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CHAPTER 3

Consistent supersymmetric decoupling of heavy scalars in $\mathcal{N} = 1$ supergravity.

3.1 Introduction

As we mentioned in the first chapter, the problem of supersymmetric decoupling of heavy fields is specially relevant when considering moduli stabilization in flux compactifications of superstring/M-theory. In 2003 Kallosh, Kachru, Linde and Trivedi (KKLT) [21] provided the first example of a mechanism to stabilize all the moduli of a compactification. In their approach all moduli were stabilized at a supersymmetric critical point of the potential, which was generated by a combination of background fluxes and non-perturbative effects. A sector of these fields, the complex structure moduli, were stabilized at some high energy scale and integrated out in a supersymmetric way, leaving behind an effective theory described by $\mathcal{N} = 1$ supergravity.

In general the solutions of these effective theories are only approximate solutions to the full theory, and are valid for energies much lower than the mass scale of the fields which are integrated out $E \ll M_H$,

$$\frac{\delta S_{eff}}{\delta L^i} = 0 \quad \Longrightarrow \quad \frac{\delta S}{\delta L^i} = 0 + \mathcal{O}(E/M_H). \tag{3.1.1}$$

Similarly, supersymmetry is only preserved approximately in the low energy effective theory, and the supersymmetry breaking terms are typically assumed to be suppressed by the masses of the heavy fields. In order to know to what extent are these approximations valid it is important to understand precisely which type of couplings ensure that the solutions of the effective theory are *exact* solutions of the full theory, and that supersymmetry is *exactly* preserved. In particular this analysis is useful to characterize the interactions between the fields appearing in the low energy effective action and the heavy sector. Such interactions have been proven to be specially relevant in inflationary models in supergravity theories [103, 104, 105], where they might leave an observable imprint on the Cosmic Microwave Background.

The aim of the present chapter is to study the consistency conditions for the consistent supersymmetric truncation of part of the fields in a $\mathcal{N}=1$ supergravity theory. The truncation is defined fixing a fraction of the fields, denoted by H^α, at homogeneous and static configuration, i.e. a covariantly constant configuration:

$$\nabla_\mu H^\alpha = 0, \tag{3.1.2}$$

so that the Lorentz symmetry is preserved. Moreover, the condition of unbroken Lorentz symmetry also implies that the field strengths corresponding to truncated gauge bosons $A_\mu^{\tilde{a}}$ must vanish too:

$$F_{\mu\nu}^{\tilde{a}} = 0 \tag{3.1.3}$$

In section 3.1 we derive the necessary conditions for the corresponding reduced theory to preserve $\mathcal{N}=1$ supersymmetry which, as we shall see, also ensure that every solution of the reduced model is also a solution of the full equations of motion. Thus the conditions we find can be classified in two blocks: those which guarantee that the equations of motion respect the truncation of any theory (not necessarily supersymmetric), and those which are specific to supersymmetric theories.

We begin reviewing the first set of conditions. The vacuum expectation value of the truncated scalar fields might break some of the gauge symmetries, therefore the corresponding gauge bosons become massive and their field strengths must vanish. The time evolution of the fields has to be consistent with the truncation:

- The kinetic terms of the truncated scalar fields must be decoupled from the kinetic terms of the surviving ones in the reduced theory.

- The kinetic terms of the truncated gauge bosons should be decoupled from kinetic terms of the surviving ones, in particular the gauge kinetic functions should have a block diagonal structure $f_{\tilde{a}b} = 0$, where the indices with a tilde correspond to the gauge fields associated to broken symmetries and the other ones to the surviving gauge fields.

The truncated fields should not reappear due to gauge interactions:

- If the gauge group of the mother theory is non-abelian, then the truncated gauge bosons should not be sourced by the surviving ones by their interactions. In particular the full gauge group should have a cross product structure of the form $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where $\mathbb{G}_h$ is the subgroup associated to the truncated gauge fields, and $\mathbb{G}_l$ corresponds to the surviving subgroup.

- The gauge fields associated to broken symmetries, $\mathbb{G}_h$, should not couple to the scalar fields surviving the truncation.

- The truncated scalar fields should not couple to the gauge fields associated to unbroken symmetries, $\mathbb{G}_l$.

The conditions which are specific to consistent supersymmetric truncations from $\mathcal{N} = 1 \to \mathcal{N} = 1$ are quite obvious: each time we truncate a field we must also truncate the whole supermultiplet which contains it. In particular if we truncate a scalar field we must truncate the whole chiral multiplet, including the auxiliary fields, leading to the usual condition on the F-terms:

$$D_H W = 0, \tag{3.1.4}$$

In the case of the gauge fields which become massive we must truncate the whole vector multiplet, including auxiliary field, and thus the moment map associated to the broken gauge symmetry should be zero

$$\mathcal{P}_h = 0. \tag{3.1.5}$$

The condition (3.1.4) can be expressed as a constraint on the Kähler function. In section 3.3 and 3.4 we discuss the form of the Kähler functions satisfying this constraint, and we study explicit examples. In section 3.5 we consider the consistency of the truncation when the truncated and surviving sectors are coupled using the following ansatz:

$$\widehat{K} = K_1(H, \bar{H}) + K_2(L, \bar{L}) \qquad \widehat{W} = W_1(H) + W_2(L), \tag{3.1.6}$$

which in *global supersymmetry* describes non-interacting sectors. Here the fields surviving the truncation have been denoted by L.

The problem of truncating a sector of a theory while preserving a fraction of the supersymmetries has been studied thoroughly in the context of reductions of $\mathcal{N}$ extended supergravity theories, to a lower $\mathcal{N}'$ supergravity theories [106], [107]. The study we present here is done in the context of $\mathcal{N} = 1$ supergravity models requiring supersymmetry to be fully preserved $\mathcal{N} = \mathcal{N}'$, and has not been published to the best of our knowledge.

This analysis is intended to be used to study the consistency of truncating a heavy sector in cosmological models, thus we use the notation H to denote the truncated scalar fields, which represent the would-be heavy scalar fields, and L to denote the surviving scalar fields, which would correspond to the light scalars. However, as we will see in chapter 4, even when all the consistency conditions are met, there are unavoidable interactions between the two sectors which might turn the ansatz for the truncation unstable, and thus the names "heavy" and "light" are no longer appropriate for the truncated sector and surviving sectors respectively.

3.2 Consistency conditions for supersymmetric integration.

Consider a general $\mathcal{N} = 1$ supergravity theory as (2.2.3),

$$\mathcal{S} = \int dx_4 \sqrt{-g}(-\tfrac{1}{2}R + \mathcal{T} + \mathcal{L}_{gauge} - V),$$

where the fields ξ^I can be split into two sets, H^α and L^i, which represent the heavy and light degrees of freedom respectively, with $\alpha = 1, \ldots, n_h$ and $i = 1, \ldots, n_l$. The kinetic terms for the scalars $\mathcal{T}$ and the gauge bosons $\mathcal{L}_{gauge}$ were defined in (2.2.4) and (2.2.6) respectively, and the scalar potential V is given by the expressions (2.2.7), (2.2.8) and (2.2.14). We will assume that the masses of the heavy fields are large compared with the energy scale of the physics we are interested in, so that they can be stabilized at an extremum of the scalar potential with a vacuum expectation value[1] H_0^α. After truncating the heavy moduli we are left with an effective field theory describing the dynamics of the light fields only. From now on, we will use hatted quantities in the full theory and unhatted quantities for the effective theory:

$$S(L, \overline{L}) = \widehat{S}(H_0, \overline{H}_0, L, \overline{L}). \tag{3.2.1}$$

For the truncation to be consistent, the equations of motion of the light fields derived from the effective theory must coincide with the equations of motion obtained from the full action. To zeroth order in the fluctuations of the heavy

[1]Note that the vacuum expectation value of the heavy scalars H_0^α is only determined up to gauge transformations.

fields the effective action S should satisfy:

$$\left.\frac{\delta\widehat{S}}{\delta L}\right|_{H_0} = \frac{\delta\widehat{S}|_{H_0}}{\delta L} = \frac{\delta S}{\delta L}, \quad \text{where} \quad \left.\frac{\delta\widehat{S}}{\delta H}\right|_{H_0} = 0, \tag{3.2.2}$$

ensuring that the fluctuations of H are not sourced by the light fields. Actually the truncation $H = H_0^\alpha$ can be seen as an ansatz to solve the full equations of motion. As we discussed in the introduction, we are interested in the case where the low energy effective theory preserves the invariance under $\mathcal{N} = 1$ local supersymmetry. Therefore we must also require the supersymmetry transformations to be compatible with the truncation of the heavy fields, which leads to conditions on the couplings and matter content of the effective theory. Since these conditions are more easily obtained than those derived from the equations of motion (3.2.2), we will start with the analysis of the supersymmetry transformations, and then we will find out the extra constraints imposed by (3.2.2).

3.2.1 Reduction of the chiral multiplets.

The supersymmetry transformations must respect the ansatz of the truncation $H = H_0^\alpha$, and therefore from (2.2.21) we find that:

$$\delta_\epsilon H^\alpha = \bar{\epsilon}_L \chi_L^\alpha = 0, \tag{3.2.3}$$

which implies that the fermionic partners of the heavy fields, χ^α, must also be truncated:

$$\chi^\alpha = 0 \quad \Longrightarrow \quad \delta_\epsilon \chi_L^\alpha = \tfrac{1}{2}\gamma^\mu \nabla_\mu H^\alpha \epsilon_R - \tfrac{1}{2}\mathrm{e}^{\frac{1}{2}\hat{K}} \widehat{G}^{\alpha\bar{J}} \mathcal{D}_{\bar{J}} \overline{\widehat{W}}|_{H_0} \epsilon_L = 0. \tag{3.2.4}$$

In order for the supersymmetry transformations to be consistent with the truncation of the heavy chiral fermions we have to set to zero the following quantities in the reduced theory :

$$\begin{aligned} \widehat{G}^{\bar{\alpha}J} \mathcal{D}_J \widehat{W}|_{H_0} &= [\widehat{G}^{\bar{\alpha}\beta} \mathcal{D}_\beta \widehat{W} + \widehat{G}^{\bar{\alpha}j} \mathcal{D}_j \widehat{W}]_{H_0} = 0\,, && (3.2.5) \\ \nabla_\mu H^\alpha &= \partial_\mu H^\alpha - k_a^\alpha (H_0, L) A_\mu^a = 0, && (3.2.6) \end{aligned}$$

which have to be satisfied for any value of the light fields L^i. Since we want to allow the possibility that superymmetry is broken by the light fields, in principle the quantity $\mathcal{D}_i \widehat{W}$, which is related to the auxiliary field of the light chiral multiplets, could have any arbitrary value. Therefore, in order to solve (3.2.5) we must require the Kähler metric to be block diagonal in the two sectors at $H^\alpha = H_0^\alpha$:

$$\widehat{G}_{\alpha\bar{\imath}}|_{H_0} = \widehat{G}_{\bar{\alpha}i}|_{H_0} = 0. \tag{3.2.7}$$

This condition ensures that the kinetic terms of the light and heavy scalar fields are decoupled. This is required for the equations of motion to respect the truncation, since otherwise the propagators of the scalars would mix the light and heavy fields. Using (3.2.7) we can see that the condition (3.2.5) reduces to:

$$\mathcal{D}_\alpha \widehat{W}|_{H_0} = (\partial_\alpha \widehat{W} + \partial_\alpha \widehat{K}\,\widehat{W})_{H_0} = 0 \tag{3.2.8}$$

When the superpotential is not vanishing, $W(H_0, L) \neq 0$, we can write this constraint in terms of the Kähler function $\widehat{G}(H, \bar{H}, L, \bar{L})$:

$$\widehat{G}_\alpha(H_0, \bar{H}_0, L, \bar{L}) = 0 \quad \text{for all} \;\; L^i. \tag{3.2.9}$$

This also implies that all the higher order derivatives of $G_\alpha|_{H_0}$ with respect to the light fields are zero, such as the components of the Kähler metric mixing light and heavy fields $G_{\alpha\bar{\imath}}$, which is precisely the constraint (3.2.7). The condition (3.2.7), when combined with (3.2.6), leads to a simplification of the kinetic terms of the scalars $\mathcal{T}$ (2.2.4)

$$\begin{aligned} \mathcal{T}|_{H_0} \;\; = \;\; & \Big[\widehat{G}_{\alpha\bar{\beta}} \nabla_\mu H^\alpha \nabla^\mu H^{\bar{\beta}} + \widehat{G}_{i\bar{\jmath}} \nabla_\mu L^i \nabla^\mu L^{\bar{\jmath}} + \widehat{G}_{\alpha\bar{\jmath}} \nabla_\mu H^\alpha \nabla^\mu L^{\bar{\jmath}} + \\ & \widehat{G}_{i\bar{\beta}} \nabla_\mu H^i \nabla^\mu L^{\bar{\beta}}\Big]_{H_0} = \widehat{G}_{i\bar{\jmath}}|_{H_0} \nabla_\mu L^i \nabla^\mu L^{\bar{\jmath}}. \end{aligned} \tag{3.2.10}$$

According to the expression we obtained for the kinetic terms of the reduced theory (3.2.10), the truncation $H^\alpha = H_0^\alpha$ must define a Kähler submanifold $\mathcal{M} \subseteq \widehat{\mathcal{M}}$ of the original scalar manifold. The metric of $\mathcal{M}$ is given by the second derivatives of the Kähler potential of the reduced theory $K(L, \bar{L})$:

$$G_{i\bar{\jmath}} = \partial^2_{i\bar{\jmath}} K(L, \bar{L}) \qquad \text{with} \qquad K(L, \bar{L}) \equiv \widehat{K}(H_0, \bar{H}_0, L, \bar{L}), \tag{3.2.11}$$

where the Kahler potential is determined only up to Kähler transformations (2.2.28). On the other hand, looking at the supersymmetry transformation of the gravitino (2.2.22),

$$\delta\psi_{\mu L} = (\partial_\mu + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \mathrm{i}\hat{A}_\mu^B)_{H_0}\epsilon_L + \tfrac{1}{2}\left[\mathrm{e}^{\hat{K}/2}\widehat{W}\right]_{H_0}\gamma_\mu \epsilon_R \tag{3.2.12}$$

we can identify the superpotential of the reduced theory, which is given by

$$W(L) = \widehat{W}(H_0, L). \tag{3.2.13}$$

Therefore we can express the reduced Kähler function $G(L, \bar{L})$ in terms of the Kähler function of the full theory as follows:

$$G(L, \bar{L}) = K(L, \bar{L}) + \log|W(L, \bar{L})|^2 = \widehat{G}(H_0, \bar{H}_0, L, \bar{L}). \tag{3.2.14}$$

Before concluding with the chiral multiplets we have to check that the condition (3.2.9), and thus also (3.2.7), is respected by the supersymmetry transformations. If this were not the case we would have to impose additional constraints to ensure consistency. Note that the equation (3.2.9) only depends on bosonic quantities, and then it could only vary under supersymmetry transformations due to the shifts of the scalar fields $\delta_\epsilon H^\alpha$ and $\delta_\epsilon L^i$

$$\delta_\epsilon G_\alpha|_{H_0} = [\nabla_\beta\, G_\alpha \delta_\epsilon H^\beta + \nabla_{\bar\beta}\, G_\alpha \delta_\epsilon H^{\bar\beta} + \nabla_i\, G_\alpha \delta_\epsilon L^i + \nabla_{\bar\imath}\, G_\alpha \delta_\epsilon L^{\bar\imath}]_{H_0}. \quad (3.2.15)$$

Here ∇_I denotes the covariant derivative involving the Levi-Civita connection of the Kähler manifold. As we have discussed at the beginning of the section, the heavy fields do not shift $\delta_\epsilon H^\alpha = 0$ in a supersymmetric background, and thus they cannot induce a change in (3.2.9). On the other hand, due to the condition (3.2.9), which is valid for any value of the light fields L^i, the covariant derivatives appearing in the last two terms of (3.2.15) vanish in the configuration $H^\alpha = H_0^\alpha$

$$\nabla_i\, G_\alpha|_{H_0} = [G_{i\alpha} - \Gamma^\beta_{i\alpha} G_\beta - \Gamma^j_{i\alpha} G_j]_{H_0} = 0 \quad (3.2.16)$$

$$\nabla_{\bar\imath}\, G_\alpha|_{H_0} = G_{\bar\imath\alpha}|_{H_0} = 0. \quad (3.2.17)$$

In particular, it is possible to prove that the components of the Kähler connection $\Gamma^j_{i\alpha}$ are zero using their relation to the metric $G_{I\bar J}$

$$\Gamma^j_{i\alpha}|_{H_0} = [G^{j\bar k} G_{\bar k i\alpha} + G^{j\bar\beta} G_{\bar\beta i\alpha}]_{H_0} = 0. \quad (3.2.18)$$

This condition ensures that the submanifold $\mathcal{M}$ that defines the reduced theory is totally geodesic (see [108]), which is required for the equations of motion to be consistent with the truncation. Gathering these results we can conclude that supersymmetry preserves the condition (3.2.9), i.e. its transformation under supersymmetry (3.2.15) is zero, and thus there is no need to impose additional constraints.

3.2.2 Reduction of the vector multiplets.

We start the discussion by studying the variation of (3.2.6) under supersymmetry transformations. Using the same argument as in the previous paragraph it is possible to show that the shifts of the chiral fields, $\delta_\epsilon L^i$ and $\delta_\epsilon H^\alpha$, do not induce variations on the condition[2] (3.2.6). Therefore, for (3.2.6) to be respected by supersymmetry transformations, we only have to impose the constraints:

$$\delta_\epsilon \left(\partial_\mu H^\alpha - k^\alpha_a(H, L)\, A^a_\mu \right)_{H_0} = -k^\alpha_a(H_0, L)\, \delta_\epsilon A^a_\mu = 0, \quad (3.2.19)$$

which should hold for every $\alpha = 1, \ldots, n_h$ and for any configuration of the light fields L^i. Squaring the previous equation we can express the constraint as

$$M_{ab}\, \delta_\epsilon A^a_\mu\, \delta_\epsilon A^{b\mu} = 0, \qquad M_{ab} \equiv (G_{\alpha\bar\beta}\, k^\alpha_a k^{\bar\beta}_b)_{H_0}, \quad (3.2.20)$$

[2]In this reasoning we must also use the fact that the supersymmetry transformations and spatial translations commute $[P_\mu, \delta_\epsilon] = 0$.

where the symmetric matrix M_{ab} is proportional to the mass matrix of the gauge bosons. The matrix M_{ab} can always be diagonalized redefining the gauge fields conveniently (2.4.23)

$$A^b = O^b_a \tilde{A}^a \qquad \tilde{k}^\alpha_b = O^a_b k^\alpha_a, \qquad \tilde{M}_{cd} = M_{ab} O^a_c O^b_d \tag{3.2.21}$$

where ${O^a}_b$ is a non-singular real matrix. If the mass matrix M_{ab} is diagonal then, choosing the Coulomb gauge $A^a_0 = 0$, the condition (3.2.20) reads

$$\sum_a M_{aa} \, (\delta_\epsilon \vec{A}^a)^2 = \sum_a (G_{\alpha\bar{\beta}} \, k^\alpha_a k^{\bar{\beta}}_a)_{H_0} \, (\delta_\epsilon \vec{A}^a)^2 = 0. \tag{3.2.22}$$

Since the expressions in parentheses are the square of the killing vectors, all the terms in the sum are positive semidefinite. Thus, for the sum to be zero all the terms have to vanish

$$(G_{\alpha\bar{\beta}} \, k^\alpha_a k^{\bar{\beta}}_a)_{H_0} \, (\delta_\epsilon \vec{A}^a)^2 = 0 \quad \text{for all } a = 1, \ldots, n_V. \tag{3.2.23}$$

Each of these equations admit two different solutions. If the gauge symmetry associated to A^a_μ remains unbroken after the stabilization of the heavy fields, i.e. $k^\alpha_a(H_0, L) = 0$ for all L^i, then we do not have to impose any more constraints. Moreover, the equation (3.2.6) does not impose any condition on the gauge field A^a_μ and we can keep the full vector multiplet in the low energy theory. Note that the massless gauge bosons are decoupled from the heavy fields in the reduced theory $k^\alpha_a(H_0, L) = 0$, as required by consistency, since otherwise the heavy fields could be sourced due to gauge interactions.

If, on the contrary, the gauge symmetry is spontaneously broken by the vacuum expectation values of the heavy fields, i.e. $k^\alpha_{\tilde{a}}(H_0, L) \neq 0$, then the supersymmetry transformations $\delta_\epsilon A^{\tilde{a}}_\mu$ must vanish:

$$\delta_\epsilon A^{\tilde{a}}_\mu = -\tfrac{1}{2} \bar{\epsilon} \gamma_\mu \lambda^{\tilde{a}} = 0. \tag{3.2.24}$$

In the following we will indicate with a tilde, i.e. $\lambda^{\tilde{a}}$, those indices that correspond to the heavy gauge bosons, while the indices without a tilde are associated to the massless bosons. This constraint implies that the fermionic partner of the massive gauge boson A^a_μ has to be truncated:

$$\lambda^{\tilde{a}} = 0 \quad \Longrightarrow \quad \delta_\epsilon \lambda^{\tilde{a}} = \tfrac{1}{4} \gamma^{\mu\nu} {F^{\tilde{a}}}_{\mu\nu} \epsilon + \tfrac{1}{2} i D^{\tilde{a}} \gamma_5 \epsilon = 0. \tag{3.2.25}$$

Therefore, in order for the supersymmetry transformations to respect the truncation of the gaugino, we have to require the following quantities to vanish in the low energy theory:

$$\begin{aligned} {F^{\tilde{a}}}_{\mu\nu} &= \partial_{[\mu} A^{\tilde{a}}_{\nu]} + f^{\tilde{a}}_{\tilde{b}\tilde{c}} A^{\tilde{b}}_\mu A^{\tilde{c}}_\nu + f^{\tilde{a}}_{bc} A^b_\mu A^c_\nu + f^{\tilde{a}}_{b\tilde{c}} A^b_{[\mu} A^{\tilde{c}}_{\nu]} = 0, && (3.2.26) \\ D^{\tilde{a}} &= \left[(\operatorname{Re} f)^{-1|\tilde{a}\tilde{b}} \mathcal{P}_{\tilde{b}} + (\operatorname{Re} f)^{-1|\tilde{a}b} \mathcal{P}_b \right]_{H_0} = 0. && (3.2.27) \end{aligned}$$

If these conditions are to be satisfied for any configuration of the massless gauge bosons A^a_μ, the following components of the structure constants have to vanish

$$f^{\tilde{a}}_{bc} = f^{\tilde{a}}_{b\tilde{c}} = 0. \tag{3.2.28}$$

If the original gauge symmetry group $\mathbb{G}$ is semi-simple these conditions imply that it must have a cross product structure $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where subgroups $\mathbb{G}_h$ and $\mathbb{G}_l$ act on the heavy and light fields respectively at the point H^α_0. Then the equation (3.2.26) reduces to

$$F^{\tilde{a}}{}_{\mu\nu} = \partial_{[\mu} A^{\tilde{a}}_{\nu]} + f^{\tilde{a}}_{\tilde{b}\tilde{c}} A^{\tilde{b}}_\mu A^{\tilde{c}}_\nu = 0, \tag{3.2.29}$$

which implies that the heavy gauge bosons have to be truncated also. The killing vectors of the light sector k^i_a, and thus also $\mathcal{P}_a$, could have any arbitrary value. Therefore, in order to satisfy the condition (3.2.27) for any value of the light fields we have to set to zero the real part of the components $f_{ab}(H, L)$ mixing the massive and massless gauge fields:

$$\mathrm{Re}\, f_{a\tilde{b}}(H_0, L) = 0 \quad \text{for any } L^i. \tag{3.2.30}$$

This condition is necessary to ensure that the kinetic terms of the massive and massless gauge bosons are decoupled, so that the dynamical evolution also respects the truncation[3]. Using the last equation we can see that the condition (3.2.27) implies that the moment maps associated to the killing vectors of the broken symmetries have to vanish in the reduced theory

$$\mathcal{P}_{\tilde{a}}(H_0, L) = 0. \tag{3.2.31}$$

Since the moment map determines the killing vectors (2.2.15), in the reduced theory the killing vectors of the broken symmetries cannot have non-vanishing components along the light directions:

$$k^i_{\tilde{a}}(H_0, L) = -\mathrm{i} G^{i\bar{\jmath}} \partial_{\bar{\jmath}} \mathcal{P}_{\tilde{a}}|_{H_0} = 0. \tag{3.2.32}$$

The last constraints are consistent with the intuition that the heavy gauge bosons should decouple from the light fields, otherwise the light fields could source the massive gauge bosons that were integrated out. Finally, using the conditions we just found for the killing vectors and the complex structure constants, it is possible to show that in order to satisfy the lie algebra (2.2.11) the killing vectors the have to satisfy:

$$k^\alpha_{\tilde{a},i}(H_0, L) = k^i_{a,\alpha}(H_0, L) = 0 \quad \text{for all } L^i \tag{3.2.33}$$

It is straightforward to check that no more constraints are necessary to ensure that supersymmetry respects the conditions derived in this section.

[3] Since the gauge kinetic functions are holomorphic and $\mathrm{Re}\, f_{a\tilde{b}}(H_0, L) = 0$, then the imaginary part of these components at the configuration $H^\alpha = H^\alpha_0$ has to be independent of the values of the light fields, in other words $\mathrm{Im}\, f_{\tilde{a}b}(H_0, L) = \mathrm{Im}\, f_{\tilde{a}b}(H_0)$.

3.2.3 Two fermion terms in the supersymmetry variations.

For simplicity, the supersymmetry transformations presented in the last chapter (2.2.21-2.2.24) only contained terms linear in the fermions, but the full supersymmetry transformations include higher order terms in the fermions. In particular, a set of supersymmetry transformations of the bosons which also contain-two fermion terms can be found in references [89]. These terms are of the form:

$$\delta_\epsilon b \sim f f \epsilon, \tag{3.2.34}$$

where b and f denote generic boson and fermion fields. By not considering these higher order terms we are overlooking important consistency conditions for the supersymmetric truncation of the heavy fields. Indeed, we will now prove that the conditions we have derived so far are insufficient to ensure that the heavy field configuration H_0^α is an extremum of the action (2.2.3). This implies that the dynamics of the system are not consistent with the truncation, or in other words, equation (3.2.2) is not satisfied.

Since the action must be extremized with respect to variations of the heavy fields for any configuration of the light fields, it is sufficient to consider each of the terms of the action independently. The gravitational term does not need to be considered because is independent of the heavy sector. It is easy to check that, due to the constraints (3.2.6), (3.2.9) and (3.2.33), the kinetic terms of the scalars, $\mathcal{T}$, are extremized when the heavy fields are integrated out in a supersymmetric way at $H^\alpha = H_0^\alpha$. On the other hand, neither the kinetic terms of the gauge fields $\mathcal{L}_{gauge}$, nor the scalar potential V, are extremized by the heavy field configuration. For instance, after using the condition $k_a^\alpha(H_0, L) = 0$ and the equations (3.2.28-3.2.31), we find the variation of the action along H^α is of the form:

$$\delta_\alpha \mathcal{S} \sim \int \tfrac{1}{4}\,\mathrm{Re}\, f_{ab,\alpha}(H_0, L)\; F^a_{\mu\nu} F^{b\mu\nu} + \tfrac{1}{2}[(\mathrm{Re}\, f^{-1ab})_{,\alpha}\, \mathcal{P}_a \mathcal{P}_b]_{H_0}, \tag{3.2.35}$$

where, as in the previous section the indices a and b run only over the massless gauge bosons. The first term arises from the variation of the kinetic terms of the gauge fields and the second one from the D-term potential V_D. Since, a priori, the field strengths and moment maps associated to the light gauge bosons are arbitrary, a consistent truncation requires that the gauge kinetic function satisfies:

$$\mathrm{Re}\, f_{ab,\alpha}(H_0, L) = (\mathrm{Re}\, f(H_0, L)^{-1ab})_{,\alpha} = 0 \qquad \text{for all } L^i. \tag{3.2.36}$$

This condition is also necessary in order for the truncation to preserve $\mathcal{N}=1$ supergravity. The reason we did not find it in the previous two sections is that it is only appears when the two-fermion terms of the supersymmetry transformations are taken into account. For instance, after using the conditions found in the previous two sections, the variations of the heavy chiral fermions χ^α still contain a non-vanishing two fermion term of the form [89]

$$\delta_\epsilon \chi_L^\alpha \sim \epsilon_L \bar{\lambda}_R^a \lambda_R^b \, [G^{\alpha\bar{\beta}} \, \bar{f}_{ab,\bar{\beta}}]_{H_0}. \qquad (3.2.37)$$

For the truncation to preserve supersymmetry, these variations have to be zero for any configuration of the light gaugini λ^a, and therefore we have to require (3.2.36) to be satisfied.

With some algebra it is possible to prove that, the conditions we found in the last sections together with (3.2.36), are sufficient to guarantee that the truncation is consistent with the dynamics of the system, i.e. that the solutions of the reduced theory are also solutions of the full equations of motion (3.2.2). In particular, in order to freeze the heavy fields consistently the mass matrix of the fluctuations around any configuration of the form (H_0^α, L^i) should be block diagonal in the heavy and light sectors:

$$\partial^2_{i\alpha} V(H_0, L) = 0, \qquad \partial^2_{i\bar{\alpha}} V(H_0, L) = 0 \quad \text{for all } L^i. \qquad (3.2.38)$$

Indeed, to integrate out the fluctuations with large masses around a given vacuum first we have to find their mass spectrum, which requires diagonalizing the mass matrix, and only after having identified the heavy modes can we set them consistently to zero. Proceeding in this way, by construction, the mass matrix at (H_0^α, L^i) is always block diagonal in the massive and light modes. If the heavy fields are truncated in a supersymmetric way, it is not hard to check that the scalar potential satisfies these requirements.

3.3 Consistency of the effective action

In the literature there are several studies about how to obtain the low energy effective action left after the supersymmetric integration of the heavy fields such as [109, 110, 111, 91]. We will now describe the general ideas involved in these approaches.

Suppose we are given a particular theory, defined by a Kähler potential $\widehat{K}$ and a superpotential $\widehat{W}$, which contains a heavy scalar sector H^α stabilized at some large energy scale. In order to integrate out the heavy fields while preserving supersymmetry the heavy field configuration has to satisfy equation (3.2.8)

$$\Big[\partial_\alpha \widehat{W}(H, L) + \partial_\alpha \widehat{K}(H, \overline{H}, L, \overline{L})\, \widehat{W}(H, L)\Big]_{H_0} \equiv \Phi(H, \bar{H}, L, \bar{L}) = 0. \qquad (3.3.1)$$

The left hand side is some function of both the heavy and the light fields, let us call it $\Phi(H, \overline{H}, L, \overline{L})$. In general, the condition $\Phi = 0$ relate the heavy and light fields. Therefore, in general, if we solve for H^α we obtain an expression of H_0^α as a function of the light fields,

$$H^\alpha = H_0^\alpha(L, \overline{L}). \tag{3.3.2}$$

The most naive approach would be to substitute this expression back into $\widehat{K}, \widehat{W}$, and identify the resulting functions as the Kähler potential and superpotential of the effective theory

$$K \equiv K\big(H(L,\bar{L}), \bar{H}(L,\bar{L}), L, \bar{L}\big), \qquad W = W\big(H(L,\bar{L}), L\big). \tag{3.3.3}$$

These two quantities would then define an effective action for the light fields

$$S(L, \overline{L}) = \widehat{S}(H_0(L,\overline{L}), \overline{H}_0(L,\overline{L}), L, \overline{L}). \tag{3.3.4}$$

An immediate concern with the consistency of this procedure, pointed out in [109], is that in general this leads to a non-holomorphic expression for the would-be effective superpotential W. A possible solution to this problem was suggested in [109, 76]: there is no conflict with the holomorphicity of the effective superpotential if $\widehat{W}$ is independent of H. The case $\widehat{W}=0$ is obvious, so let us consider $\widehat{W} \neq 0$. Then, it is always possible to perform a Kähler transformation that makes $\widehat{W}$ constant

$$\widehat{K} \to \widehat{K} + \log \widehat{W} + \log \overline{\widehat{W}} \equiv \widehat{G} \qquad\qquad \widehat{W} \to 1 \ . \tag{3.3.5}$$

In this so called Kähler gauge, eq. (3.2.8) becomes equivalent to (3.2.9)

$$\widehat{G}_\alpha(H, \overline{H}, L, \overline{L}) = 0, \tag{3.3.6}$$

from which we can extract $H^\alpha = H_0^\alpha(L,\overline{L})$. Then we make the previous substitution directly into the Kähler invariant function without having to deal with non-holomorphic superpotentials:

$$G = \widehat{G}(H_0(L,\overline{L}), \overline{H}_0(L,\overline{L}), L, \overline{L}). \tag{3.3.7}$$

Although this approach is apparently satisfactory, it is not entirely consistent yet. The origin of these difficulties can be traced back to the expression (3.3.2). The whole approach can only be consistent with our original assumptions provided the heavy fields H^α have no dependence on the light fields. Indeed, in section 3.2.1 the condition (3.2.8) was derived supposing that the heavy fields H^α were stabilized at the *constant* configuration H_0^α. If the constraint (3.2.9) requires the heavy fields to depend on the light fields, then the approach is not self-consistent. This means that either the heavy fields have not been identified correctly, or the Kähler function $\widehat{G}$ is not suitable to truncate the heavy fields in a supersymmetric way.

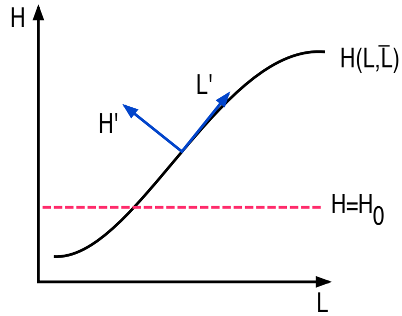

Figure 3.1 *– The solid line represents schematically a general solution of (3.3.1), i.e. (3.3.2), which defines the scalar manifold of the reduced theory. Only solutions where H^α is independent of L^i (dashed line) are consistent with a supersymmetric integration.*

To clarify the situation, let us consider the meaning of equation (3.3.2). This equation defines the scalar manifold of the reduced theory, $\mathcal{M} \subseteq \widehat{\mathcal{M}}$, which characterizes the kinetic terms in the truncated action. The submanifold $\mathcal{M}$ can be parametrized in the following way

$$\xi^I = \big(L^i, H^\alpha(L, \bar{L})\big), \tag{3.3.8}$$

where the fields ξ^I are a parametrization of the full target space $\widehat{\mathcal{M}}$. In section 3.2.1 we proved that $\mathcal{M}$ must be a Kähler submanifold of the full scalar manifold. However, the previous expression only defines a complex submanifold of $\widehat{\mathcal{M}}$ provided that, at least locally, the fields H^α can be expressed as holomorphic functions of L^i [112], so that we have $H^\alpha = H^\alpha(L)$.

This field dependence implies that the *true* light directions on field space are no longer characterized by L^i, but instead they are given by a combination of L^i and H^α. Similarly, the heavy fields can no longer be identified with H^α, since these fields are still allowed to fluctuate as long as they satisfy (3.3.2), and thus they are not the ones to be truncated. This situation is represented schematically in figure 3.1. The solid line of the figure represents the scalar manifold of the reduced theory, which is defined by the solution (3.3.2). The "light directions" in field space are those along the solid curve, such as L'^i. At a particular point of the curve, the "heavy directions" are those orthogonal to the curve with respect to the Kähler metric (3.2.7), such as H'^α. In general, unless the expression for H^α in (3.3.2) is independent from L^i, the fields to be integrated out H'^α do not coincide with H^α, and therefore the condition (3.3.1) is not the one that should be solved.

3.4 Constraints in the sgoldstino direction

In this section we would like to discuss the problem of finding a consistent supersymmetric truncation from a more geometrical perspective. In particular, we would like to study the following problem: given a particular Kähler manifold, characterized by a Kähler potential $K(\xi, \bar{\xi})$, we want to determine the class of superpotentials $W(\xi)$ that allow the supersymmetric integration of *some* sector of the theory.

In order to have a valid $\mathcal{N} = 1$ supergravity low energy theory, the corresponding reduced scalar manifold $\mathcal{M} \subseteq \widehat{\mathcal{M}}$ has to be a Kähler manifold, and moreover, it must be totally geodesic in order to preserve supersymmetry and be consistent with the equations of motion. Actually a simple consistency test to see if a theory admits a consistent supersymmetric reduction, is to check whether the vacuum manifold of the scalar potential, or any of its submanifolds, is a totally geodesic Kähler submanifold of $\widehat{\mathcal{M}}$.

On the other hand, $\mathcal{M}$ is a Kähler submanifold iff it is given locally by the set of zeros of a collection of n_H holomorphic functions $f^1, \ldots, f^{n_H}$ defined in $\widehat{\mathcal{M}}$ [112]:

$$f^\alpha(\xi^I) = 0, \quad \text{for all} \quad \alpha = 1, \ldots, n_H. \tag{3.4.1}$$

Equivalently it can be defined locally as the image of a a holomorphic map $f : U \subseteq \mathbb{C}^{n-n_H} \to \widehat{\mathcal{M}}$, which is precisely the type of description we used in the previous section:

$$\xi^I = \big(\xi^i, \xi^\alpha(\xi^i)\big), \tag{3.4.2}$$

where ξ^i parametrize the open set $U \subseteq \mathbb{C}^{n-n_H}$, and ξ^I are a set of coordinates in $\widehat{\mathcal{M}}$. According to the expression (3.4.1), locally it is possible to define the heavy fields by the change of coordinates

$$H^\alpha = f^\alpha(\xi^I), \tag{3.4.3}$$

since the truncation of these fields is precisely what defines the reduced manifold.

At each point of the scalar manifold $\widehat{\mathcal{M}}$ the derivatives of the Kähler function G_I define a direction in field space, which is known as the *sgoldstino*. The condition for unbroken supersymmetry (3.2.9) can be written as

$$G^\alpha = G^{I\bar{J}}\, G_{\bar{J}}(\xi, \bar{\xi})\, \partial_I f^\alpha(\xi) = 0 \qquad \text{for all } \xi^I \text{ such that} \qquad f^\alpha(\xi) = 0. \tag{3.4.4}$$

This condition simply means that the functions $f^\alpha(\xi)$ have to remain constant and thus, without loss of generality, zero along the flow defined by the sgoldstino direction. In other words, the sgoldstino direction should be parallel to the reduced manifold in all its points. Therefore, in order to define a consistent supersymmetric truncation we just have to choose a superpotential such that the vector field defined by the sgoldstino is parallel to one of the totally geodesic submanifolds of $\widehat{\mathcal{M}}$. Moreover, from our discussion in section 3.2.1 it follows that any Kähler submanifold defined by (3.4.1) that satisfies (3.4.4), is completely geodesic.

Actually, given a theory with a scalar manifold characterized by $K(\xi,\bar{\xi})$, and any totally geodesic submanifold $\mathcal{M}$, *locally* it is always possible to find a superpotential consistent with the supersymmetric reduction $\widehat{\mathcal{M}} \to \mathcal{M}$. If the reduced manifold is defined in a particular patch by the functions $f^\alpha(\xi)$ as in (3.4.1) then, as we will show, the most general expression for the superpotential that allows the supersymmetric integration of the heavy sector is given by

$$W(\xi) = W_0(\xi)\,\mathrm{e}^{-\gamma_\alpha f^\alpha}, \tag{3.4.5}$$

where $W_0(\xi)$ is an arbitrary holomorphic function, and $\gamma_\alpha(\xi)$ are a set of *holomorphic* functions determined by the Kähler potential, the reduced manifold $\mathcal{M}$, and W_0.

Let us begin the proof showing that we can always find a set of holomorphic functions $\gamma(\xi)$ so that (3.4.5) satisfies the constraint (3.4.4). First note that one can locally choose the independent functions $f^\alpha(\xi)$ to be the first coordinates of a regular coordinate system $\xi^I = (H^\alpha, L^i)$ (3.4.3). With this parametrization the Kähler invariant function reads

$$G(H,\bar{H},L,\bar{L}) = K(H,\bar{H},L,\bar{L}) + \log|W_0(H,L)|^2 - \gamma_\alpha H^\alpha - \bar{\gamma}_\alpha \bar{H}^\alpha. \tag{3.4.6}$$

If we choose the functions $\gamma_\alpha(H,L)$ to be independent of the heavy fields, the constraint on the sgoldstino direction (3.4.4) is simply

$$\partial_\alpha G(H,\bar{H},L,\bar{L})|_{H=0} = \Big[\partial_\alpha K(H,\bar{H},L,\bar{L}) + \frac{\partial_\alpha W_0(H,L)}{W_0(H,L)}\Big]_{H=0} - \gamma_\alpha(L) = 0. \tag{3.4.7}$$

For the superpotential (3.4.5) to be holomorphic, the functions $\gamma_\alpha(L)$ must be holomorphic themselves. An immediate concern is the apparent dependence of the first term on $\bar{L}^i$. However, since the reduced manifold is totally geodesic, the terms mixing light and heavy fields on the metric must vanish over the reduced manifold, what implies that $K_{\alpha\bar{\imath}}|_{H=0} = 0$ for every light field configuration L^i. Therefore all the terms of the previous equation depend holomorphically on the light fields, and we can solve consistently for $\gamma_\alpha(L)$

$$\gamma_\alpha(L) = \partial_\alpha K|_{H=0} + \frac{\partial_\alpha W_0(0,L)}{W_0(0,L)}. \tag{3.4.8}$$

Now we shall prove that the ansatz (3.4.5) is the most general solution to the constraint on the sgoldstino direction (3.4.4). Consider the Taylor expansion of

$\widehat{G}$ on the heavy fields around a configuration $H^\alpha = 0$:

$$\begin{aligned}\widehat{G}(H, \bar{H}, L, \bar{L}) &= g_0(L, \bar{L}) + g_\alpha(L, \bar{L})\, H^\alpha + g_{\bar{\alpha}}(L, \bar{L})\, H^{\bar{\alpha}} + g_{\alpha\beta}(L, \bar{L})\, H^\alpha H^\beta + \\ & \quad g_{\bar{\alpha}\bar{\beta}}(L, \bar{L})\, H^{\bar{\alpha}} H^{\bar{\beta}} + g_{\alpha\bar{\beta}}(L, \bar{L})\, H^\alpha H^{\bar{\beta}} + \mathcal{O}(3) \qquad (3.4.9)\end{aligned}$$

The most general Kähler function which solves the condition (3.4.4), or equivalently (3.2.9), satisfies $g_\alpha = g_{\bar{\alpha}} = 0$, and has a expansion of the form

$$\begin{aligned}\widehat{G}(H, \bar{H}, L, \bar{L}) &= g_0(L, \bar{L}) + g_{\alpha\beta}(L, \bar{L})\, H^\alpha H^\beta + g_{\bar{\alpha}\bar{\beta}}(L, \bar{L})\, H^{\bar{\alpha}} H^{\bar{\beta}} + \\ & \quad g_{\alpha\bar{\beta}}(L, \bar{L})\, H^\alpha H^{\bar{\beta}} + \mathcal{O}(3). \qquad (3.4.10)\end{aligned}$$

It is straightforward to check that the Taylor expansion of the Kähler function associated to the ansatz for the superpotential (3.4.6) is of this form and, moreover, the coefficients of this expansion are unconstrained, since they depend only on the functions $K(\xi, \bar{\xi})$ and $W_0(\xi)$ which are arbitrary, and not on $\gamma_\alpha(\xi)$ or $f^\alpha(\xi)$. In particular, this implies that (3.4.6) and (3.4.10) are equivalent and thus, as we already anticipated in the previous section, the superpotential (3.4.5) characterizes the most general theory consistent with the supersymmetric integration of a heavy sector.

In this discussion we have only investigated the problem locally, but for a complete analysis we should also consider global issues, and in particular whether the superpotential (3.4.5) is well defined over the whole Kähler manifold. The complete characterization of this solution would also require the understanding of the type of holomorphic functions $f^\alpha(\xi)$ that define the totally geodesic submanifolds of a generic Kähler manifold $\widehat{\mathcal{M}}$, but that is beyond the scope of this thesis.

An example: consistent reduction with $\widehat{\mathcal{M}} = \mathbb{C}^2$.

Let us consider an example, the complex euclidean space $\mathbb{C}^2$. We choose the Kähler potential to be given by:

$$K(\xi, \bar{\xi}) = \xi^1\, \bar{\xi}^1 + \xi^2\, \bar{\xi}^2. \qquad (3.4.11)$$

Since this space has complex dimension 2 the totally geodesic submanifolds coincide with the geodesics. The geodesics are straight lines in $\mathbb{C}^2$, and those defining a Kähler submanifold are configurations of the form (3.4.1)

$$\alpha_1\, \xi^1 + \alpha_2\, \xi^2 + \beta = 0. \qquad (3.4.12)$$

We will show how to find superpotentials W compatible with supersymmetric reductions to these submanifolds.

The condition that the sgoldstino should be parallel to the reduced manifold (3.4.4) translates into a constraint for the superpotential

$$\bar{\alpha}_1 \partial_1 W + \bar{\alpha}_2 \partial_2 W - \bar{\beta} W = 0, \qquad (3.4.13)$$

which must be satisfied for configurations satisfying (3.4.12). Defining the new coordinates H and L

$$H = \alpha_1 \xi^1 + \alpha^2 \xi^2 + \beta, \qquad L = \bar{\alpha}_2 \xi^1 - \bar{\alpha}_1 \xi^2. \tag{3.4.14}$$

we see that we can rewrite the equation which defines the reduced theory (3.4.12) as $H = 0$, and thus H can be identified as the field to be truncated. The field L has been chosen so that it parametrizes the orthogonal direction to the reduced manifold. In terms of these variables the constraint (3.4.13) reduces to

$$\partial_H W(L,H) = \gamma_0 W(L,H) \qquad \text{where} \qquad \gamma_0 = \bar{\beta}/(|\alpha_1|^2 + |\alpha_2|^2). \tag{3.4.15}$$

The most general superpotential consistent with the reduction defined in (3.4.12) is given by

$$W(L,H) = W_0(L,H)\, \mathrm{e}^{\gamma(L)H} \qquad \text{with} \qquad \gamma(L) = \gamma_0 + \partial_H \log W_0|_{H=0}. \tag{3.4.16}$$

Here $W_0(L,H)$ is an arbitrary holomorphic function. In particular, a large class of superpotentials are those where W_0 only depends on the field L

$$W(L,H) = W_0(L)\mathrm{e}^{\gamma_0 H}. \tag{3.4.17}$$

3.5 Solving the constraints.

Now we would like to discuss some specific situations where the Kähler potential and the superpotential, $\widehat{K}$ and $\widehat{W}$, allow for a consistent supersymetric decoupling of the heavy fields. As we have discussed in section 3.3 the only consistent solution to the condition of vanishing F-terms (3.3.1) is of the form

$$H^\alpha = H_0^\alpha(L, \overline{L}) = \text{const}, \tag{3.5.1}$$

where the heavy fields have no dependence on the light sector. Although this condition might seem obvious, it is not empty and, for instance, we will see in the next section that it *is not satisfied* generically for couplings of the form $K = K_1 + K_2$ and $W = W_1 + W_2$.

In the two following situations the decoupling condition (3.5.1) does hold:

1. The consistency condition is trivially satisfied if the function $\Phi(H, \overline{H}, L, \overline{L})$ defined in equation (3.3.1) has no explicit dependence on the light fields. If that is the case integrating this equation we recover the condition found in [75]

$$\partial_\alpha \widehat{G} = \Phi(H, \overline{H}) \to \widehat{G} = \widehat{G}_1(H, \overline{H}) + \widehat{G}_2(L, \overline{L}). \tag{3.5.2}$$

   In the Taylor expansion of this Kähler function (3.4.10), except the zero order term $g_0(L, \bar{L})$, all the other coefficients are independent of the light fields. This ansatz and is particularly interesting because it allows a detailed stability analysis of the heavy fields, even when the light sector is in a supersymmetry breaking vacuum. We will come back to this problem in the next chapter.

2. On the other hand, this requirement is too restrictive. A different possibility is when the function $\Phi(H,\overline{H},L,\overline{L})$ factorizes:

$$\Phi(H,\overline{H},L,\overline{L}) = \Phi_1(H,\overline{H},L,\overline{L})\ \Phi_2(H,\overline{H}) = 0 \tag{3.5.3}$$

in which case we just solve $\Phi_2 = 0$ to get constant H_0^α. For example, this condition holds if $\widehat{G}$ has the following functional form:

$$\widehat{G} = f(L,\overline{L},h(H,\overline{H})) \tag{3.5.4}$$

since in that case eq. (3.2.9) is replaced by

$$\partial_\alpha h(H,\overline{H}) = 0. \tag{3.5.5}$$

As the heavy fields only appear in $\widehat{G}$ through the function $h(H,\bar{H})$, the coefficients of its Taylor expansion must satisfy certain constraints. For instance, to second order the expansion is of the form

$$\begin{aligned}\widehat{G}(H,\bar{H},L,\bar{L}) \quad = \quad & g_0(L,\bar{L}) + \partial_h^2 f|_{H_0}(L,\bar{L})\,\Big[\,h_{\alpha\beta}\,\hat{H}^\alpha\hat{H}^\beta + \\ & h_{\bar{\alpha}\bar{\beta}}\,\hat{H}^{\bar{\alpha}}\hat{H}^{\bar{\beta}} + h_{\alpha\bar{\beta}}\,\hat{H}^\alpha\hat{H}^{\bar{\beta}}\Big] + \mathcal{O}(3).\end{aligned} \tag{3.5.6}$$

where the coefficients $h_{\alpha\beta}$, $h_{\bar{\alpha}\bar{\beta}}$ and $h_{\alpha\bar{\beta}}$ are independent of L^i.

The first situation, eq. (3.5.2), is a special case of eq. (3.5.5), with Φ_1 constant. In both cases, the same condition that makes $\widehat{G}_H|_{H_0} = 0$ also implies that the Kähler metric and the Hessian of V are block diagonal for any Φ_1. Indeed, from equation (3.5.5) we find that

$$\widehat{G}_{LH}|_{H_0} = \partial_L\partial_g f(L,\overline{L},g(H,\overline{H}))\partial_H g(H,\overline{H})|_{H_0} = 0 \tag{3.5.7}$$

and further all mixed derivatives with only one derivative with respect to the heavy field vanish. As V_{LH} always contains terms $\propto \widehat{G}_H$ or $\propto (\partial_L)^n\widehat{G}_H$, which vanish, the Hessian of V is block diagonal[4].

[4]Note that it is always possible to diagonalize the Kähler metric or the Hessian of V at one point, but it is not necessarily the case that both diagonalizations are compatible, as we have here.

3.6 An example of non-decoupling: separable K and W.

In supergravity is impossible to describe completely decoupled systems, since gravity couples to everything. Weakly coupled systems are often described by combining two sectors using separable Kälher potential and superpotential. Given two systems characterzed by the Kähler potentials $K_1(H,\bar{H})$ and $K_2(L,\bar{L})$ and the superpotentials $W_1(H)$ and $W_2(L)$, the ansatz reads

$$\begin{aligned} \widehat{K} &= K_1(H,\overline{H}) + K_2(L,\bar{L}) & (3.6.1)\\ \widehat{W} &= W_1(H) + W_2(L) & (3.6.2) \end{aligned}$$

This way of combining systems ensures that, in the flat limit $M_p \to \infty$, the two systems appear completely decoupled in the F-term potential of the full theory [113]. The case of the D-term potential is more involved since it depends on the details of the gauging. If we write the Planck masses explicitly in the expression of the F-term potential the equation (2.2.8) becomes:

$$V_F = \mathrm{e}^{K/M_p^2}\left(K^{I\bar{J}}\mathcal{D}_I W \mathcal{D}_{\bar{J}}\bar{W} - M_p^{-2}\,|W|^2\right), \tag{3.6.3}$$

where the Kahler covariant derivative is given by

$$\mathcal{D}_I W = \partial_I W - M_p^{-2}\,\partial_I K\, W. \tag{3.6.4}$$

Then, substituting the ansatz (3.6.2) in the previous expression, and taking the limit $M_p \to \infty$, the F-term potential becomes

$$V_F = V_1 + V_2 \quad \text{where} \quad V_i = K_i^{I\bar{J}}\partial_I W_i \partial_{\bar{J}}\bar{W}_i. \tag{3.6.5}$$

The ansatz (3.6.2) is not suitable to describe the couplings to a sector that is integrated out while preserving supersymmetry. For instance, the ansatz (3.6.2) does not satisfy the decoupling condition in general. Suppose equation (3.2.9) admits a constant solution $H^\alpha = H_0^\alpha$. Then

$$0 = \partial_\alpha W_1|_{H_0} + \partial_\alpha K_1|_{H_0}[W_1(H_0) + W_2(L)], \tag{3.6.6}$$

which only holds if

$$\begin{aligned} \partial_\alpha K_1|_{H_0} &= 0 \;\Rightarrow\; \partial_\alpha W_1|_{H_0} = 0 \\ \partial_\alpha K_1|_{H_0} &\neq 0 \;\Rightarrow\; W_2(L) = -\frac{\partial_\alpha W_1|_{H_0}}{\partial_\alpha K_1|_{H_0}} - W_1(H_0). & (3.6.7) \end{aligned}$$

Another way to see this: since $\widehat{\mathcal{D}}_\alpha \widehat{W} = 0$ does not factorize, the (Kähler-gauge covariant) requirement that it is independent of the light fields is (see also [114])

$$\widehat{\mathcal{D}}_i(\widehat{\mathcal{D}}_\alpha \widehat{W}) = 0. \tag{3.6.8}$$

Inserting the ansatz (eq. 3.6.1) then gives

$$\partial_\alpha K_1|_{H_0} \partial_i W_2 = 0. \tag{3.6.9}$$

Unless $K_1(H, \overline{H})$ has no linear terms or $W_2(L) =$ constant, the condition will not be met. However, if $W_2(L) =$ constant (e.g. no scale models [115, 116]) then equation (3.5.2) holds and $\widehat{W}$ is trivially a product. On the other hand, we can always expand $K_1(H, \overline{H})$ around H_0 and remove the linear terms by a Kähler transformation (2.2.28), but this spoils the separability of the superpotential (3.6.2).

In other words, if two sets of fields have separable Kähler functions $K = K_1(\text{heavy}) + K_2(\text{light})$, the addition of their superpotentials does not respect the decoupling condition except in special cases.

3.7 Discussion

In this chapter we have considered the circumstances that allow for a consistent supersymmetric truncation a set of scalar fields H^α in $\mathcal{N}=1$ supergravity theories. The truncation is defined fixing the fields H^α at a covariantly constant configuration $H^\alpha = H_0^\alpha$:

$$\nabla_\mu H^\alpha = [\partial_\mu H^\alpha - k_{\tilde{a}}^\alpha A_\mu^{\tilde{a}}]_{H_0} = 0, \tag{3.7.1}$$

so that the Lorentz symmetry is unbroken in the reduced theory. Some of the gauge symmetries might be broken by the vacuum expectation value of the truncated fields ($k_{\tilde{a}}|_{H_0} \neq 0$), we denote the associated gauge boson by $A_\mu^{\tilde{a}}$.

Our discussion is based on two crucial assumptions about the reduced theory describing the dynamics of the fields surviving the truncation L^i. First we have required that solutions of the reduced action should to be solutions of the full action. And second we asked the truncated theory to preserve exactly the invariance under $\mathcal{N}=1$ local supersymmetry. These two conditions consitute the definition of a consistent supersymmetric truncation.

In order to preserve supersymmetry each truncated field has to be removed from the reduced theory together with the whole supermultiplet which contains it. Otherwise the truncated fields could be sourced by supersymmetry transformations.

- The scalar fields H^α have to be truncated together with their fermionic partners, χ^α, which must be set to zero. In order to truncate fully the chiral multiplets the F-terms must also be set to zero in the reduced theory, and thus the first derivative of the Kähler function $\widehat{G}$ along the directions of truncated fields must vanish too:

$$H^\alpha = H_0^\alpha, \qquad \chi^\alpha = 0, \qquad \widehat{G}_\alpha|_{H_0} = 0 \quad \text{for all } L^i, \tag{3.7.2}$$

- The gauge bosons $A_\mu^{\tilde{a}}$ associated to the symmetries broken by the vacuum expectation value of the truncated fields $H^\alpha = H_0^\alpha$ have to be truncated together with the whole vector multiplet. These gauge bosons should be at a pure gauge configuration and therefore the field strengths have to vanish. The gaugini $\lambda^{\tilde{a}}$ and the D-terms (or equivalently the moment-maps $\mathcal{P}_{\tilde{a}}$) must also be set to zero in the reduced theory:

$$F_{\mu\nu}^{\tilde{a}} = 0, \qquad \lambda^{\tilde{a}} = 0, \qquad \mathcal{P}_{\tilde{a}}|_{H_0} = 0 \quad \text{for all } L^i, \tag{3.7.3}$$

We now turn to the conditions which ensure that every solution of the reduced theory must be a solution of the complete model. In particular this must be true for very large momenta, where the interactions can be neglected and the model behaves as a free theory. In that limit we find that the kinetic terms of the truncated fields have to be decoupled from the kinetic terms of the surviving ones:

- If the evolution of the scalar fields is described by a of non linear sigma model, the target space of the reduced theory has to be totally a geodesic submanifold of the complete target space. In particular the sigma model metric has to be block diagonal in the two sectors:

$$G_{i\alpha}(H_0, \bar{H}_0, L, \bar{L}) = 0. \tag{3.7.4}$$

  Moreover, the condition (3.7.2) implies that the sgoldstino direction must be parallel to the reduced manifold at all of its points.

- The kinetic terms of the surviving gauge bosons have to be decoupled from the kinetic terms and of those which become massive at the configuration $H^\alpha = H_0^\alpha$. This implies that the real part of the gauge kinetic functions should not have components mixing truncated and surviving gauge fields gauge fields in the reduced theory:

$$Ref_{a\tilde{b}}(H_0, L) = 0, \quad \text{and} \quad Ref_{ab,\alpha}(H_0, L) = 0 \quad \text{for all } L^i. \tag{3.7.5}$$

  The last condition needs to be imposed for the dynamics to be consistent with the truncation. If this condition is not met a gaugino condensation in the reduced theory could induce supersymmetry breaking in the truncated sector (see [113]).

The truncated fields should not be sourced due to gauge interactions of the surviving fields, which leads to the following conditions:

- The non abelian gauge interactions between gauge bosons should respect the truncation. Then, if the gauge symmetry group $\mathbb{G}$ is semi-simple, it must have a cross product structure $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where subgroup $\mathbb{G}_h$ is broken at the configuration $H^\alpha = H_0^\alpha$, while the subgroup $\mathbb{G}_l$ remains unbroken. Therefore the structure constants of the group satisfy:

$$f^{\tilde{a}}_{bc} = f^{a}_{\tilde{b}c} = f^{\tilde{a}}_{b\tilde{c}} = f^{a}_{\tilde{b}\tilde{c}} = 0, \tag{3.7.6}$$

  where the indices with a tilde run over the generators of $\mathbb{G}_h$, and those without it run over the generators $\mathbb{G}_l$.

- In the reduced theory the fields in the chiral multiplets surviving the truncation should not couple to the truncated gauge bosons. Thus the killing vectors of the broken gauge symmetries, those associated to $\mathbb{G}_h$, should not have components along L^i in the reduced theory:

$$k^i_{\tilde{a}}(H_0, L) = 0, \quad \text{and they satisfy} \quad k^\alpha_{\tilde{a},i}(H_0, L) = 0 \quad \text{for all } L^i. \tag{3.7.7}$$

  The second condition ensures the decoupling of the surviving chiralini from the massive gauge bosons.

- The fields in the truncated chiral multiplets should be decoupled from the gauge bosons which survive the truncation A^a_μ. Thus the killing vectors of the surviving gauge symmetries, i.e. those associated to $\mathbb{G}_l$, should not have components along H^α in the reduced theory:

$$k^\alpha_a(H_0, L) = 0, \quad \text{and they satisfy} \quad k^i_{a,\alpha}(H_0, L) = 0 \quad \text{for all } L^i. \tag{3.7.8}$$

  The second condition ensures the decoupling of the truncated chiralini from the gauge fields in the reduced theory.

These conditions guarantee that the Hessian of the scalar potential of the full theory $\widehat{V}$, is block diagonal in the two sectors, which is required for the dynamics of the system to be consistent with the truncation of the heavy fields. Interestingly, the conditions for supersymmetric decoupling also provide a mechanism to embed BPS solutions of the reduced theory into the full theory without destroying their BPS character.

The condition (3.7.2) restricts the type of couplings, defined by the Kähler potential $\widehat{K}$ and the superpotential $\widehat{W}$, which are compatible with the supersymmetric truncation of a sector of the theory. We have shown that given a theory with a scalar manifold $\widehat{\mathcal{M}}$ and any totally geodesic submanifold $\mathcal{M}$, *locally* the most general superpotential consistent with the supersymmetric reduction $\widehat{\mathcal{M}} \to \mathcal{M}$ is of the form:

$$W(\xi) = W_0(\xi)\, \mathrm{e}^{-\gamma_\alpha f^\alpha}. \tag{3.7.9}$$

Here the holomorphic functions f^α characterize the reduced manifold by the solution to the set of equations $f^\alpha(\xi) = 0$, $W_0(\xi)$ is an arbitrary holomorphic function, and $\gamma_\alpha(\xi)$ are a set of holomorphic functions determined by the Kähler potential and W_0. We have also presented a broad class of Kähler functions $\widehat{G}$ that that allow for a consistent supersymmetric truncation of the would-be heavy fields,

$$\widehat{G}(H, \bar{H}, L, \bar{L}) = f(L, \bar{L}, h(H, \bar{H})).$$

From the point of view of model building, the decoupling condition provides a simple consistency test. For instance, we have shown that when the truncated and surviving sectors are coupled using the ansatz,

$$\widehat{K}(H, \bar{H}, L, \bar{L}) = K_h(H, \bar{H}) + K_l(L, \bar{L}), \qquad \widehat{W}(H, L) = W_h(H) + W_l(L)$$

in general the would-be heavy fields cannot be truncated while preserving supersymmetry. There are stringy scenarios which approximately satisfy the decoupling condition in the form (3.5.4), such as some LARGE volume compactifications [117] [118] [119].

Despite our notation, is important to stress that the conditions we have presented here *do not* ensure the stability of the truncated sector. Actually, in the next chapter we discuss the stability of the truncated sector for a particular class of couplings between the truncated and surviving sectors, and we show that the field configuration that defines the truncation might become unstable depending of the values of the surviving fields. In that case the truncated sector can no longer be considered the “heavy” sector.