}} = -\tfrac{1}{2} g_{XY} \partial_\mu q^X \partial^\mu q^Y. \tag{6.4.1}$$

The metric g_{XY} is computed from the vielbein f_X^{iA} as

$$g_{XY} = f_X^{iA} f_{YiA}, \tag{6.4.2}$$

where $f_{XiA} = (f_X^{iA})^*$. We denote by f_{iA}^X and $f^{XiA} = (f_{iA}^X)^*$ the inverse of the vielbein as a $4n_H \times 4n_H$ matrix and its complex conjugate, thus

$$f_Y^{iA} f_{iA}^X = \delta_Y{}^X. \tag{6.4.3}$$

It can be shown that the vielbein and its inverse satisfy the following reality conditions:

$$f_{iA}^X = f^{XjB} \mathbb{C}_{BA} \epsilon_{ji}, \qquad f^{XiA} = \varepsilon^{ij} \mathbb{C}^{AB} f_{jB}^X. \tag{6.4.4}$$

[3] For a review of quaternionic geometry see [148, 139, 149]. In particular, we shall use the conventions of appendix B of [149].

In our conventions, the matrices ϵ^{ij} and $\mathbb{C}^{AB}$ read:

$$\varepsilon^{ij} = \mathrm{i}\sigma^2, \quad \mathbb{C}^{AB} = \varepsilon^{ij} \otimes \mathbb{1}_{st}, \quad i,j = 1,2, \quad s,t = 1,\dots,n_H, \tag{6.4.5}$$

and $\mathbb{C}_{AB}$ the inverse of $\mathbb{C}^{AB}$. In the previous equation, the indices $A, B = 1,\dots,2n_H$ has been decomposed into $A \equiv (i,t), B \equiv (j,s)$ where $i,j = 1,2$ and $t,s = 1,\dots,n_H$.

The reality condition for the veilbein f_X^{iA} can be translated into the following property of the 2×2 matrices $f_X^t \equiv (f_X^t)^i{}_j$

$$(f_X^t)^* = \sigma^2 f_X^t \sigma^2. \tag{6.4.6}$$

This implies that f^t can be seen as n_H one-forms with quaternion entries[4] written in the representation where the quaternionics units are $(\mathbb{1}_2, -\mathrm{i}\sigma^x)$ where $x = 1,2,3$ [5]. We can then say that f_X^t is a *quaternionic vielbein* as at each point of the scalar manifold $\mathcal{M}_Q$, the $4n_H$ real scalar fields q^X can be organized into n_H quaternions q^t :

$$q^t = f_X^t q^X. \tag{6.4.7}$$

In a quaternionic manifold it is possible to define a triplet J^x $(x = 1,2,3)$ of complex structures :

$$(J^x)_X{}^Y = -\mathrm{i} f_X^{iA} (\sigma^x)_i{}^j f_{jA}^Y \tag{6.4.8}$$

which satisfy the multiplication table of quaternionic units

$$J^x J^y = -\delta^{xy} \mathbb{1}_{4n_H} + \varepsilon^{xyz} J^z. \tag{6.4.9}$$

Any linear combination of the form $\tilde{J} = a_x J^x$ also defines a complex structure $(J^2 = -\mathbb{1})$ provided that

$$\|\vec{a}\|^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 = 1. \tag{6.4.10}$$

It follows that at each point of the manifold there is a sphere of complex structures which are related to each other by SU(2) rotations. Note that the quaternionic vielvein f_X^{iA} contains an index i associated to the SU(2) R-symmetry. Therefore, from the definition of the complex structures (6.4.8), it is easy to see to check that the SU(2) R-symmetry can be identified with the SU(2) that rotates the complex structures.

The complex structures are covariantly constant with respect to an SU(2) connection $\omega_{Xi}{}^j = \mathrm{i}\omega_X^x (\sigma^x)_i{}^j$

$$\nabla_X \vec{J} \equiv \nabla_X^{\mathrm{LC}} \vec{J} + 2\vec{\omega}_X \times \vec{J} = 0, \tag{6.4.11}$$

[4] The set of quaternions is defined by $\mathbb{H} = \{q_0 1 + q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k} | q_i \in \mathbb{R}\}$, with the elements of the basis satisfying $\mathrm{i.j} = \mathrm{k}$, together with all cyclic permutations and $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$.

[5] Any 2×2 matrix q satisfying the condition $q^* = \sigma^2 q \sigma^2$ can be written as a quaternion $q = q^0 \mathbb{1} - \mathrm{i} q^x \sigma^x$ with $q^0, q^x \in \mathbb{R}$ and σ^x are the Pauli matrices.

where $\nabla^{\rm LC}$ is the Levi-Civita covariant derivative on the quaternionic manifold. In quaternionic manifolds the SU(2) curvature

$$\vec{\mathcal{R}}_{XY} \equiv 2\partial_{[X}\vec{\omega}_{Y]} + 2\vec{\omega}_X \times \vec{\omega}_Y \tag{6.4.12}$$

is proportional to the quaternionic structure $\vec{J}_{XY}$:

$$\mathrm{R}_{XY} = \frac{1}{4n_H} g_{XY}\mathrm{R}, \quad \vec{\mathcal{R}}_{XY} = \frac{1}{2}\nu\vec{J}_{XY}, \quad \nu = \frac{1}{4n_H(n_H+2)}\mathrm{R}, \tag{6.4.13}$$

with $R_{XY} = R^{Z}_{XZY}$. Here the constant ν is proportional to the gravity coupling constant $\nu = -\kappa^2$. Since we work in units in which $\kappa = 1$, that is $\nu = -1$.

6.5 Isometries, gauging and scalar potential

In $\mathcal{N} = 2$ supergravity coupled to vector multiplets and hypermultiplets, the only way to generate a scalar potential is to promote some of the symmetries of the scalar manifold to be local symmetries. This implies a choice of the Killing vectors of the scalar manifolds and a choice of vector fields that will be used as gauge fields in the covariant derivatives.

In this thesis we will only consider abelian gauging of the symmetries of $\mathcal{M}_Q$. The gauged symmetry is defined by the transformation with parameters α^Λ:

$$\delta q^X = k^X_\Lambda \alpha^\Lambda, \tag{6.5.1}$$

where k^X_Λ are the Killing vectors that we will gauge with the vector fields A^Λ_μ. To gauge a symmetry all the derivatives of the hyperscalars have to be extended to covariant derivatives. The gauge field is taken from the vector multiplets:

$$\nabla_\mu q^X = \partial_\mu q^X - k^X_\Lambda A^\Lambda_\mu. \tag{6.5.2}$$

In order to preserve supersymmetry the action has to be corrected adding the following scalar potential

$$\mathcal{V} = (g_{\alpha\bar{\beta}} k^\alpha_\Lambda k^{\bar{\beta}}_\Sigma + 2 g_{XY} k^X_\Lambda k^Y_\Sigma)\mathrm{e}^{\mathcal{K}} \bar{Z}^\Lambda Z^\Sigma + 4\left(U^{\Lambda\Sigma} - 3\mathrm{e}^{\mathcal{K}}\bar{Z}^\Lambda Z^\Sigma\right)\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma. \tag{6.5.3}$$

where $\mathcal{P}^{ij}_\Lambda = \mathcal{P}^x_\Lambda(\mathrm{i}\sigma^x)^{ij}$ and $\mathcal{P}_{\Lambda|ij} = \mathcal{P}^x_\Lambda(\mathrm{i}\sigma^x)_{ij}$ are the *moment maps* [148, 139, 149] related to the Killing vectors k^X_Λ of the quaternionic-Kähler manifold, and k^α_Λ are the Killing vectors of the special manifold. We also have used the definition

$$U^{\Lambda\Sigma} \equiv \mathrm{e}^{\mathcal{K}} g^{\alpha\bar{\beta}} \mathcal{D}_\alpha Z^\Lambda \mathcal{D}_{\bar{\beta}} Z^\Sigma. \tag{6.5.4}$$

As the scalar fields of vector multiplets transform in the adjont representation of the gauge group, the Killing vectors $k^\alpha_\Lambda, k^{\bar{\alpha}}_\Lambda$ of the special manifold vanish for Abelian gauging. In particular, the sector proportional to $g_{\alpha\bar{\beta}} k^\alpha_\Lambda k^{\bar{\beta}}_\Sigma$ of the scalar potential is not present for abelian gauging.

6.5.1 Moment map and Fayet-Iliopoulos terms in $\mathcal{N} = 2$ supergravity

The triplet of moment maps $\mathcal{P}^x_\Lambda$ appearing in the scalar potentials are defined as a solution of the following equation [148, 139, 149]:

$$\nabla_X \vec{\mathcal{P}}_\Lambda \equiv \partial_X \vec{\mathcal{P}}_\Lambda + 2\vec{\omega}_X \times \vec{\mathcal{P}}_\Lambda = \frac{1}{2} k^X_\Lambda \vec{J}_{XY}. \tag{6.5.5}$$

In contrast with $\mathcal{N} = 1$ supergravity, in $\mathcal{N} = 2$ due to the non-trivial SU(2) connection, the of triplet moment maps cannot be shifted by arbitrary constants. Actually the freedom to choose the magnitude of the Fayet-Iliopoulos terms is only present in the absence of hypermultiplets, $n_H = 0$. Indeed, thanks to the identity satisfied by any moment map $\mathcal{P}^x_\Lambda$ [150]:

$$\nabla^X \nabla_X \vec{\mathcal{P}}_\Lambda = 2n_H \vec{\mathcal{P}}_\Lambda, \tag{6.5.6}$$

they are uniquely defined by

$$4n_H \vec{\mathcal{P}}_\Lambda = -\vec{J}_Y{}^Z \nabla_Z k^Y_\Lambda. \tag{6.5.7}$$

The uniqueness of $\vec{\mathcal{P}}_\Lambda$ implies in particular that the following equation

$$\nabla_X \vec{\eta} = 0, \tag{6.5.8}$$

has no nontrivial solutions. Otherwise, $\vec{\mathcal{P}}_\Lambda + \vec{\eta}$ would be another solution of equation (6.5.5). Moreover, if there is a non-trivial solution, the integrability condition $[\nabla_X, \nabla_Y]\vec{\eta} = 0$ implies that the SU(2) curvature vanishes and therefore that the SU(2) curvature is trivial. This is clearly not the case for a quaternionic-Kähler manifold as we can see from (6.4.13).

The moment map can also be described in another way. A Killing vector preserves the connection $\vec{\omega}_X$ and Kähler two forms $\vec{J}$ only modulo an SU(2) rotation. Denoting by $\mathcal{L}_\Lambda$ the Lie derivative with respect to k_Λ, we have that the gauge transformations of the SU(2) connection and the Kähler two form are

$$\begin{aligned} \delta_\Lambda \vec{\omega}_X &= (\mathcal{L}_\Lambda \vec{\omega})_X = -\tfrac{1}{2}\nabla_X \vec{r}_\Lambda, \\ \delta_\Lambda \vec{J} &= (\mathcal{L}_\Lambda \vec{J})_X = \vec{r}_\Lambda \times \vec{J}, \end{aligned} \tag{6.5.9}$$

Here $\vec{r}_\Lambda$ is known as an SU(2) *compensator*. The SU(2) curvature of a quaternionic manifold is non-trivial and therefore it is impossible to get rid of the compensator $\vec{r}_\Lambda$ by a redefinition of the SU(2) connections. The moment map can be expressed in terms of the triplet of connections $\vec{\omega}$ and the compensator $\vec{r}_\Lambda$ in the following way [148]:

$$\vec{\mathcal{P}}_\Lambda = k^Y_\Lambda \vec{\omega}_Y + \tfrac{1}{2}\vec{r}_\Lambda, \tag{6.5.10}$$

which is analogous to the relation between the moment map and the compensator in $\mathcal{N} = 1$ supergravity (2.2.16).

6.6 SU(2) R-symmetry transformations

In section 6.4 we introduced the SU(2) R-symmetry, which rotates the triplet of complex structures of the quaternionic manifold. Actually, the $\mathcal{N}=2$ supergravity action is invariant under the R-symmetry. The SU(2) vector quantities transform under these rotations as follows (see [151]):

$$\delta_l \vec{J} = \vec{l} \times \vec{J}, \qquad \delta_l \vec{\mathcal{P}}_\Lambda = \vec{l} \times \vec{\mathcal{P}}_\Lambda, \tag{6.6.1}$$

where $\vec{l}(q)$ is an arbitrary triplet of functions on the hyperscalars. Notice that the the gaugini and the gravitini have an SU(2) index and therefore they are also affected by these reparametrizations. In particular the gravitini transform as

$$\begin{aligned} \delta_l \psi_{\mu i} &= -\tfrac{\mathrm{i}}{2} \vec{l} \,.\, \vec{\sigma}_i{}^j \psi_{\mu j}, & (6.6.2)\\ \delta_l \psi^i_\mu &= \tfrac{\mathrm{i}}{2} \psi^j_\mu \,\vec{l} \,.\, \vec{\sigma}_j{}^i. & (6.6.3) \end{aligned}$$

The SU(2) connection $\vec{\omega}_X$ is the gauge field associated to the R-symmetry, and transforms as

$$\delta_l \vec{\omega}_X = -\tfrac{1}{2} \nabla_X \vec{l} = -\tfrac{1}{2} \partial_X \vec{l} - \vec{\omega}_X \times \vec{l} \tag{6.6.4}$$

In general a change in the U(1) gauge symmetry associated to a killing vector k_Λ induces an SU(2) rotation. This can be seen comparing the gauge transformations of the Kähler forms and the SU(2) connection (6.5.9) with (6.6.1) and (6.6.4). Indeed, the compensator $\vec{r}_\Lambda$ can be identified with the triplet of functions $\vec{l}$ characterizing the SU(2) rotations induced by the change of U(1) gauge.

As a consequence, in order to have a non zero contribution from the compensator in the moment map (6.5.10) the gauge symmetry must induce an SU(2) rotation, in other words we have to gauge a U(1) subgroup of the R-symmetry. Moreover, whenever the compensator is non-vanishing the gravitini must transform under the gauge symmetry

$$\begin{aligned} \delta_\Lambda \psi_{\mu i} &= -\tfrac{\mathrm{i}}{2} \vec{r}_\Lambda \,.\, \vec{\sigma}_i{}^j \psi_{\mu j}, & (6.6.5)\\ \delta_\Lambda \psi^i_\mu &= \tfrac{\mathrm{i}}{2} \psi^j_\mu \,\vec{r}_\Lambda \,.\, \vec{\sigma}_j{}^i. & (6.6.6) \end{aligned}$$

In general the compensator is field dependent but it might contain a constant contribution which would act as a U(1) charge for the gravitini. Such a contribution would be the $\mathcal{N}=2$ supergravity analog of the FI-term.

Notice the strong similarities between this discussion and the one in section 5.2 for $\mathcal{N}=1$ supergravity, where showed that the $\mathcal{N}=1$ compensator also characterizes the gauge transformations of the gravitino.

6.7 $\mathcal{N}=2$ Supersymmetry transformations

The supersymmetry transformations involve the geometrical objects that we just discussed: the moment map, the Killing vectors and the metric of the scalar manifold. For a bosonic configuration, the $N=2$ supersymmetry transformations of the left-handed fermionic fields are:

$$\begin{aligned}
\delta\psi^i_\mu &= D_\mu\epsilon^i + \tfrac{1}{4}\gamma^{\rho\sigma}T^-_{\rho\sigma}\varepsilon^{ij}\gamma_\mu\epsilon_j + \gamma_\mu S^{ij}\epsilon_j,\\
\delta\lambda^\alpha_i &= \not{D}z^\alpha\epsilon_i - \tfrac{1}{2}\mathrm{e}^{\frac{\mathcal{K}}{2}}g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar Z^\Lambda\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}F^{-\Sigma}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j - N^\alpha_{ij}\epsilon^j,\\
\delta\zeta^A &= \tfrac{1}{2}\mathrm{i}f^{Ai}_X\not{D}q^X\epsilon_i - \mathcal{N}^{iA}\varepsilon_{ij}\epsilon^j.
\end{aligned} \tag{6.7.1}$$

Here the fermionic shifts $S^{ij}, \mathcal{N}^{iA}$ and N^α_{ij} are given by

$$S^{ij} \equiv -\mathrm{e}^{\mathcal{K}}\mathcal{P}^{ij}_\Lambda Z^\Lambda, \qquad \mathcal{N}^{iA} \equiv -\mathrm{i}\mathrm{e}^{\mathcal{K}}f^{iA}_X k^X_\Lambda \bar Z^\Lambda, \tag{6.7.2}$$

$$N^\alpha_{ij} \equiv \varepsilon_{ij}\mathrm{e}^{\mathcal{K}}k^\alpha_\Lambda\bar Z^\Lambda - 2\mathrm{e}^{\mathcal{K}}\mathcal{P}_{\Lambda|ij}\bar{\mathcal{D}}_{\bar\beta}\bar Z^\Lambda g^{\alpha\bar\beta}, \tag{6.7.3}$$

and the covariant derivatives are

$$\begin{aligned}
D_\mu\epsilon^i &\equiv \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{mn}\gamma_{mn}\right)\epsilon^i + \tfrac{1}{2}\mathrm{i}A^{\mathcal{SK}}_\mu\epsilon^i + V_{\mu j}{}^i\epsilon^j,\\
D_\mu z^\alpha &= \partial_\mu z^\alpha - A^\Lambda_\mu k^\alpha_\Lambda,\\
D_\mu q^X &= \partial_\mu q^X - A^\Lambda_\mu k^X_\Lambda.
\end{aligned} \tag{6.7.4}$$

The SU(2) connection $V_{\mu i}{}^j$ is related to the quaternionic-Kähler SU(2) connection and gets a contribution from the moment map when isometries of the quaternionic-Kähler manifold have been gauged:

$$V_{\mu i}{}^j = \partial_\mu q^X\omega_{Xi}{}^j - A^\Lambda_\mu\mathcal{P}_{\Lambda|i}{}^j. \tag{6.7.5}$$

$A^{\mathcal{SK}}_\mu$ is the U(1) connection of the Special Kähler manifold:

$$A^{\mathcal{SK}}_\mu = \tfrac{\mathrm{i}}{2}\left(\partial_{\bar\alpha}K\partial_\mu z^{\bar\alpha} - \partial_\alpha K\partial_\mu z^\alpha\right). \tag{6.7.6}$$

In the case of gauging in the vector multiplet sector, this is modified by a scalar moment map similar to the SU(2) connection. The dressed graviphoton is given by

$$T^-_{\mu\nu} = \mathrm{e}^{\frac{\mathcal{K}}{2}}F^{-\Lambda}_{\mu\nu}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}Z^\Sigma. \tag{6.7.7}$$

6.8 Consistent reduction of supersymmetry

For phenomenological reasons it is important to have mechanisms to reduce the number of supersymmetries of extended supergravity models down to $\mathcal{N}=1$. There are many ways to realize a reduction of the number of supersymmetries of a given theory. One can break supersymmetry explicitly by introducing supersymmetry breaking terms in the Lagrangian. Two other methods that appear naturally in the study of string inspired supergravity theories are *spontaneous supersymmetry breaking* and *consistent reduction of supersymmetry* [106, 107, 135].

The consistent reduction of supersymmetry is closely related to the consistent truncations of $\mathcal{N} = 1$ supergravity theories we discussed in chapter 3. Recall that these truncations of $\mathcal{N} = 1$ theories consist on a reduction of the number of fields of the model while preserving both the $\mathcal{N} = 1$ supersymmetry transformations and the equations of motion. That is, the solutions to the equations of motion derived from the reduced action $\mathcal{S}$ should also solve the equations derived from the original action $\widehat{\mathcal{S}}$

$$\frac{\delta \mathcal{S}}{\delta L^X} = 0 \quad \Longrightarrow \quad \frac{\delta \widehat{\mathcal{S}}}{\delta L^X} = 0, \tag{6.8.1}$$

where L^X represent the fields surviving the truncation. In the present section we review consistent reductions of $\mathcal{N} = 2$ supergravity models down to $\mathcal{N} = 1$, which also preserve the equations of motion, but where the reduced action is only invariant under $\mathcal{N} = 1$ supergravity.

The method of spontaneous breaking of supersymmetry is less restrictive than consistent reductions. Indeed, the solutions of the reduced theory are only required to solve approximately the equations of motion of the original action at energies much lower than the supersymmetry breaking scale, $E/M_{SSB} \ll 1$:

$$\frac{\delta \mathcal{S}}{\delta L^X} = 0 \quad \Longrightarrow \quad \frac{\delta \widehat{\mathcal{S}}}{\delta L^X} = 0 + \mathcal{O}(E/M_{SSB}), \tag{6.8.2}$$

In supergravity theories coming from string theory, spontaneous supersymmetry breakings arrive naturally from compactification with fluxes and/or torsion. In such a case, one ends up with gauged supergravity theories (or with a superpotential in $\mathcal{N} = 1$ supergravity) in which some of the gravitini become massive. Consistent truncations, on the other hand do not require any fermionic masses. Although they might appear quite artificial at first look from a purely supergravity point of view, consistent reductions are naturally realized in string theory, for example by models containing orbifolds and/or orientifolds. Moreover some spontaneous supersymmetry breaking are also consistent reduction in the sense that all solutions of the equations of motion of the reduced theory are also solutions of the mother theory.

6.8.1 Consistency conditions.

Consistent truncation of supersymmetry in the context of supergravity is not a trivial task. Actually, $\mathcal{N} = 2$ supergravity theories cannot be seen in general as a special case of a $\mathcal{N}=1$ supergravity. The conditions ensuring a consistent truncation in supergravity have been analyzed carefully in [106, 107, 135] following a procedure quite similar to the one presented in chapter 3, but here we will just summarize their results.

The most obvious incompatibility between $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity is the different number of supersymmetry generators and gravitini, $\mathcal{N}$. Without loss of generality, we shall consider consistent truncations in which the first supersymmetry generator, (ϵ_1, ϵ^1), is the one that respects the truncation. Then, in the reduced theory we must truncate the second gravitino

$$\epsilon_2 = \epsilon^2 = \psi_{\mu 2} = \psi_\mu^2 = 0, \tag{6.8.3}$$

since $(\psi_{\mu 2}, \psi_\mu^2)$ the is the gauge field associated the second generator (ϵ_2, ϵ^2), and then we can make the identifications

$$\epsilon_L = \epsilon^1, \qquad \psi_{\mu L} = \psi_\mu^1. \tag{6.8.4}$$

The $\mathcal{N}=2$ supersymmetry transformations of the gravitini involve the graviphoton $T_{\mu\nu}$, which is inconsistent with the $\mathcal{N}=1$ supersymmetry transformations, and therefore it has to be truncated

$$T_{\mu\nu} = 0. \tag{6.8.5}$$

The n_H hypermultiplets of $\mathcal{N}=2$ supergravity models cannot be seen as $2n_H$ chiral multiplets. Indeed, the kinetic terms of the scalar fields of n_H hypermultiplets define a quaternionic geometry and in general a quaternionic manifold is not even a complex manifold. Therefore the scalar manifold does not qualify to describe the kinetic terms of $\mathcal{N}=1$ chiral multiplets as it is supposed to be Kähler-Hodge. The consistency of the truncation requires that each hypermultiplet has to be fully truncated or reduce to a unique chiral multiplet, and the surviving fields should parametrize a Kähler-Hodge submanifold $\mathcal{M}_\mathcal{Q}^{KH}$ of the original quaternionic-Kähler manifold $\mathcal{M}_\mathcal{Q}$. The U(1) connection of $\mathcal{M}_\mathcal{Q}^{KH}$ is determined by ω_X^3, the only one of the components of the quaternionic SU(2)-connection that can survive the truncation:

$$\omega_X^1 = \omega_X^2 = 0. \tag{6.8.6}$$

Here the scalar fields of the $\mathcal{N}=2$ vector multiplets define a special manifold. A special manifold is a Kähler-Hodge manifold and therefore one would expect that n_V $\mathcal{N}=2$ vector multiplets can be seen as n_V gauge multiplets of $\mathcal{N}=1$ supergravity together with n_V chiral multiplets. However, this is not in general the case. Indeed, the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ appearing in the kinetic terms of $\mathcal{N}=2$ gauge fields is not restricted to be a holomorphic function whereas this is mandatory in $\mathcal{N}=1$ supergravity coupled to gauge fields. For instance, in a $\mathcal{N}=2 \to \mathcal{N}=1$ consistent truncation, a vector multiplet has to be completely truncated or reduce to a gauge or a chiral multiplet. After the truncation, in the reduced $\mathcal{N}=1$ supergravity we can identify the following quantities

$$D^a = 2\,\mathrm{Im}\,\mathcal{N}^{-1|ab}(\mathcal{P}_b^0 + \mathcal{P}_b^3), \qquad W = 2Z^b(P_b^1 - \mathrm{i}\mathcal{P}_b^2), \tag{6.8.7a}$$

$$\lambda_L^a = -\lambda_2^b\,\mathrm{e}^{\frac{\mathcal{K}}{2}}\mathcal{D}_b Z^a, \qquad f_{ab} = \mathrm{i}\mathcal{N}_{ab}, \tag{6.8.7b}$$

which, in particular, implies the surviving components of the matrix $\mathcal{N}$ should be holomorphic. To write these equations we have decomposed the $\mathcal{N}=2$ vector indices Λ as $\Lambda \to (a, \tilde{a})$, with $a = 1, \cdots, n_G$ running over the n_G gauge multiplets that survive the truncation to the $\mathcal{N}=1$ theory, and $\tilde{a} = 0, n_G+1, \cdots, n_V$ over the complementary indices, which label the chiral multiplets of the truncated theory coming from $\mathcal{N}=2$ vector multiplets.

The scalar manifold of the reduced $\mathcal{N}=1$ theory is a direct product $\mathcal{M}_{\mathcal{SK}}^{KH} \times \mathcal{M}_{\mathcal{Q}}^{KH}$ where $\mathcal{M}_{\mathcal{SK}}^{KH}$ is the reduced manifold coming from the scalar manifold M_V of vector multiplets:

$$\mathcal{M} = \mathcal{M}_{\mathcal{SK}} \times \mathcal{M}_{\mathcal{Q}} \Longrightarrow \mathcal{M}_{KH} = \mathcal{M}_{\mathcal{SK}}^{KH} \times \mathcal{M}_{\mathcal{Q}}^{KH}. \tag{6.8.8}$$

6.9 $\mathcal{N}=1$ FI terms from $\mathcal{N}=2$ supergravity.

In the previous chapters we showed that in $\mathcal{N}=1$ supergravity the FI-terms, which appear as a constant contribution to the moment map, are related to the charge of the gravitino under U(1) gauge transformations. Actually the gravitino U(1) charge is proportional to the value of the moment map at the fixed point of the killing vector. Moreover, given the gauge couplings of the scalar fields, i.e. the killing vector, the $\mathcal{N}=1$ moment map is only determined up to an arbitrary real constant and thus the gravitino U(1) charge is still a free parameter of the theory.

However, this is not the case for $\mathcal{N}=2$ supergravity theories with hypermultiplets. As we discussed in section 6.5.1 in that case the triplet of moment maps $\vec{\mathcal{P}}$ is completely determined, and thus we are not allowed to shift it with an arbitrary constant vector $\vec{\eta}$. This is due to the nontrivial SU(2) curvature of the quaternionic-Kähler manifold. Nevertheless it still makes sense to ask whether if the moment map $\vec{\mathcal{P}}$, contains a constant contribution which could act as an FI-term. One way to identify such a constant contribution is using a consistent reduction of supersymmetry from $\mathcal{N}=2$ down to a $\mathcal{N}=1$ theory, where we can identify the magnitude of the FI-term using the arguments presented in chapter 5. In particular in this section we will show how to construct $\mathcal{N}=2$ theories which can be truncated down to an $\mathcal{N}=1$ theory containing a non-vanishing FI-term.

Consider a killing vector k_1 associated to a compact isometry of a simply connected quaternionic manifold $\mathcal{M}_{\mathcal{Q}}$. Suppose that the killing vector induces a rotation of the complex structures such that the corresponding compensator is constant

$$\delta_{k_1}\vec{J}=\vec{r}_{k_1}\times\vec{J},\qquad r^x_{k_1}=\delta^x_3. \tag{6.9.1}$$

Since in simply connected manifolds any compact isometry always admits fixed points, it is possible to find a field configuration where $k_1=0$, and thus the moment map reads

$$\vec{\mathcal{P}}_{k_1}|_{k_1=0}=\left[k_1^X\vec{\omega}_X+\tfrac{1}{2}\vec{r}_{k_1}\right]_{k_1=0}=\tfrac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{6.9.2}$$

Then, we can consider a reduced theory defined by the condition $k_1=0$, which in components is given by

$$k_1^X(q^Y)=0,\qquad X=1,\dots,4n_H. \tag{6.9.3}$$

Since the truncation implies that the moment map $\vec{\mathcal{P}}_{k_1}$ is a constant, the reduced theory can not be invariant under $\mathcal{N}=2$ supersymmetry. Indeed, in order to obtain a non-trivial compensator we need to gauge the SU(2) R-symmetry group. Since the R-symmetry acts non-trivially in all the hypermultiplets, the condition $k_1=0$ leads to constraints that affect all of them. In particular, within each hypermultiplet at least half of the fields have to be truncated, and the reduced scalar manifold is a Kähler-Hodge submanifold of $\mathcal{M}_\mathcal{Q}$ [90]. Therefore, the reduced theory defined by $k_1=0$ cannot be described by $\mathcal{N}=2$ supergravity, but it is consistent with a reduction of supersymmetry $\mathcal{N}=2\to\mathcal{N}=1$.

If we gauge the isometry associated to the killing vector $k=\eta\,k_1$, according to equations (6.8.7b), the corresponding reduced $\mathcal{N}=1$ theory has a zero superpotential and a constant moment map, that is an FI-term, given by the third component of $\vec{\mathcal{P}}_{k_1}$

$$\vec{\mathcal{P}}_{k_1}|_{k_1=0}=\begin{pmatrix}0\\0\\\frac{\eta}{2}\end{pmatrix}\qquad\Longrightarrow\qquad D^{k_1}\propto\eta. \tag{6.9.4}$$

The mechanism that we have just discussed was first presented in [73] in a model where $\mathcal{N}=2$ supergravity is coupled to one vector multiplet and one chiral multiplet. After truncation it yields a $\mathcal{N}=1$ supergravity theory coupled to a gauge and a chiral multiplet and admitting a D-term potential endowed with a constant FI term. The FI term was used to construct the first example of a half-BPS cosmic string solutions of $\mathcal{N}=2$ supergravity. As we shall see in the examples below, it is possible to consider more general scenarios, where a theory with a field dependent compensator also leads to a constant FI-term in the reduced theory.

Using the property of consistent truncations, the construction of [73] can be described as the embedding of a $\mathcal{N}=1$ half-BPS D-term cosmic string solution in a $\mathcal{N}=2$ supergravity theory. Indeed, any solution of the reduced $\mathcal{N}=1$ theory is also a solution of the mother $\mathcal{N}=2$ theory. One can alternatively consider the use of consistent truncation in [73] as a trick to simplify the $\mathcal{N}=2$ BPS equations so that they look similar to those of a $\mathcal{N}=1$ supergravity model.

6.9.1 Example: the quaternionic space $\frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)}$.

We will now present two explicit realizations of the mechanism we just described. In order to study how FI-terms arise in the reduced $\mathcal{N}=1$ theory we only need to consider the the quaternionic manifold and its isometries. We will consider a model with a single hypermultiplet with the kinetic terms charaterized by the quaternionic manifold

$$\mathcal{M}_{\mathcal{Q}} = \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)} \cong \frac{\mathrm{Sp}(1,1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}. \qquad (6.9.5)$$

We define the scalars in the hypermultiplets $\{h, b_1, b_2, b_3\}$ so that the corresponding quaternionic vielbein and SU(2) connection are given by

$$f^{iA} = \tfrac{1}{\sqrt{2}}(\mathrm{d}h\mathbb{1}_2 + \mathrm{i}\mathrm{e}^{-h}\sigma^x \mathrm{d}b_x), \qquad \omega^x = -\tfrac{1}{2}\mathrm{e}^{-h}\mathrm{d}b_x. \qquad (6.9.6)$$

Therefore, the kinetic terms of the hyperscalars read:

$$\mathcal{T} = -\partial_\mu h\,\partial^\mu h - \mathrm{e}^{-2h}\,\partial^\mu \vec{b}\,\partial_\mu \vec{b}, \qquad \vec{b} = (b_1, b_2, b_3). \qquad (6.9.7)$$

We will study the effect of gauging the isometries of the quaternionic manifold characterized by the following killing vectors:

$$\begin{aligned} k_1 &= 2b_1\frac{\partial}{\partial b_2} - 2b_2\frac{\partial}{\partial b_1}, \\ k_2 &= 4b_3\frac{\partial}{\partial h} + 4b_1 b_3\frac{\partial}{\partial b_1} + 4b_2 b_3\frac{\partial}{\partial b_2} + 2[b_3^2 - b_1^2 - b_2^2 + 1 - \mathrm{e}^{2h}]\frac{\partial}{\partial b_3} \end{aligned} \qquad (6.9.8)$$

These two killing vectors induce a rotation of the complex structures, and therefore the corresponding compensator is non-trivial (6.5.9):

$$\vec{r}_{k_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{r}_{k_2} = 2\begin{pmatrix} -b_2 \\ b_1 \\ -\mathrm{e}^h \end{pmatrix}. \qquad (6.9.9)$$

The moment maps can be computed using the equations (6.5.10),

$$\vec{\mathcal{P}}_{k_1} = \begin{pmatrix} \mathrm{e}^{-h}b_2 \\ -\mathrm{e}^{-h}b_1 \\ 1 \end{pmatrix}, \qquad \vec{\mathcal{P}}_{k_2} = \begin{pmatrix} -2b_2 - 2b_1 b_3\mathrm{e}^{-h} \\ 2b_1 - 2b_2 b_3 \mathrm{e}^{-h} \\ -\mathrm{e}^{-h}[b_3^2 - b_1^2 - b_2^2 + 1] - \mathrm{e}^h \end{pmatrix}. \qquad (6.9.10)$$

Field independent compensator.

From (6.9.9), we can see that the gauging of the killing vector $\eta_1 k_1$ leads to the situation described in the previous paragraph. We can define a reduction of supersymmetry consistent with the condition $k_1 = 0$, which implies $b_1 = b_2 = 0$. The kinetic terms of the reduced theory read:

$$\mathcal{T} = -\partial_\mu h\,\partial^\mu h - \mathrm{e}^{-2h}\,\partial^\mu b_3\,\partial_\mu b_3, \tag{6.9.11}$$

and defining $S = \mathrm{e}^h + \mathrm{i}b_3$ they can be derived from the Kähler potential

$$K(S,\bar{S}) = -\log(S+\bar{S}). \tag{6.9.12}$$

Actually, the scalar manifold of the reduced $\mathcal{N}=1$ supergravity theory is $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$, which describes the axio-dilaton system that we studied in previous chapter. According to the identifications (6.8.7b) the reduced $\mathcal{N}=1$ theory has a vanishing superpotential, and the compensator $\vec{r}_{k_1}$ leads to a constant contribution in the moment map of the reduced theory, a FI-term:

$$\mathcal{P}|_{\mathcal{N}=1} = 2\mathcal{P}^3_{k_1}|_{k_1=0} = 2\eta_1. \tag{6.9.13}$$

Note that, as the killing vector k_1 vanishes identically after the truncation, there is no gauged symmetry in the reduced action. However, if we gauge a symmetry $k = \eta_1 k_1 + k_3$, where k_3 acts non-trivially on h and b_3, we can end in a $\mathcal{N}=1$ supergravity action with an abelian gauge symmetry and a constant FI-term.

Field dependent compensator.

Consider now the gauging of $\eta_2\,k_2$. In the reduced theory defined by $b_1 = b_2 = 0$, k_2 still acts non trivially on h and b_3:

$$k_2|_{\mathcal{N}=1} = 4\eta_2\,b_3\frac{\partial}{\partial h} + 2\eta_2\,[b_3^2 + 1 - \mathrm{e}^{2h}]\frac{\partial}{\partial b_3}, \tag{6.9.14}$$

which can written in terms of S as

$$k_2|_{\mathcal{N}=1} = \mathrm{i}2\eta_2(1-S^2)\frac{\partial}{\partial S}. \tag{6.9.15}$$

Thus the killing vector k_2 represents the same isometry we used in sections 5.4 and 5.4.2 to obtain constant FI-terms from field dependent moment maps (5.4.6) after the consistent truncation of the axio-dilaton. Actually, after the truncation $\mathcal{N}=2 \to \mathcal{N}=1$ the moment map of this killing vector reduces to

$$\mathcal{P}|_{\mathcal{N}=1} = 2\mathcal{P}^3_{k_2}|_{b_1=b_2=0} = -2\eta_2\,[\mathrm{e}^{-h}(b_3^2+1) + \mathrm{e}^h] = -4\eta_2\frac{1+|S|^2}{(S+\bar{S})}. \tag{6.9.16}$$

As we discussed in the previous chapter, when the field S is truncated at $S = 1$, the fixed point of $k_2|_{\mathcal{N}=1}$, this moment map also becomes an effective FI-term:

$$\mathcal{P}(S = 0)|_{\mathcal{N}=1} = -4\eta_2. \tag{6.9.17}$$

Note that, although the compensator associated to k_2 was not a constant in the original $\mathcal{N} = 2$ theory, it has led to a constant FI-term in the reduced theory. Thus, the mechanisms to generate FI-terms from consistent reductions of $\mathcal{N} = 2$ theories are more general that the one discussed in [73].