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CHAPTER 2

$\mathcal{N}=1$ supergravity and supersymmetric cosmic strings.

2.1 Introduction.

In the previous chapter we already introduced the concept of *supersymmetry*, the symmetry that transforms bosons and fermions into each other. In particular, we discussed how global supersymmetry turned out to provide a natural explanation for the Hierarchy problem, justifying the smallness of the Higgs mass as compared with the GUT scale. In supersymmetric theories the quadratically divergent renormalization corrections to the Higgs mass are absent, since the fermionic and the bosonic contributions cancel each other. This particular example illustrates an important property of supersymmetric theories: they have a better ultraviolet behavior than non-supersymmetric ones.

Originally local supersymmetry, or *supergravity*, was conceived as a theory that could cure the non-renormalizability of quantum gravity, providing a natural framework for the unification of the four fundamental forces. However, nowadays it is mainly considered as an effective field theory describing the light degrees of freedom of a more fundamental theory of quantum gravity, string/M-theory.

This chapter will provide an introduction to the basic concepts in $\mathcal{N}=1$ supergravity. In the first section we will discuss the main features of the bosonic part of the $\mathcal{N}=1$ supergravity action, and the rest of the chapter is dedicated to the discussion of supersymmetric configurations in $\mathcal{N}=1$ supergravity, i.e. those configurations which leave unbroken all, or at least some, of the supersymmetry generators. In section 2.4 we study the stability of vacua which fully preserve supersymmetry. This is relevant for the construction of superstring inspired models describing late time cosmology with low energy supersymmetry breaking. In particular we present a derivation of the well known result that all supersymmetric vacua are stable. This analysis will serve us to introduce the techniques required in chapter 4 to study the stability of consistently decoupled heavy fields.

We will close the chapter reviewing another class of supersymmetry preserving configurations: *half-BPS cosmic strings*, which appear in many cosmological scenarios based on superstrings. These cosmic strings solutions break only half of the original supersymmetries of the model. In particular, we will present the method to obtain the cosmic string solutions from the condition of unbroken supersymmetry. This type of calculation is essential for the last two chapters of this thesis where we discuss supersymmetric cosmic strings in $\mathcal{N}=2$ supergravity.

2.2 Overview of $\mathcal{N}=1$ supergravity.

A detailed review of $\mathcal{N}=1$ supergravity can be found in [18], but in this thesis we will follow the notation and conventions of [84, 85]. The *superPoincaré* algebra characterizes the symmetry underlying supergravity theories. The super-Poincaré algebra is the result of extending the regular Poincaré algebra with the $\mathcal{N}$ generators of the supersymmetry transformations[1] Q, which, schematically, act on the fermionic ($|F>$) and bosonic ($|B>$) states as follows:

$$Q\,|F>=|B> \qquad Q\,|B>=|F> \tag{2.2.1}$$

As supersymmetry relates bosons and fermions, consistency requires the supercharges Q to transform as spinors under the Lorentz group. In particular, in 4-dimensional spacetime with Minkowski signature the supercharges are Majorana spinors, which can be decomposed into two chiral spinors of opposite chirality. The anticommutator of two supercharges yields the generator of translations[2], P^μ (see [84]):

$$\{Q_\alpha, Q_\beta\} = \tfrac{1}{2}(\gamma_\mu \mathcal{C}^{-1})_{\alpha\beta} P^\mu. \tag{2.2.2}$$

[1]The algebra can also contain extra generators commuting with all the supercharges, the central charges.

[2]The operator $\mathcal{C}$ is the charge conjugation matrix, and is related to Majorana conjugation: $\bar\lambda \equiv \lambda^T \mathcal{C}$, [84].

As we anticipated in the previous chapter, this implies that any theory of local supersymmetry is also invariant under local space-time translations, and therefore it also describes gravity.

In supersymmetric theories there is always a group of rotations of the $\mathcal{N}$ supercharges which commutes with the Lorentz transformations and leaves invariant the supersymmetry algebra, the *R-symmetry group* H_R. The R-symmetry is the automorphism group of the supersymmetry algebra, which in 4-dimensions with Minkowski signature is given by $H_R = \mathrm{U}(\mathcal{N})$. As we shall see, it plays an important role in constraining the type of interactions between the different fields in a theory. In the present case, $\mathcal{N} = 1$ supergravity, the R-symmetry is given by $H_R = U(1)$.

Unlike in Grand Unified Theories where all the particles of a given representation have the same spin, the irreducible representations of the super-Poincare algebra, the *supermultiplets*, involve particles of different spin. For unextended supergravity, $\mathcal{N} = 1$, the supermultiplets consist of only two particles which differ in spin 1/2. We will consider theories that involve the following multiplets:

- The chiral multiplet: contains one complex real scalar ξ^I and one Weyl fermion, the chiralino χ^I, where the label runs in $I = 1, \ldots, n_C$ for n_C chiral multiplets.

- The vector multiplet: it consists of one Weyl fermion, called gaugino λ^a and one gauge field A_μ^a. Here $a = 1, \ldots, n_V$ labels n_V different vector multiplets.

- The graviton multiplet : it contains a spin 2 field representing the space-time metric $g_{\mu\nu}$, and one spin 3/2 gravitino, ψ_μ.

Each chiral and vector multiplet also involves a real auxiliary field, F^I and D^a respectively. These fields have no kinetic terms in the action, and thus they are usually expressed in terms of the scalars ξ^I after solving their algebraic equations motion.

In supergravity the gravitino plays the role of the gauge field for local supersymmetry. More generally, $\mathcal{N}$ extended supergravity theories contain $\mathcal{N}$ gravitini, all of which belong to the corresponding graviton multiplet.

Some string compactifications appear to be described in a natural way by a type of multiplets not introduced here, the tensor (linear) multiplets (see for example [86]), however it has been proven that these multplets admit a dual description interms of chiral multiplets (see [87]). Thus, the type of multiplets considered here is sufficient to obtain a complete description of $\mathcal{N} = 1$ supergravity.

Chiral multiplets $(0, \frac{1}{2})$	scalars chiralini	ξ^I χ^I	$I = 1, \ldots, n_C$
Vector multiplet $(\frac{1}{2}, 1)$	gauge fields gaugini	A_μ^a λ^a	$a = 1, \ldots, n_V$
Gravity multiplet $(\frac{3}{2}, 2)$	vielbein gravitino	e_μ^m ψ_μ	$\mu, m = 0, \cdots, 3$

Table 2.1 *– Field content of $\mathcal{N}=1$ supergravity coupled to n_V vector multiplets and n_C chiral multiplets.*

2.2.1 Bosonic sector of the action without gauge couplings.

In this thesis we will focus on the study of bosonic configurations, therefore we only review the features of the bosonic sector of the action. We will first consider actions with no gauge couplings between the vectors and the scalars, and we leave for the next section the discussion on gauged supergravity actions. We will work in units of the reduced Plank mass $M_p^{-2} = 8\pi G = 1$.

The bosonic part of a $\mathcal{N}=1$ supergravity action with n_C chiral multiplets and n_V vector multiplets reads,

$$S = \int d^4x\sqrt{-g}(-\tfrac{1}{2}R + \mathcal{T} + \mathcal{L}_{\text{gauge}} - V). \tag{2.2.3}$$

The action can be decomposed in four different contributions: the standard gravitational term $\frac{1}{2}R$, the kinetic terms for the scalars $\mathcal{T}$, the kinetic terms for the gauge bosons $\mathcal{L}_{gauge}$, and the scalar potential V.

In general, the kinetic terms of the scalars in a supergravity theory are characterized by a non-linear sigma model. In the particular case of ungauged $\mathcal{N}=1$ supergravity with n_C chiral multiplets they are given by:

$$\mathcal{T} = -G_{I\bar{J}}\,\partial_\mu\xi^I\partial^\mu\xi^{\bar{J}}, \qquad I, J = 1, \ldots, n_c. \tag{2.2.4}$$

The target space of the sigma model $\mathcal{M}$ is called the scalar manifold. In a geometric description of supergravity the scalar fields are usually interpreted as coordinates on the scalar manifold, and the quantities $G_{I\bar{J}}$ as the metric on $\mathcal{M}$. In this picture the invariance of the action under supersymmetry transformations leads to constraints in the geometry of the scalar manifold. In the present case, $\mathcal{N}=1$ supergravity, $\mathcal{M}$ should be a Kähler-Hodge manifold (see [88]), and thus the metric $G_{I\bar{J}}$ can be expressed locally as the derivatives of a real function of the scalar fields, the *Kähler potential* $K(\xi,\bar{\xi})$:

$$G_{I\bar{J}}(\xi,\bar{\xi}) = \partial_I \partial_{\bar{J}} K(\xi,\bar{\xi}). \tag{2.2.5}$$

The kinetic terms of the gauge fields A^a_μ, $a=1,\ldots,n_V$, are given in terms of the (holomorphic) *gauge kinetic functions* $f_{ab}(\xi)$:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}(\operatorname{Re} f_{ab}) F^a{}_{\mu\nu} F^{b\mu\nu} + \frac{1}{4\sqrt{-g}}(\operatorname{Im} f_{ab}) F^a{}_{\mu\nu}\epsilon^{\mu\nu\rho\sigma} F^b{}_{\rho\sigma}, \tag{2.2.6}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu$ is the field strength corresponding to the gauge boson A^a_μ.

The scalar potential has two contributions, the F-term potential, V_F, and the $D-term$ potential, V_D:

$$V = V_F + V_D. \tag{2.2.7}$$

The scalar potential appears in the action after solving the the algebraic equations of motion of the auxiliary fields F^I, and D^a in terms of the scalars ξ^I. The form of the F-term potential is very constrained by supersymmetry. Actually, it can be expressed in terms of the Kähler potential $K(\xi,\bar{\xi})$ and holomorphic function $W(\xi)$, known as the *superpotential*:

$$V_F = \mathrm{e}^K \left(K^{I\bar{J}} \mathcal{D}_I W \mathcal{D}_{\bar{J}} W - 3|W|^2 \right). \tag{2.2.8}$$

Here the Kähler covariant derivatives are given by: $\mathcal{D}_I W = \partial_I W + \partial_I K\, W$. The D-term potential is only non-vanishing in the presence of gauge couplings, and thus we will discuss it in the next section.

2.2.2 Gauging of global symmetries.

As in the case of non-supersymmetric theories we can couple the scalars and gauge bosons by promoting to local the global symmetries of the action. Any global symmetry transformation must leave invariant the kinetic terms of the scalars, and therefore, in the geometrical picture, it can be associated to an isometry of the scalar manifold. The transformation of the scalars under an isometry trasformation is characterized by the killing vectors $k_a(\xi) = k^I_a\, \partial_I$:

$$\delta_{gauge}\xi^I = k_a{}^I(\xi)\alpha^a, \qquad a = 1,\ldots,n_V, \tag{2.2.9}$$

which, in $\mathcal{N}=1$ supergravity, have a holomorphic dependence on the scalars. Here α^a are the gauge parameters. We say that the configuration ξ_0 is a *fixed point* of k_a when

$$k_a^I(\xi_0)=0 \quad \text{for all} \quad I=1\dots n_C. \tag{2.2.10}$$

The set of killing vectors of a scalar manifold generate a representation of its isometry group $\mathbb{G}$, and they satisfy the following Lie bracket relations:

$$[k_a,k_b]^I=k_{b,J}^I k_a^J-k_{a,J}^I k_b^J=f_{ab}{}^c k_c^I, \tag{2.2.11}$$

where $f_{ab}{}^c$ are the *structure constants* of $\mathbb{G}$.

In order to promote the isometries of the scalar manifold to local symmetries the derivatives of the scalars in (2.2.4) have to be substituted by covariant derivatives, which in the case of non-linear sigma models read:

$$D_\mu\xi^I=\partial_\mu\xi^I-A_\mu^a k_a^I(\xi). \tag{2.2.12}$$

Note that this expression reduces to the usual covariant derivative in the case of $U(1)$ transformations $k(\xi)=\mathrm{i}\xi\,\partial_\xi$:

$$\delta_{gauge}\xi=\mathrm{i}\xi\,\alpha \qquad \Longrightarrow \qquad D_\mu\xi=\partial_\mu\xi-\mathrm{i}A_\mu^a\xi. \tag{2.2.13}$$

In order to keep invariance under supersymmetry transformations after the gauging of isometries, the bosonic action has to be supplemented with an extra term: the D-term potential V_D. In terms of the auxiliary fields of the vector multiplets, D^a, it reads:

$$V_D=\tfrac{1}{2}(\mathrm{Re}\,f_{ab})D^aD^b=\tfrac{1}{2}(\mathrm{Re}\,f)^{-1|ab}\mathcal{P}_a\mathcal{P}_b, \qquad D^a=(\mathrm{Re}\,f)^{-1|ab}\mathcal{P}_b. \tag{2.2.14}$$

The *moment maps* $\mathcal{P}_a(\xi,\bar\xi)$ are real functions which determine the killing vector through their derivatives:

$$\partial_{\bar J}\mathcal{P}_a(\xi,\bar\xi)=\mathrm{i}k_a^I G_{I\bar J}. \tag{2.2.15}$$

Integrating this equation we can find a expression for the moment map:

$$\mathcal{P}_a=\mathrm{i}\Big(\,k_a^I(\xi)\partial_I K(\xi,\bar\xi)-3r_a(\xi)\,\Big), \tag{2.2.16}$$

where holomorphic functions $r_a(\xi)$, which are called the *compensators*. The constant factor multiplying the compensators, $-3\mathrm{i}$, is for convenience. Using the previous expression it is easy to see that the compensators characterize the gauge transformations of the Kähler potential $K(\xi,\bar\xi)$:

$$\delta_a K(\xi,\bar\xi)=k_a^I(\xi)\partial_I K(\xi,\bar\xi)+k^{\bar I}(\bar\xi)\partial_{\bar I}K(\xi,\bar\xi)=3r_a(\xi)+3\bar r_a(\bar\xi). \tag{2.2.17}$$

This relation fixes the compensators up to an imaginary constant, which implies that the moment map itself it is only determined up to a constant shift. In other words, if $\mathcal{P}_a$ is a moment map associated to the killing vector k_a^I, so is

$$\mathcal{P}_a+\eta_a \quad \text{for any constant } \eta_a. \tag{2.2.18}$$

The constant shift η_a of the moment map is called the *Fayet-Iliopoulos* term (FI). A more detailed analysis shows that the freedom to add a FI term to the moment map is restricted to the case of abelian symmetries (see [18]).

It can also be shown that the superpotential, $W(\xi)$, transforms under the killing vectors $k_a^I(\xi)$ as:

$$\delta_a W(\xi) = k_a^I(\xi)\partial_I W(\xi) = -3r_a(\xi)W(\xi). \tag{2.2.19}$$

When the action involves a non vanishing superpotential, it follows from (2.2.19) that the compensators are completely determined by the superpotential and the killing vectors. Conversely, given the geometry of the scalar manifold and the gauge couplings, (2.2.19) can be seen as a set of constraints on the form of the superpotential.

Global symmetries of the action must also leave invariant the kinetic terms of the gauge bosons, and thus the gauge kinetic functions $f_{ab}(\xi)$ must transform in the following way[3]:

$$\delta_{gauge} f_{ab}(\xi) = (f_{ab,I}k_c^I + f_{da}f_{bc}^d + f_{db}f_{ac}^d)\alpha^c = \mathrm{i}c_{abc}\,\alpha^c \tag{2.2.20}$$

This means that the full gauge transformation of the gauge kinetic function amounts to a shift on the Peccei-Quinn symmetry. This expression can be interpreted as a set of constraints on the allowed form of the gauge kinetic functions, similar to (2.2.19) for the superpotential, which depends on the gauge group and the gauge couplings.

2.2.3 Supersymmetry transformations

Omitting the transformations of the auxiliary fields, the $\mathcal{N}=1$ supersymmetry transformations linear in the fermions are given by[4]

$$\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma^a\psi_\mu, \quad \delta W_\mu^a = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^a, \quad \delta\xi^I = \bar{\epsilon}_L\chi^I, && (2.2.21)\\
\delta\psi_{\mu L} &= (\partial_\mu + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \tfrac{\mathrm{i}}{2}A_\mu^B)\epsilon_L + \tfrac{1}{2}\mathrm{e}^{K/2}W\gamma_\mu\epsilon_R && (2.2.22)\\
\delta{\chi^I}_L &= \tfrac{1}{2}\gamma^\mu\nabla_\mu\xi^I\epsilon_R - \tfrac{1}{2}\,\mathrm{e}^{\frac{1}{2}K}K^{I\bar{J}}\mathcal{D}_{\bar{J}}\bar{W}\,\epsilon_L && (2.2.23)\\
\delta\lambda^a &= \tfrac{1}{4}\gamma^{\mu\nu}{F^a}_{\mu\nu}\epsilon + \tfrac{1}{2}iD^a\gamma_5\epsilon. && (2.2.24)
\end{aligned}$$

Here ϵ is the parameter of the supersymmetry transformations, and γ^μ represent the gamma matrices as usual. The subscripts R and L of the fermions stand for right and left chirality respectively:

$${\chi^I}_R = \tfrac{1}{2}(1-\gamma^5){\chi^I}_R \qquad {\chi^I}_L = \tfrac{1}{2}(1+\gamma^5){\chi^I}_L \tag{2.2.25}$$

[3]Note that each index of the gauge kinetic functions $f_{ab}(\xi)$ transforms in the adjoint representation of the gauge group.

[4]The complete supersymmetry transformations can be found in [75, 89].

The gauge boson A^B_μ is a composite field called the U(1) connection, and is expressed in terms of the other fields as follows:

$$A^B_\mu = \tfrac{\mathrm{i}}{2}[\partial_{\bar I}K\partial_\mu\xi^{\bar I} - \partial_I K\partial_\mu\xi^I] + A^a_\mu\mathcal{P}_a. \tag{2.2.26}$$

This field appears in the covariant derivative of the gravitino

$$D_{[\mu}\psi_{\nu]} = \left(\partial_{[\mu} + \tfrac{1}{4}\omega_{[\mu}^{ab}\gamma_{ab} + \tfrac{1}{2}\mathrm{i}A^B_{[\mu}\gamma_5\right)\psi_{\nu]}. \tag{2.2.27}$$

In the presence of a constant FI term η_a the moment map $\mathcal{P}_a$ has a constant contribution, and thus the gravitino acquires a U(1) charge under one of the abelian gauge symmetries.

2.2.4 Kähler transformations

The $\mathcal{N}=1$ supergravity action and the supersymmetry transformations are invariant under Kähler transformations, which act on the Kähler potential and the superpotential in the following way:

$$K(\xi,\bar\xi) \longrightarrow K(\xi,\bar\xi) + h(\xi) + \bar h(\bar\xi) \qquad W(\xi) \longrightarrow W(\xi)\mathrm{e}^{-h(\xi)}, \tag{2.2.28}$$

where $h(\xi)$ is an arbitrary holomorphic function of the scalars. Actually, if the superpotential is non-zero, the full supersymmetry action and the supersymmetry transformations can be expressed in terms of a single Kähler invariant quantity, the *Kähler function* $G(\xi,\bar\xi)$:

$$G(\xi,\bar\xi) = K(\xi,\bar\xi) + \log|W(\xi)|^2 \quad \text{for} \quad W(\xi)\neq 0. \tag{2.2.29}$$

In particular, the metric of the scalar manifold can be expressed in terms of the derivatives of the Kähler function, since

$$G_{I\bar J} = \partial_I\partial_{\bar J}K = \partial_I\partial_{\bar J}G. \tag{2.2.30}$$

The F-term and D-term potentials are given by[5]:

$$\begin{aligned} V_F &= G_{I\bar J}F^I F^{\bar J} - 3e^G = \mathrm{e}^G(G^{I\bar J}G_I G_{\bar J} - 3), && (2.2.31)\\ V_D &= \frac{1}{2}\,\mathrm{Re}(f_{ab})D^aD^b = \tfrac{1}{2}(\mathrm{Re}\,f)^{-1|ab}k^I_a G_I k^{\bar J}_b G_{\bar J}, && (2.2.32)\end{aligned}$$

where the auxiliary fields read:

$$\begin{aligned} F^I &= e^{G/2}G^{I\bar J}G_{\bar J} && (2.2.33)\\ D^a &= i(\mathrm{Re}\,f)^{-1ab}k^I_b G_I = -i(\mathrm{Re}\,f)^{-1ab}k^{\bar I}_b G_{\bar I}. && (2.2.34)\end{aligned}$$

[5] We will denote with subscripts the derivatives with respect to the scalars: $\partial_I G = G_I$, $\partial^2_{IJ}G = G_{IJ}$, etc ... . The matrix $G^{I\bar J}$ is defined as the inverse of the metric $G_{I\bar K}G^{\bar J K} = \delta^J_I$, $G_{K\bar J}G^{K\bar I} = \delta^{\bar I}_{\bar J}$, and we use the matrix $G_{I\bar J}$ and its inverse to lower and raise indices.

The two expressions given for the D-terms are equivalent because, for the action to be invariant under gauge transformations, the Kähler function $G(\xi,\bar{\xi})$ should also be invariant, and thus:

$$\delta_{\text{gauge}} G = (k_a^I G_I + k_a^{\bar{I}} G_{\bar{I}})\alpha^a = 0, \quad \text{for all } \alpha^a. \tag{2.2.35}$$

In order for the Kähler transformations to leave invariant the fermionic part of the action, we must also require the fermions to transform in the following way:

$$\psi_\mu \to \exp[-\tfrac{\mathrm{i}}{2}\gamma_5 \operatorname{Im} h]\,\psi_\mu, \quad \chi^I \to \exp[\tfrac{\mathrm{i}}{2}\gamma_5 \operatorname{Im} h]\,\chi^I, \quad \lambda^a \to \exp[-\tfrac{\mathrm{i}}{2}\gamma_5 \operatorname{Im} h]\,\lambda^a. \tag{2.2.36}$$

However, these conditions are not sufficient to ensure the action to be well-defined over the whole scalar manifold $\mathcal{M}$, there are consistency problems that might arise globally. In particular the fermions which transform under the Kähler transformations can only be well-defined globally provided the Kähler form,

$$\Omega = \frac{\mathrm{i}}{2\pi} G_{I\bar{J}}\, d\xi^I \wedge d\xi^{\bar{J}} = \frac{\mathrm{i}}{2\pi}\partial_I \partial_{\bar{J}} K\, d\xi^I \wedge d\xi^{\bar{J}}, \tag{2.2.37}$$

defines an even cohomology, so that

$$c_1 = \tfrac{1}{2}[\mathcal{K}] \in \mathbb{Z}. \tag{2.2.38}$$

Therefore we can only define consistently an $\mathcal{N}=1$ supergravity action over Kähler manifolds satisfying this constraint, the so called *Kähler-Hodge* manifolds [90].

2.3 Supersymmetric critical points.

In the present thesis we will deal with supersymmetric configurations, which are those where the supersymmetry transformations of all fields are zero for some non-vanishing value of the supersymetry parameter ϵ. In particular we will focus in two types of supersymmetric configurations: *supesymmetric vacua of the scalar potential*, and *half-BPS cosmic string solutions.* In the rest of this section and in the following one we will focus in supersymmetry preserving vacua, and we will leave the discussion on half-BPS cosmic strings for section 2.5.

In a bosonic background, where all the fermions are set to zero, the supersymmetry transformations of the bosons are all vanishing, and therefore the conditions of unbroken supersymmetry can be derived from the transformations of the fermions alone. Moreover, in a homogeneous background which preserves Lorentz invariance ($\nabla^\mu \xi^I = F^a{}_{\mu\nu} = 0$), a set of necessary conditions for unbroken supersymmetry is (2.2.23):

$$\mathcal{D}_I W = 0 \qquad \text{for all} \qquad I = 1, \ldots, n_C. \tag{2.3.1}$$

Equivalently, if $W(\xi) \neq 0$, this condition can be written in terms of the Kähler function as:

$$\partial_I G(\xi,\bar{\xi}) = 0 \qquad \text{for all} \qquad I = 1,\ldots,n_C. \tag{2.3.2}$$

Note that although it is always possible to break supersymmetry spontaneously by non-vanishing F-terms (2.2.23) and zero D-terms (2.2.24), the relations (2.2.33) and (2.2.34) imply that non-vanishing D-terms necessarily require non-vanishing F-terms, and therefore supersymmetry can never be broken by D-terms alone [91].

Supersymmetric configurations are also called *supersymmetric critical points* of the scalar potential, since any configuration satisfying (2.3.2) is always a critical points of the scalar potential, as can be easily checked from (2.2.32). Moreover, the result (2.3.2) implies, together with the expression for the scalar potential (2.2.7) and (2.2.32), that supersymmetric critical points ξ_0^I with non vanishing superpotential $W(\xi_0) \neq 0$ always have a negative vacuum energy, i.e. they are AdS critical points:

$$V(\xi_0) = -3e^{G(\xi_0)} < 0. \tag{2.3.3}$$

Interestingly supersymmetric critical points are always perturbatively stable, regardless of being local minima, maxima or saddle points of the scalar potential. The reason is that in an AdS background a fluctuation with a tachyonic mass might still be stable as long as it satisfies the Breitenlohner-Freedman bound [92]:

$$m^2 \geq \frac{3}{4}\, V(\xi_0), \tag{2.3.4}$$

and this condition is always fulfilled by supersymmetric critical points, as we shall prove in the next section.

2.4 Stability of supersymmetric critical points

In this section we study the stability properties of a supersymmetric critical point in a completely general setup. We take the action to be characterized by a Kähler potential $G(\xi,\bar{\xi})$, and we allow for an arbitrary gauge coupling defined by the gauge kinetic functions $f_{ab}(\xi)$ and Killing vectors $k(\xi)_a^I$. We will only assume that the superpotential is non-vanishing, $W(\xi) \neq 0$, so that the Kähler function $G(\xi,\bar{\xi})$ is well defined. We will relate the stability of the supersymmetric vacua to the curvature of the Kähler function, and in particular we will show that maxima of the scalar potential always correspond to minima of the Kähler function.

2.4.1 Analysis of the Kähler function

We begin by studying the character of the critical points of the Kähler function, which is a simple calculation and will serve us to introduce the technique we will use later in the analysis of the scalar potential. The Taylor expansion of the Kähler potential $G(\xi^I, \xi^{\bar{I}})$ around a generic point ξ_0^I, reads:

$$\begin{aligned} G \;\; = \;\; & G(\xi_0) + G_I(\xi_0)\hat{\xi}^I + G_{\bar{I}}(\xi_0)\hat{\xi}^{\bar{I}} + G_{I\bar{J}}(\xi_0)\, \hat{\xi}^I \hat{\xi}^{\bar{J}} + \\ & \tfrac{1}{2} G_{IJ}(\xi_0)\, \hat{\xi}^I \hat{\xi}^J + \tfrac{1}{2} G_{\bar{I}\bar{J}}(\xi_0)\, \hat{\xi}^{\bar{I}} \hat{\xi}^{\bar{J}} + \ldots, \end{aligned} \tag{2.4.1}$$

where we define $\hat{\xi}^I = \xi^I - \xi_0^I$. If the point ξ_0^I is a supersymmetric critical point then the first order terms vanish $G_I(\xi_0) = 0$. In order to know if ξ_0^I corresponds to a minimum, a maximum or a saddle point the Kähler function we need to find the eigenvalues of its Hessian evaluated at the critical point

$$\begin{pmatrix} G_{I\bar{J}} & G_{IJ} \\ G_{\bar{I}\bar{J}} & G_{\bar{I}J} \end{pmatrix}_{\xi_0} . \tag{2.4.2}$$

This expression simplifies considerably by redefining the ξ^I fields so that they have canonical kinetic terms at the critical point, $G_{I\bar{J}}(\xi_0) = \delta_{I\bar{J}}$. With this choice of coordinates the equation for the eigenvalues, g, takes the form:

$$\det \begin{pmatrix} (1-g)\mathbb{1} & X \\ X^\dagger & (1-g)\mathbb{1} \end{pmatrix} = 0 \tag{2.4.3}$$

where we have used the matrix notation $X = X^T = G_{IJ}(\xi_0)$ and $\mathbb{1} = \delta_{I\bar{J}}$. Using a known property of determinants

$$\det \begin{pmatrix} M & P \\ Q & N \end{pmatrix} = \det (MN - QP)\,, \quad \text{provided that} \;\; QN = NQ, \quad \det(M) \neq 0, \tag{2.4.4}$$

for square submatrices M, N, P, Q, we can see that g is a solution of (2.4.3) if and only if it also solves:

$$\det \left((1-g)^2 - X^\dagger X \right) = 0. \tag{2.4.5}$$

Strictly speaking this equation was derived for $g \neq 1$, but it is not difficult to check that it also gives the correct solution for $g = 1$. In order to solve this equation we use the freedom of field redefinition once more. Requiring that the fields have canonical kinetic terms is not enough to fix the choice of fields completely, we can still redefine the fields by a constant unitary transformation of the form $\tilde{\xi}^I = U^I{}_J\, \xi^J$. Under this field redefinition the matrix X and the combination $X^\dagger X$ transform as:

$$X = U^T \tilde{X} U, \qquad X^\dagger X = U^\dagger\, (\tilde{X}^\dagger \tilde{X})\, U, \quad \text{where} \quad U = U^I{}_J, \tag{2.4.6}$$

and therefore we can use this freedom to transform the Hermitian combination $X^\dagger X$ into a real diagonal matrix. The eigenvalues of $X^\dagger X$ are necessarily nonnegative, and we will denote them by $|x_\lambda|^2$, with λ labeling the p different eigenspaces. Moreover, the symmetry of X implies that $X^\dagger X = (XX^\dagger)^*$, thus in the basis that makes $X^\dagger X$ diagonal we have:

$$X^\dagger X = XX^\dagger = Diag(|x_1|^2 \mathbb{1}_{n_1}, \ldots, |x_p|^2 \mathbb{1}_{n_p}), \qquad |x_\lambda|^2 \geq 0, \tag{2.4.7}$$

where n_λ is the dimension of the eigenspace of eigenvalue $|x_\lambda|^2$. Note also that in this particular basis the matrices X and $X^\dagger X$ commute, which implies that X should be block diagonal in each of the eigenspaces of $X^\dagger X$:

$$X = Diag(X_1, \ldots, X_p) \qquad \text{with} \qquad X_\lambda^\dagger X_\lambda = |x_\lambda|^2 \mathbb{1}_{n_\lambda}. \tag{2.4.8}$$

This means that the eigenvalue problem decouples for the m different eigenspaces of $X^\dagger X$, and therefore the equation (2.4.5) takes a very simple form:

$$\prod_{\lambda=1}^{p} \left[(1 - g_\lambda)^2 - |x_\lambda|^2\right]^{n_\lambda} = 0, \tag{2.4.9}$$

which we can solve easily giving the eigenvalues

$$g_{\pm\lambda} = 1 \pm |x_\lambda|. \tag{2.4.10}$$

which have multiplicity n_λ. The different possibilities for the character of the critical point ξ_0^α are summarized in the following table:

$$\begin{array}{rll} \text{Local minimum} & |x_\lambda| < 1 & \text{for all } \lambda = 1, \ldots, p \\ \text{Saddle point} & |x_\lambda| > 1 & \text{for some or all } \lambda \end{array} \tag{2.4.11}$$

The result (2.4.10) also indicates that, for each eigenvalue of $X^\dagger X$ that satisfies $|x_\lambda|^2 = 1$, the Kähler function has a flat direction and a local minimum (a trough) along one of the complex directions ξ^I.

2.4.2 Analysis of the scalar potential with vanishing D-terms

We will now analyze how the maxima and saddle points of the Kähler function relate to the different types of supersymmetric critical points of the scalar potential. This is especially interesting because the Kähler function is much easier to study than the scalar potential. We will demonstrate a remarkable result: the minima of the Kähler function are in one to one correspondence with the supersymmetric AdS maxima of the scalar potential. We start assuming that there are no gauge interactions, and in the next section we will prove this result in full generality.

The stability analysis of a supersymmetric critical point of the scalar potential can be done using the same techniques of the previous subsection. Consider first its Taylor expansion around a supersymmetric critical point ξ_0^I:

$$V = V(\xi_0) + \tfrac{1}{2} V_{IJ}(\xi_0)\, \hat{\xi}^I \hat{\xi}^J + \tfrac{1}{2} V_{\bar{I}\bar{J}}(\xi_0)\, \hat{\xi}^{\bar{I}} \hat{\xi}^{\bar{J}} + V_{I\bar{J}}(\xi_0)\, \hat{\xi}^I \hat{\xi}^{\bar{J}} + \ldots\,, \tag{2.4.12}$$

where the second derivatives of the potential evaluated at the point $\xi^I = \xi_0^I$ can be calculated from (2.2.32) using that $G_I(\xi_0) = 0$:

$$\begin{aligned} V_{IJ}(\xi_0) &= -G_{IJ}(\xi_0)\mathrm{e}^{G(\xi_0)} && (2.4.13)\\ V_{I\bar{J}}(\xi_0) &= \mathrm{e}^{G(\xi_0)}\left[G^{R\bar{S}}G_{RI}G_{\bar{S}\bar{J}} - 2G_{I\bar{J}}\right]_{\xi_0}. && (2.4.14)\end{aligned}$$

In order to determine the character of the critical point ξ_0^I we need to find the eigenvalues of the Hessian of the potential, which gives the mass spectrum of the fluctuations around ξ_0^I. As in the previous subsection we define the fields ξ^I so that they have canonical kinetic terms, $G_{I\bar{J}}(\xi_0) = \delta_{I\bar{J}}$, and the Hermitian matrix $X^\dagger X$ becomes diagonal. With this choice the Hessian has the simple form:

$$\begin{pmatrix} V_{I\bar{J}} & V_{IJ} \\ V_{\bar{I}\bar{J}} & V_{\bar{I}J} \end{pmatrix}_{\xi_0} = \mathrm{e}^{G(\xi_0)}\begin{pmatrix} XX^\dagger - 2\,\mathbb{1} & -X \\ -X^\dagger & X^\dagger X - 2\,\mathbb{1} \end{pmatrix}. \qquad (2.4.15)$$

Since in the basis we have chosen $X^\dagger X = XX^\dagger$ it is easy to check that this matrix also satisfies the first of the conditions necessary to apply (2.4.4), and we find that the equation for the spectrum of masses m^2 reads:

$$\det\left((X^\dagger X - (2 + e^{-G(\xi_0)}\, m^2)\,\mathbb{1})^2 - X^\dagger X\right) = 0. \qquad (2.4.16)$$

In order to use (2.4.4) we also need to assume that the following matrix is non singular,

$$\det\left(X^\dagger X - (2 + e^{-G(\xi_0)}\, m^2)\,\mathbb{1}\right) \neq 0, \qquad (2.4.17)$$

but after some algebra it is possible to prove that (2.4.16) also gives the right result in the singular case. As in the previous section we can use that X has the block diagonal form (2.4.8) to show that the eigenvalue problem can be decomposed in each of the eigenspaces of $X^\dagger X$. Using this fact the eigenvalue equation (2.4.16) can be written as

$$\prod_{\lambda=1}^{p}\left[(|x_\lambda|^2 - 2 - e^{-G(\xi_0)}\, m^2)^2 - |x_\lambda|^2\right]^{n_\lambda} = 0. \qquad (2.4.18)$$

Therefore the spectrum of masses at the supersymmetric critical point is given by:

$$m^2_{\pm\lambda} = \mathrm{e}^{G(\xi_0)}(|x_\lambda|^2 - 2 \pm |x_\lambda|) = \mathrm{e}^{G(\xi_0)}\left(\left(|x_\lambda|^2 \pm \tfrac{1}{2}\right)^2 - \tfrac{9}{4}\right). \qquad (2.4.19)$$

Each of these energy levels contains n_λ different excitations with the same mass. The set of quantities $|x_\lambda|$ determine which type of extremum the supersymmetric critical point ξ_0^I is:

$$\begin{aligned} &|x_\lambda| > 2 \quad \text{for all } \lambda \quad \Rightarrow \quad \text{local AdS minimum,}\\ &|x_\lambda| < 1 \quad \text{for all } \lambda \quad \Rightarrow \quad \text{local AdS maximum,} \qquad (2.4.20)\end{aligned}$$

and any other combination corresponds to AdS saddle points ($|x_\lambda| = 1, 2$ give flat directions). The result (2.4.19) provides a proof of the stability of all supersymmetric critical points, regardless of the possible negative curvature of the potential. Since all these critical points are AdS, the perturbative stability is determined by the Breitenlohner-Freedman bound (2.3.4), which is always satisfied as is clear from (2.4.19):

$$m^2 \geq -\frac{9}{4}e^{G(\xi_0)} = \frac{3}{4}\, V(\xi_0), \tag{2.4.21}$$

Now we already have at hand all the results we need to check the claim we made at the beginning of this subsection. Comparing equations (2.4.11) and (2.4.20) we see immediately that the supersymmetric AdS maxima of the potential always coincide with the minima of the Kähler function.

2.4.3 Analysis of the scalar potential with non-vanishing D-terms

Let us now generalize the result of the previous subsection to the case where the gauge couplings are turned on. In this case we have to add to the scalar potential the contribution from D-terms (2.2.34). In order to calculate the new contributions to the Hessian of the scalar potential around the critical point ξ_0^I we must find the derivatives of the D-term potential at this point. Using that $G_I(\xi_0) = 0$ we find:

$$\begin{aligned} V_{D|IJ}(\xi_0) &= \tfrac{1}{2}(Ref(\xi_0))^{-1\,ab} k_a^R(\xi_0) k_b^{\bar{S}}(\bar{\xi}_0)[G_{IR}G_{J\bar{S}} + G_{JR}G_{I\bar{S}}]_{\xi_0}, \\ V_{D|I\bar{J}}(\xi_0) &= \tfrac{1}{2}(Ref(\xi_0))^{-1\,ab} k_a^R(\xi_0) k_b^{\bar{S}}(\bar{\xi}_0)[G_{IR}G_{\bar{J}\bar{S}} + G_{\bar{J}R}G_{I\bar{S}}]_{\xi_0} \end{aligned} \tag{2.4.22}$$

As we have done previously we will define the scalar fields ξ^I so that they have trivial kinetic terms $G_{I\bar{J}} = \delta_{I\bar{J}}$ and the Hermitian matrix $X^\dagger X$ is diagonal. In the case of the D-term potential we can simplify the calculations even further making use of the freedom we have to define the gauge fields A_μ^a. Actually, the action is invariant under constant linear transformations of the gauge fields $A_\mu^b = O_a^b\, \tilde{A}_\mu^a$, with O_a^b any non-singular real matrix, provided that the gauge kinetic functions f_{ab} and the Killing vectors k_a^I transform as follows:

$$\tilde{f}_{cd} = O_c^a\, O_d^b\, f_{ab} \qquad \tilde{k}_b^I = O_b^a\, k_a^I. \tag{2.4.23}$$

In particular note that the gauge covariant derivatives and the Yang-Mills terms do not transform under these redefinitions since:

$$A_\mu^a k_a^I = \tilde{A}_\mu^b \tilde{k}_b^I, \qquad (\mathrm{Re} f_{ab}) F_{\mu\nu}^a F^{a\,\mu\nu} = (\mathrm{Re}\tilde{f}_{cd}) \tilde{F}_{\mu\nu}^c \tilde{F}^{d\,\mu\nu}. \tag{2.4.24}$$

Then we can always use this freedom to turn $\mathrm{Re}\, f_{ab}$ into a matrix proportional to the identity

$$\mathrm{Re}\, f_{ab} = e^{G(\xi_0)}\, \delta_{ab}, \tag{2.4.25}$$

where the overall factor has been chosen for convenience. Using these conventions, and defining the matrix $k = k_a^I(\xi_0)$, we can write the Hessian of the D-term potential as:

$$\begin{pmatrix} V_{D|I\bar{J}} & V_{D|IJ} \\ V_{D|\bar{I}\bar{J}} & V_{D|\bar{I}J} \end{pmatrix}_{\xi_0} = \tfrac{1}{2}\mathrm{e}^{G(\xi_0)} \begin{pmatrix} kk^\dagger + X\,kk^\dagger X^\dagger & X\,kk^\dagger + k^* k^T\,X \\ X^\dagger\,k^* k^T + kk^\dagger\,X^\dagger & k^* k^T + X^\dagger\,k^* k^T\,X \end{pmatrix}, \tag{2.4.26}$$

which has to be added to (2.4.15) in order to get the Hessian of the full scalar potential.

Before we continue the calculation let us derive a useful property of the Killing vectors k. We mentioned in section (2.2.4) that the Kähler function $G(\xi^I)$ has to be invariant under gauge transformations. In particular in a Taylor expansion of the Kähler function around ξ_0 (2.4.1) all the terms have to be invariant under gauge transformations order by order in $\hat{\xi} = \xi - \xi_0$. From the gauge transformation of the order one terms in the expansion we find the condition:

$$\Big(G_{IJ}(\xi_0)\,k_a^J(\xi_0) + G_{I\bar{J}}(\xi_0)\,k_a^{\bar{J}}(\xi_0) + G_J(\xi_0)\,\partial_I k_a^J\Big)\,\hat{\xi}^I \alpha^a = 0, \tag{2.4.27}$$

which has to be satisfied for all values of the gauge parameters α^a, and the fluctuations $\hat{\xi}^I$. Since $G_I(\xi_0) = 0$, then with our field definitions and in matrix notation this condition reads simply:

$$k^* = -Xk. \tag{2.4.28}$$

An immediate consequence of this requirement is that the Killing vectors are eigenvectors of the matrix $X^\dagger X$ with eigenvalue $g_\lambda = 1$:

$$X^\dagger X\,k = -X^\dagger k^* = k, \tag{2.4.29}$$

since $X^T = X$. This means that the matrices $kk^\dagger$ and kk^T have all the entries zero except in the block that corresponds to the eigenspace of eigenvalue $|x_\lambda|^2 = 1$ of $X^\dagger X$. As we saw in the previous section the eigenvalue problem of the Hessian decouples in the different eigenspaces of the matrix $X^\dagger X$. We have just proven that the corrections introduced by the D-term potential respect this decoupling and moreover, that the corrections only affect the eigenspace with eigenvalue $|x_\lambda|^2 = 1$. Therefore we can use the results of the previous section for all the eigenspaces with $|x_\lambda| \neq 1$ to find the corresponding eigenvalues. In the remaining of this section we will just focus on solving the eigenvalue problem in the eigenspace where $|x_\lambda|^2 = 1$, which we label by $\lambda = 1$. In order to keep notation simple, we will use the matrices X_1 and k_1 to represent the submatrices corresponding to the eigenspace $\lambda = 1$, thus:

$$X_1^\dagger X_1 = X_1 X_1^\dagger = \mathbb{1}_{n_1}. \tag{2.4.30}$$

Notice that, since the hermitian matrix $k_1 k_1^\dagger$ transforms under scalar field redefinitions in the same way as $X_1 X_1^\dagger$, we can use the residual freedom to choose the eigenvectors in the eigenspace with $|x_\lambda|^2 = 1$ to turn $k_1 k_1^\dagger$ into a real diagonal matrix:

$$k_1 k_1^\dagger = Diag(|k_1|^2, \ldots, |k_{n_1}|^2), \tag{2.4.31}$$

where n_1 is the dimension of the eigenspace with $|x_\lambda|^2 = 1$. The final expression for the Hessian of the full potential restricted to this eigenspace can be obtained from (2.4.26) and (2.4.15)

$$\begin{pmatrix} V_{I\bar{J}} & V_{IJ} \\ V_{\bar{I}\bar{J}} & V_{\bar{I}J} \end{pmatrix}_{\lambda=1} = \mathrm{e}^{G(\xi_0)} \begin{pmatrix} -\mathbb{1} + k_1 k_1^\dagger & (-\mathbb{1} + k_1 k_1^\dagger)\, X_1 \\ (-\mathbb{1} + k_1 k_1^\dagger)\, X_1^\dagger & -\mathbb{1} + k_1 k_1^\dagger \end{pmatrix}, \tag{2.4.32}$$

where we have used the properties (2.4.28), (2.4.31) and (2.4.30) in order to simplify (2.4.26). It is easy to check that the matrices X_1 and $k_1 k_1^\dagger$ commute, therefore we can use equation (2.4.4) in order to find the equation for the mass spectrum m^2, which reads:

$$\prod_{i=1}^{n_1} \left(\left(|k_i|^2 - 1 - e^{-G(\xi_0)}\, m^2 \right)^2 - (|k_i|^2 - 1)^2 \right) = 0, \tag{2.4.33}$$

after having substituting the diagonal form of $k_1 k_1^\dagger$ (2.4.31). The solutions to this equations, together with the results we found in the previous section, which apply for $|x_\lambda| \neq 1$ are summarized below:

$$\begin{aligned} m^2_{\pm\lambda} &= e^{G(\xi_0)} \left(\left(|x_\lambda \pm \tfrac{1}{2}| \right)^2 - \tfrac{9}{4} \right) && \text{if } |x_\lambda|^2 \neq 1, \\ m^2_{+1i} &= 2e^{G(\xi_0)} \left(|k_i|^2 - 1 \right) && \text{if } |x_\lambda|^2 = 1, \\ m^2_{-1i} &= 0 && \text{if } |x_\lambda|^2 = 1. \end{aligned} \tag{2.4.34}$$

The quantities $|k_i|^2$ determine the mass of the gauge bosons at the supersymmetric critical point ξ_0, therefore we can see that the breaking of gauge symmetries can only improve the stability of vacuum. This result agrees with the analysis of Gomez-Reino and Scrucca of the stability of uplifted vacua in [93].

The fact that the Killing vectors are associated to the eigenvalues $|x_\lambda|^2 = 1$ should not be surprising. On the one hand the eigenvalues $|x_\lambda|^2 = 1$ are always related to marginally stable directions $m^2 = 0$. And on the other hand we know that the potential has to be invariant under gauge transformations, thus each Killing vector has to be naturally associated with a flat direction of the potential, which appear in the spectrum as massless fluctuations (the would-be Goldstone bosons that disappear due to the Higgs mechanism).

In view of the result (2.4.34) we can argue that the presence of non-vanishing gauge couplings does not modify the conclusion of the previous section, the minima of the Kähler function $G(\xi^I, \xi^{\bar{I}})$ are always in one to one correspondence with the supersymmetric AdS maxima of the scalar potential.

2.5 Supersymmetric Cosmic Strings and FI-terms

In this section we will leave aside supersymmetric vacua, and we will focus on a different type of supersymmetry preserving configurations: *supersymmetric cosmic strings*.

We are going to discuss these solutions in the context of a supersymmetric extension of the Abelian-Higgs model, and to show that they preserve half of the supersymmetries of the system. This review is based on the work by Dvali *et al.* [61], Binetruy *et al.* [75], and Becker *et al.* [57].

The model we review here describes the dynamics of one chiral multiplet, which involves the scalar field acting as a Higgs ϕ and its fermionic partner ζ, one vector multiplet containing the gauge boson A_μ and a gaugino λ, and the graviton multiplet consiting of the gravitino ψ_μ and the graviton itself, represented by the vielbein e_μ^m.

As we discussed in section 2.2, in $\mathcal{N}=1$ supergravity the action is defined in terms of the Kähler potential, the superpotential, the gauge kinetic functions and the gauge couplings (the killing vectors). We choose the Kähler potential $K=\phi\bar{\phi}$, so that the scalar has trivial kinetic terms as in the Abelian-Higgs model, and we set the superpotential to zero. For simplicity, we will take the gauge kinetic function to be a real constant, which can be set to one via field redefinitions, $f(\phi)=1$. In order to couple the gauge boson to the scalar field as in the Abelian-Higgs model we gauge the U(1) symmetry of the scalar manifold:

$$\delta_{gauge}\phi = \mathrm{i}g\phi\,\alpha, \tag{2.5.1}$$

which is associated to the usual covariant derivatives $D_\mu\phi=\partial_\mu\phi-\mathrm{i}g\phi A_\mu$. Since this symmetry is abelian we can include a FI term $g\eta$ in the theory, and thus the D-term potential is given by:

$$V_D=\tfrac{1}{2}D^2 \quad \text{where} \quad D=g\eta-g|\phi|^2. \tag{2.5.2}$$

The bosonic sector of the resulting supergravity action reads:

$$e^{-1/2}\mathcal{L}=-\tfrac{1}{2}R-D_\mu\phi\,D^\mu\bar{\phi}-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}g^2(|\phi|^2-\eta)^2\,, \tag{2.5.3}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Note that, ignoring the gravitational term, this action is a particular case of the one considered in section 1.2.2, with the parameters of the theory satisfying $2\lambda=g^2$, i.e. the BPS bound. Therefore the vacuum manifold has the same topology $\pi(\mathcal{V})=\mathbb{Z}$, and there must be topological cosmic strings solutions.

2.5.1 The BPS-equations.

We are interested in cosmic string solutions which leave unbroken a fraction of the supersymmetries, therefore we are looking for a field configuration such that the supersymmetry transformations are zero for some non-vanishing value of the supersymmetry parameter ϵ.

Since we focus on purely bosonic solutions, with all the fermions set to zero, such as those found in section 1.2.2, it is sufficient to study the supersymmetry transformations of the fermions (the rest are vanishing). In our particular model the supersymmetry transformations reduce to:

$$\begin{aligned} \delta\psi_{\mu L} &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{mn}(e)\gamma_{mn} + \tfrac{1}{2}\mathrm{i}A^B_\mu\right)\epsilon_L\,, && (2.5.4)\\ \delta\zeta_L &= \tfrac{1}{2}\not{D}\phi\,\epsilon_R\,, && (2.5.5)\\ \delta\lambda &= \tfrac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\epsilon + \tfrac{1}{2}\mathrm{i}\gamma_5 D\,\epsilon\,, && (2.5.6)\end{aligned}$$

where the gravitino U(1) connection A^B_μ is given by:

$$A^B_\mu = \frac{1}{2}\mathrm{i}\left[\phi\,\partial_\mu\bar{\phi} - \bar{\phi}\,\partial_\mu\phi\right] + gA_\mu\eta\,. \qquad (2.5.7)$$

The condition to have a vanishing supersymmetry transformation of the chiral field (2.5.5) on the string background reads:

$$(\gamma^1 D_1 + \gamma^2 D_2)\phi\,\epsilon_R = 0. \qquad (2.5.8)$$

If we multiply both sides of the equation from the left by $\not{D}\phi$ we obtain:

$$\left((D_1\phi)^2 + (D_2\phi)^2\right)\epsilon_R = 0 \quad\Longleftrightarrow\quad (D_1 \pm \mathrm{i}D_2)\phi = 0, \quad \text{with} \quad \epsilon_R \neq 0. \quad (2.5.9)$$

The condition we just obtained for the chiral field configuration is precisely the first of the BPS equations (1.3.28) derived in section 1.2.2. Note that, although space-time will be curved in general, we are working on a locally Lorentzian frame where the metric is just η_{ab} in order to simplify the manipulations of the gamma matrices.

Substituting the last result back into equation (2.5.8) we obtain a projector condition for the supersymmetry generator:

$$(\gamma^1 \pm \mathrm{i}\gamma^2)\,\epsilon_R = 0, \quad \text{provided that} \quad D_1\phi \neq 0. \qquad (2.5.10)$$

The constraint for the left handed supersymmetry generator can be obtained acting with the charge conjugation operator C on the previous expression [94]:

$$(\gamma^1 \mp \mathrm{i}\gamma^2)\,\epsilon_L = 0. \qquad (2.5.11)$$

After some manipulations of the gamma matrices we can write the projector conditions for ϵ_L and ϵ_R together in a single expression:

$$\gamma^{12}\epsilon = \mp \mathrm{i}\gamma^5\epsilon. \tag{2.5.12}$$

This condition implies that only half of the supersymmetries will be left unbroken in the string background. To understand this we decompose ϵ into ϵ_+ and ϵ_-, which satisfy the conditions:

$$\gamma^{12}\epsilon_+ = \mathrm{i}\gamma^5\epsilon_+, \qquad \gamma^{12}\epsilon_- = -\mathrm{i}\gamma^5\epsilon_-. \tag{2.5.13}$$

If we choose the upper sign in (2.5.9), then ϵ_- leaves the supersymmetries unbroken while the transformations generated by ϵ_+ are broken, and viceversa.

Let's turn to the supersymmetry transformation of the gaugino (2.5.6). If we consider a static straight cosmic string solution along the z-axis ($\mu = 3$) the only non-vanishing component of the field strength $F_{\mu\nu}$ is F_{12}, therefore the supersymmetry transformation of the gaugino (2.5.6) is simply:

$$\delta\lambda = \tfrac{1}{2}(\gamma^{12}F_{12} + \mathrm{i}\gamma_5 D)\,\epsilon. \tag{2.5.14}$$

This expression should be equal to zero for any supersymmetry transformation satisfying the projection condition (2.5.12):

$$(\gamma^{12}F_{12} + \mathrm{i}\gamma_5 D)\,\epsilon = \mathrm{i}(\mp F_{12} + D)\gamma_5\,\epsilon = 0, \tag{2.5.15}$$

and thus, for $\epsilon \neq 0$, the field configuration must satisfy:

$$F_{12} \mp D = 0, \tag{2.5.16}$$

which is identical to the second of the BPS equations (1.3.28) found in section 1.2.2.

In order to characterize the space time metric in the background of the cosmic string we need an extra equation, which can be obtained requiring the the supersymmetry transformation of the gravitino to be vanishing:

$$\left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{mn}(e)\gamma_{mn} + \tfrac{1}{2}\mathrm{i}A^B_\mu\right)\epsilon_L = 0 \tag{2.5.17}$$

The last equation together with (2.5.9), (2.5.16) and the boundary conditions on the fields (which are specified below) determines the cosmic string solution. To solve them we proceed as in section 1.3.28 and we use the following ansatz for static straight cylindrically symmetric cosmic string:

$$\phi(r,\theta) \;=\; \sqrt{\eta}f(r)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad A_\theta = \frac{n}{gr}v(r), \tag{2.5.18}$$

where we are using polar coordinates $\{r,\theta\}$, and $f(r)$ and $v(r)$ are real functions. The most general cylindrically symmetric ansatz for the metric written in polar coordinates is given by [26]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + C^2(r)\mathrm{d}\theta^2\,. \tag{2.5.19}$$

The conditions we obtained above for the chiral field (2.5.9) and the gauge field (2.5.16) are written in a locally lorentzian frame, such as the one defined by the vielbein

$$e^1 = dr \qquad \text{and} \qquad e^2 = C(r)d\theta. \tag{2.5.20}$$

In these coordinates equations (2.5.9) and (2.5.16) read:

$$(D_r \pm \mathrm{i}C(r)^{-1}D_\theta)\phi = 0 \tag{2.5.21}$$
$$C(r)^{-1}\,F_{r\theta}\,\mp D\;=\;0\,, \tag{2.5.22}$$

which after substituting the ansatz (2.5.18) lead to

$$\begin{aligned} C(r)f'(r) &= \pm n f(r)\,[1-v(r)]\,, \\ v'(r) &= \pm\frac{g^2\eta}{n}C(r)\left[1-f^2(r)\right]. \end{aligned} \tag{2.5.23}$$

For the solution to have finite energy the functions $f(r)$ and $v(r)$ must approach to 1 at spatial infinity $r\to\infty$, and if we require the solution to be regular at the origin then both functions have to vanish for $r\to 0$.

The gravitino equation (2.5.17) characterizes the profile function of the metric $C(r)$. For the choice of space-time viebein given in (2.5.20) the only non-vanishing components of the spin connection ω_μ^{mn} are[6]:

$$\omega_r{}^{12} = 0\,, \qquad \omega_\theta{}^{12} = -C'(r)\,, \tag{2.5.24}$$

and therefore we write the gravitino equation in polar coordinates reads:

$$\partial_r\epsilon = 0\,, \qquad \left[\partial_\theta - \tfrac{1}{2}C'(r)\gamma_{12} + \tfrac{1}{2}\mathrm{i}A_\theta^B\right]\epsilon_L = 0\,, \tag{2.5.25}$$

where we have used that ω_r^{mn} and A_r^B are both vanishing. In order to solve this equation we choose the supersymmetry generator to have the following angular dependence[7]:

$$\epsilon_L(\theta) = \mathrm{e}^{\mp\frac{1}{2}\mathrm{i}\theta}\epsilon_{0L}. \tag{2.5.26}$$

[6]The components of the spin connection $\omega^{mn} \equiv \omega^m{}_p\eta^{pn}$ can be obtained solving the torsion-free condition $de^m + \omega^m{}_n \wedge e^n = 0$, and $\omega^{mn} = -\omega^{nm}$ [88].

[7]This ansatz will ensure that the in the absence of strings the metric reduces to $\eta_{\mu\nu}$ which represents a Minkowsky space-time.

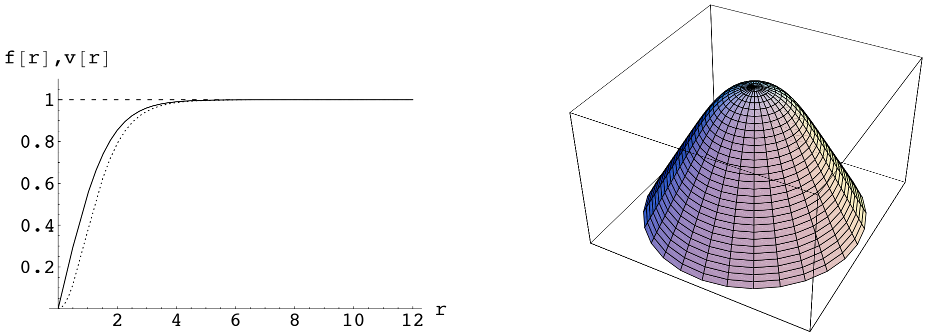

Figure 2.1 *– LEFT: Profile functions (2.5.18) $f(r)$ (solid line) and $v(r)$ (dotted line) characterizing the field configuration of a supersymmetric D-term string, with $n = g = 2\eta = 1$. RIGHT: Embedding of the metric on the plane orthogonal to the string in three dimensions. Far away from the string, which is located at the tip of surface, the metric approaches that of a cone with deficit angle $\Delta = 2\pi n\eta$.*

where ϵ_{0L} is some constant spinor which satisfies the condition (2.5.12). Substituting the ansatz in (2.5.25), we can see that it will vanish provided that the metric profile function $C(r)$ satisfies

$$1 - C'(r) = \pm A_\theta^B \,, \tag{2.5.27}$$

where the expression for the azimutal component of the gravitino U(1) connection, A_θ, can be obtained from (2.5.7) after using the ansatz (2.5.18).

$$A_\theta^B = n\eta f^2(r) + \frac{n}{g} v(r) D = n\eta v(r) + n\,\eta f^2 \left[1 - v(r)\right] . \tag{2.5.28}$$

The supersymmetric cosmic string configurations are the solutions of the system of ordinary differential equations (2.5.23) and (2.5.27). Any solution of these equations is also a solution of the full set of equations of motion, and in particular t-t component of the Einstein equations can be derived from the gravitino equation (2.5.17) [57].

Regular solutions can only be found provided that we choose the upper sign for $n > 0$ and the lower sign for $n < 0$. In figure 2.1 we have plotted a solution to these equation for the particular values of the parameters $n = g = 2\eta = 1$.

Far away from the core of the string $r \to \infty$ equation (2.5.27) reduces to

$$C'(r) = 1 - n\eta \quad \Longrightarrow \quad C(r) \sim (1 - n\eta) r, \tag{2.5.29}$$

and therefore the metric on the plane orthogonal to the string approaches that of a cone

$$\mathrm{d}s^2 \approx -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r(1 - n\eta)\mathrm{d}\theta^2 \,, \quad \text{for } r \to \infty, \tag{2.5.30}$$

with a deficit angle $2\pi n\eta$. With the Plank masses back the deficit angle reads

$$\Delta = 2\pi n\eta M_p^{-2} = 16\pi^2 nG\eta. \tag{2.5.31}$$

2.5.2 String tension

In a theory including gravity, for time independent configurations, we can use the following definition of the energy:

$$E = \int_M \sqrt{det\, g}\left(\frac{M_p^2}{2}R - L_{\text{matter}}\right) + M_p^2 \int_{\partial M} \sqrt{det\, h}\, K\,. \tag{2.5.32}$$

The last term is the Gibbons-Hawking surface term [95], and K is the Gaussian curvature at the boundary (on which the metric is h). Thus, in the Abelian-Higgs model, the energy of the cosmic string is:

$$\begin{aligned}\mu_{\text{string}} \quad &= \quad \int \sqrt{\det g}\ \mathrm{d}r\mathrm{d}\theta\left[D_\mu\phi D^\mu\bar{\phi} + \frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} + \tfrac{1}{2}D^2 + \tfrac{1}{2}R\right] \\ &\quad + \int \mathrm{d}\theta\sqrt{\det h}\ K\bigg|_{r=\infty} - \int \mathrm{d}\theta\sqrt{\det h}\ K\bigg|_{r=0}\,,\end{aligned} \tag{2.5.33}$$

where the sums over μ,ν run only over r,θ. Here the surface term has been evaluated at the boundaries, which are at $r=\infty$ and $r=0$. Following [96] we choose a highly symmetric ansatz for the metric (2.5.19):

$$\sqrt{\det g} = C(r)\,, \qquad \sqrt{\det g}\,R = 2C''\,, \qquad \sqrt{\det h}\,K = -C'. \tag{2.5.34}$$

The symmetries of the metric allow us to rearrange the expression for string tension in similar way as we did in section 1.3.2 for a non-gravitational model. Indeed, using the identities (1.3.26) and substituting the ansatz for the metric (2.5.19), the equation (2.5.33) can be rewritten as follows

$$\begin{aligned}\mu_{\text{string}} \quad &= \quad \int \mathrm{d}r\mathrm{d}\theta\, C(r)\left\{\,|(D_r\phi \pm \mathrm{i}C^{-1}\,D_\theta)\phi|^2 \,+\, \tfrac{1}{2}\left[F_{12} \mp D\right]^2\right\} \,+ \\ &\quad + \quad \int \mathrm{d}r\mathrm{d}\theta\ \left[\partial_r\left(C' \pm A^B_\theta\right) \mp \partial_\theta A^B_r\right] - \int \mathrm{d}\theta\, C'\bigg|_{r=\infty} + \int \mathrm{d}\theta\, C'\bigg|_{r=0}\,,\end{aligned} \tag{2.5.35}$$

In the first line, we can recognise the terms between brackets as the l.h.s. of the BPS equations (2.5.22), and therefore they are vanishing. Moreover, the first term in the second line also vanishes in the string background, due to the gravitino BPS equation (2.5.27). The only surviving contribution to the energy are the two boundary terms, which after using (2.5.27) and (2.5.28) read:

$$\mu_{\text{string}} = 2\pi\left(C'|_{r=0} - C'|_{r=\infty}\right) = \pm 2\pi n\eta\,. \tag{2.5.36}$$

As mentioned in the previous sub section the BPS equations can only solved if we choose the signs in the BPS equations so that $\pm n = |n|$. Thus all the solutions have positive energy according to (2.5.36). Note that we just obtained the same string tension as in the non-supersymmetric model considered in 1.3.2, and thus these supersymmetric strings saturate the same BPS energy bound. If we write the deficit angle in terms of the string tension we also recover the expression we anticipated in section 1.5

$$\Delta = 8\pi G \, \mu_{\text{string}}. \tag{2.5.37}$$

The approach to obtain the string tension that we have reviewed here relies on the possibility to rewrite the energy as a sum of squares, as in the non-gravitational case. However, in order to do it we had to choose a highly symmetric ansatz for the metric, i.e. (2.5.19). There are more sophisticated techniques which make use of a spinorial definition for the total energy, and do not require imposing symmetries on the space-time metric. This technique, developed by Nester [97] and Witten [98], has been used in order to study the stability of cosmic strings solutions in supergravity models in three [99] and four space-time dimensions [100], [101].

2.5.3 F-term strings.

For completeness, let us consider another possibility to construct cosmic string solutions in $\mathcal{N} = 1$ supergravity. The scalar potential of the supersymmetric Abelian-Higgs model discussed in the last section is a pure D-term potential, it has no contribution from the F-terms. We have seen that the Fayet-Ililipoulos term plays an essential role because it is responsible for driving the spontaneous breaking of the gauge symmetry, and the tension of the string is proportional to its value.

If we are to construct local cosmic string solutions we cannot avoid the D-term contribution to the potential, due to the coupling between the Higgs and the gauge boson, however it is possible to construct a model which admits cosmic strings solutions in $\mathcal{N} = 1$ supergravity in the absence of a FI term. This can be done including a $F-$term contribution to the potential to break the gauge symmetry. However, it was proven in [61] that this type of strings, which are called F-term strings, necessarily break all the supersymmetries of the original model. Following [61] we discuss this statement using an example.

Consider a theory with three chiral multiplets involving the scalar fields ϕ_0, ϕ_+ and ϕ_-, with the Kähler potential given by:

$$K = \phi_0 \, \bar{\phi}_0 + \phi_+ \, \bar{\phi}_+ + \phi_- \, \bar{\phi}_-, \tag{2.5.38}$$

thus with trivial kinetic terms, and minimally coupled to a U(1) gauge field A_μ

with charges $g_0 = 0$, $g_+ = g$, and $g_- = -g$ respectively:

$$\delta_{gauge}\phi_0 = 0, \qquad \delta_{gauge}\phi_\pm = \pm \mathrm{i} g \phi_\pm \, \alpha, \qquad D = -g|\phi|^2, \tag{2.5.39}$$

where we have set the FI-term to zero, and we have choosen a constant gauge kinetic function $f(\phi_0, \phi_\pm) = 1$ as in the previous section. Moreover, if we use the superpotential

$$W(\phi_0, \phi_+, \phi_-) = \sqrt{\lambda}\phi_0(\phi_+\phi_- - \eta), \tag{2.5.40}$$

the total scalar potential has a vacuum manifold defined by the conditions

$$\phi_0 = 0, \qquad |\phi_+|^2 = |\phi_-|^2, \qquad \phi_+\phi_- - \eta = 0, \tag{2.5.41}$$

which is isomorphic to S^1, and thus the theory admits topological string solutions. In this case the constant η is what plays the role of the FI-term.

For the special choice of parameters $2\lambda = g^2$ it is possible to find a cosmic string solution with the same tension as the D-term string (2.5.36), and thus saturating the BPS bound. In that case the string configuration must satisfy $|\phi_+|^2 = |\phi_-|^2 = \eta^2$ and $\phi_0 = 0$.

To check that these strings break all the supersymmetries of the model we can look at the gaugino supersymmetry transformations (2.2.24). In our example at the center of the string the fields are in the configuration $\phi_0 = \phi_+ = \phi_- = 0$, and therefore the D auxiliary field is also vanishing. This implies that in order to preserve supersymmetry the following condition must be satisfied at the center of the string:

$$\delta\lambda|_{core} = \tfrac{1}{4}\gamma^{\mu\nu}F_{\ \mu\nu}|_{core}\, \epsilon = 0. \tag{2.5.42}$$

It can be seen that, since there is nothing to compensate the magnetic term, this expression cannot be zero for any non zero value of the supersymmetry parameter, and therefore the string solution breaks all the supersymmetries.

This conclusion is also true for the globally supersymmetric Abelian-Higgs model [54] since the vacuum manifold of the scalar potential has the same topology, and the supersymmetry transformation of the gaugino contains the same terms.

It was shown in [102] that, for certain values of the coupling constants, F-term strings can also be BPS saturated objects in the sense that they preserve half of the supersymmetries of the model. For that particular choice of the parameters the model becomes invariant under a second supersymmetry, and actually the F-term string breaks only half of the $\mathcal{N}=2$ supersymmetries.