Chapter 7

Conclusions

Now that I have presented all results in detail, it is time to review to where this has led us. First, let us reflect on the obtained results.

7.1 Summary of results

In one sentence each, the main chapters can be summed up as follows:

(3) The vortex-unbinding transition causes the demise of current conservation, and is a valid description for systems of any dimension larger than 2;

(4) All electrodynamic phenomena related to Abrikosov vortices are comprised in a single equation for the vortex world sheet;

(5) There is a new state of matter, called type-II Mott insulator, which features quantized vortices of electric current, that is directly accessible in experiment;

(6) The gauge symmetry due to an emerging local conservation law corresponds to a superposition of all possible configurations of topological defects.

7.2 Outlook

These quantitative and qualitative outcomes are of course the main results of this thesis. However, they also provoke thought on some of the deeper principles surrounding ordered systems, gauge symmetry and the universality of vortex-unbinding transitions.

7.2.1 The Landau paradigm

As mentioned in §§2.1,2.2.1, ever since Landau employed an *order parameter* to describe the formation of the superfluid, this has been the prevailing *modus operandi* in the theory of ordered states and (continuous) phase transitions. This sometimes goes under the name of the Landau–Ginzburg–Wilson paradigm. The most important property of the order parameter is that it is a local variable, namely a function on every point in space.

But starting with the Kosterlitz–Thouless transition, several phase transitions have been identified that seem to fall outside of this characterization. Thus, the KT transition is sometimes said to be a phase transition without spontaneous symmetry breaking, or a phase transition of infinite order (instead of second order). It is also often claimed that there is no order parameter for the phase across the transition. The theme of a different kind of order really caught on with the advent of the quantum Hall effect. The distinct quantum Hall states are characterized by a quantum number called the Chern number, which is *topological*, meaning it can only be defined for the system as a whole. From it has evolved the study of topological order, with Xiao-Gang Wen as one of the major pioneers [120]. He argues that also the concept of symmetry groups is too restricted to capture all phase transitions and should be extended to *projective symmetry groups* [121].

To distinguish one ordered state from another it is necessary to define some quantity which will differ for distinct ordered states. The Landau order parameter fulfils this task, but because it is local, it seems not universal enough to capture for instance topological order. It is often said that "topological order is beyond the Landau paradigm". However throughout this thesis and emphasized in chapter 6, we have seen that what is local in one description is extremely non-local in the dual description. For example, the Mott insulator is a vortex condensate, where the density of the vortex liquid is the order parameter that obtains an expectation value, and the associated phase variable is broken spontaneously. From the phase variable point of view, all correlations are lost, but in dual language it is just condensation of a local order parameter. As such, it may well be that the topological orders for which currently no local order parameter can be defined, will be made to do so by a suitable duality transformation. If this turns out to be achievable, then possibly the Landau paradigm will hold up and the intuition-pleasing notion of a local order parameter can survive.

This is also the attitude taken in Hopf symmetry breaking [20–26]. A Hopf algebra or quantum group is a mathematical generalization of an ordinary symmetry group, that treats topological defects and particle excitations on equal footing. Both symmetry breaking and symmetry restoration by defect condensation are contained within this formalism. This should be fertile ground for further explorations of these matters.

7.2.2 Quantum liquid crystals

As mentioned in the introduction §1.2, the original topic of this thesis was to be the quantum liquid crystals. Next to the correspondence between quantum nematics and linearized gravity (§6.2), electronic quantum liquid crystals have received much attention lately, because they seem to be present in the pseudogap phase of underdoped cuprates (§5.1.2). Therefore, they make up an interesting and relevant topic in its own right.

The difference between classical and quantum liquid crystals is that the latter can have superpositions of Burgers vector orientations (§6.2.1) as defect condensates. Therefore it admits many more ground states than the classical varieties. This leaves room for surprises, and it would be very useful to have a complete mathematical classification of all possible quantum condensates, analogous to the group symmetry scheme for regular ordered states. There are 17 so-called wallpaper groups of infinite tilings of the spatial plane. These would correspond to all possible two-dimensional crystals (quantum crystals are 2+1 dimensional and do not suffer from the Mermin–Wagner theorem). Melting such crystals by dislocation proliferation leads to the corresponding quantum liquid crystals.

The Hopf algebra or quantum group formalism advertises its ability to perform this classification [23–25]. It is well suited to handle the non-commutativity present in the marriage between translations and rotations, and also automatically provides the classification of topological defects in each liquid crystal phase. The downside is that there is no recipe to list all possible inequivalent quantum superposition condensates, so they must be guessed and written out by hand. There may be also some mathematical intricacies related to the infiniteness of the lattice groups (cf. Ref. [122]).

Carrying this out for all condensates of all 17 groups seems a daunting task. My advice would be to start with the square and triangular lattice (p4m and p6m in crystallography language) which seem most relevant to experiment.

7.2.3 Vortex duality and fermions

All of this work so far has concerned bosonic physics only. Even the superconductor was treated exclusively as a Bose condensate of Cooper pairs. The main reason is that fermions are in fact much harder to describe theoretically, all of it related to their minus signs (see e.g. Ref. [123]). Still, given the very general considerations presented here, and the relation between disorder in the real and order in the dual variables stressed in chapter 6, one cannot help but think that the defect-mediated melting should prevail also in systems other than purely bosonic.

There are several ways that may indeed establish vortex condensation in fermionic physics. One way could be to somehow to impose the Pauli-exclusion interactions as additional constraints in the path integral. This had led to some interesting insights into quantum criticality [124], but does not reveal how to continue this line of thought. Another possibility may be to let the vortices in a bosonic (phase) field take care of the phase transition, and wire in the fermionic physics by a separate particle species. This is the approach take in the many slave-particle models available on the market, see e.g. Ref. [58]. Fascinatingly, a very recent work literally employs the vortex duality in the *fermionic* Mott insulator by mode expansion, where only the $k = 0$ mode relates to the vortex condensate and all finite-momentum modes concern the fermionic spin dynamics [94]. It would also be interesting to see whether such principles may be applied to the topological Mott insulators that are in vogue these days [125, 126].

7.2.4 Quantum vs. classical

We conclude with perhaps the most fundamental question in the realm of condensed matter physics: Where is the border between classical and quantum phenomena, if it exists at all?

It is certainly not true that quantum mechanics only manifests is itself at energies $\hbar/\tau > k_{\mathrm{B}}T$, where τ is some characteristic time scale. Not only is for instance light an inherently quantum mechanical wave/particle, but all materials in everyday life only exist because of quantum mechanics. What I mean is that it is not enough to have atoms + Coulomb interaction + Pauli exclusion. A crystal is rigid because of spontaneous symmetry breaking, leading to long-range correlations communicated by Goldstone modes. Even disordered systems such as glasses are dominated by this effect on other length scales. Only truly simple liquids such as liquid nitrogen seem to fall outside of this world, and even there Van der Waals interactions are quantum mechanical in origin.

In fact, a superconductor, often referred to as a macroscopic quantum system, is not any different in this respect from a crystal. It is just an internal instead of an external (spacetime) symmetry that is spontaneously broken. Again I take the point of view that *any* ordered system must be some sort of condensate, and as such must be distinguishable from a completely unordered system, presumably by a symmetry property. Thus a superconductor is at the basic level as classical as a crystal, or a crystal is as quantum mechanical as a superconductor, whichever one prefers. The Goldstone mode conveys the rigidity of the condensate. Spontaneous symmetry breaking is caused by coupling to an enormous amount of just-above-zero-energy states called the thin spectrum, and is therefore unavoidable in many-body physics (see e.g. [127, 128]).

We touched upon this theme in the very last paragraph of §6.3: can we distinguish the 'magnetic monopole' excitations in spin ice from true Dirac monopoles? In fact, the experiments available at the moment cannot resolve the difference between time-averaged and ensemble-averaged results, and therefore see only the 'classical' consequences of the monopoles. This once more suggests that the specialities of quantum mechanics lie in the role of time. In the condensates this is not noticeable. We may conjecture that the true dividing line between classical media, including superconductors, and quantum stuff would manifest itself in some way by the role of time. Perhaps whenever Wick rotation to imaginary time is possible, the 'true' quantum features are obscured, or irrelevant for the final outcome.

Surely, in high-energy particle physics Wick rotation is part of the standard toolbox, and those processes are definitely purely quantum mechanical. Or are they not? One could argue that the calculation of the outcomes of a scattering process from the infinite past to the infinite future is just as classical a result as the propagation sound waves in a crystal, in the sense that the special properties of time play no role. If this is so—and I cannot at all claim to have in any way established or even corroborated it—then only things like anomalies are ‘truly’ quantum.

The classicalness of all condensates also underlies much confusion surrounding the quantum measurement problem or “collapse of the wave function”. Nobody disputes that the statistical interpretation of quantum mechanics, where the probability amplitudes are meaningful in many repetitions of the same experiment, is extremely accurate. This is the ensemble average. But what happens for each individual experiment and whether quantum mechanics has anything to say about this, is still unclear. The problem arises in how to couple a single particle like a free electron to a condensate such as a photographic plate. The ground state wave function of the condensate is as classical as can be, and not only does the quantum information of the electron get lost in the huge amount of degrees of freedom, the fact that spontaneous symmetry breaking has already taken place prevents a straightforward coupling of the quantum particle to the classical system. A very interesting proposal is that gravity prevents the quantum superposition of a heavy enough object by the mismatch of deformed spacetimes [129]. Moreover, the deeper mechanism behind this seems to be the special role of time [130].

It is my opinion that the mysteries of the peculiar role of time are largely unresolved. It has not only bearing on things like quantum gravity, but definitely also on the divide between quantumness and classicalness in the everyday world around us.