Chapter 4

Electrodynamics of Abrikosov vortices

Having learned that Bose-Mott insulators near the quantum phase transition support vortex excitations, we would like to study those objects in electrically charged systems: the superconductor to charged Bose-Mott insulator transition. But first it is necessary to fully understand how the electromagnetic field comes into play, and therefore this chapter is dedicated to the vortex world sheet formalism in the superconducting state. Here the topological defects are of course the well-studied Abrikosov vortices we encountered in §2.1.2. It will prove to be an interesting subject in its own right.

The study of the matter formed from Abrikosov vortices in type-II superconductors constitutes a vast and mature research subject. This subject is crucial for the technological applications of superconductivity [67] but it has also proven to be a fertile source for fundamental condensed matter physics research. The elastic and hydrodynamical properties of matter formed from vortices can be very easily tuned by external means and it has demonstrated to be an exceedingly fertile model system to study generic questions regarding crystallization, the effects of background quenched disorder and so forth [68, 69]. Especially after the discovery of the cuprate high-T_c superconductors it became also possible to study the fluids formed from vortices. Because of the strongly two-dimensional nature of the superconductivity in the cuprates, the Abrikosov vortex lattice becomes particularly soft and it melts easily due to thermal motions at temperatures that are much below the mean field H_{c2}-line [70].

Many phenomena in this field are of a dynamical nature, associated with the fact that vortices are in motion. This includes the vortex flow, the magnetic field penetration and the flux creep, but also the large Nernst effect of the vortex fluid and, perhaps most spectacularly, the use of cuprate vortices as source of terahertz radiation [71, 72]. This vortex dynamics is analogous to the magnetohydrodynamics of electrically charged plasmas in the sense that the forces exerted on vortices are exclusively of electromagnetic origin, while in turn the vortex matter backreacts on the electromagnetic fields. The phenomena that arise are rather thoroughly understood starting from the AC and DC Josephson relations as well as the Maxwell equations as the force equations in this "vortex magnetohydrodynamics".

Although the computations explaining these phenomena are certainly correct, they are of a rather improvised, ad hoc nature, at least compared to the Landau–Lifshitz style [73] of deriving the usual magnetohydrodynamics from first principles. In this chapter we show that with the use of the vortex world sheets in 3+1 dimensional spacetime, all of the phenomena related to the electrodynamics of vortices in superconductors can be captured in one concise equation. Furthermore the electrodynamics of stringlike objects in the absence of monopole sources has very special features, turning the Maxwell field strength itself into a gauge field.

We shall show quickly how the vortex world sheet current arises in the relativistic Ginzburg–Landau model. Then we take a small theoretical detour to explore the electrodynamics of two-form sources in general. After that the rigorous vortex duality is derived for charged superfluids, and finally we shall present the equations of motion that contain all the electrodynamical phenomena related to moving Abrikosov vortices. We conclude with a short outlook.

4.1 The vortex world sheet in relativistic superconductors

We will now show how the vortex world sheet appears from the Ginzburg–Landau equations. In §4.2, we shall derive the more generic coupling of a vortex current to electromagnetic fields.

Before we write down the partition function let us stress that it may be less familiar to researchers in the field of superconductivity, since it will be fully relativistic. In particular it will have a squared time-derivative, whereas most works start with a single time-derivative term. The latter applies to systems which are diffusion-limited. Of course, in actual superconductors vortices are accompanied by such diffusion processes. However, the relativistic action is necessary to derive the vortex world sheet. Furthermore processes such as Thomas–Fermi screening are in fact ballistic. Finally the validity of this relativistic approach is verified by the results of §4.4. If one wishes to consider diffusion processes, an appropriate term can be added to the Lagrangian at will.

In this chapter we find it convenient to stay in real time, because we do not need to carry out the vortex proliferation, and because we can compare directly to other known results. The partition function associated with the relativistic Ginzburg–Landau action deep in the superconducting state is (cf. §§2.1.2,2.4.7),

$$Z = \int \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}/\hbar \int \mathrm{d}^4x\, \mathscr{L}}, \tag{4.1}$$

$$\mathscr{L} = -\frac{1}{4\mu_0} F_{\mu\nu}^2 - \frac{\hbar^2}{2m^*}\rho_\mathrm{s}(\partial_\mu^\mathrm{ph}\varphi - \frac{e^*}{\hbar}A_\mu^\mathrm{ph})^2. \tag{4.2}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength; $\mathscr{F}(A_\mu)$ denotes an appropriate gauge fixing condition; φ is the superconducting phase related to the order parameter $\Psi = \sqrt{\rho_\mathrm{s}}\mathrm{e}^{\mathrm{i}\varphi}$; ρ_s is the superfluid density; m^* and e^* are the mass and charge of a Cooper pair; and most importantly, one must take great care to differentiate between the two velocities in the problem, namely the velocity of light c pertaining to the photon field A_μ, and the phase velocity in the superconductor c_ph. Therefore we have defined $\partial_\mu = (\partial_0, \nabla)$, $\partial_0 = \frac{1}{c}\partial_t$ and $\partial_\mu^\mathrm{ph} = (\partial_0^\mathrm{ph}, \nabla)$, $\partial_0^\mathrm{ph} = \frac{1}{c_\mathrm{ph}}\partial_t$. Furthermore $A_\mu = (-\frac{1}{c}V, \mathbf{A})$ and $A_\mu^\mathrm{ph} = (-\frac{1}{c_\mathrm{ph}}V, \mathbf{A})$. The last form is dictated by gauge invariance of the second term in Eq. (4.2).

We shall for the moment proceed in the relativistic limit where $c_\mathrm{ph} = c$, for simplicity. The equations of motion then follow from variation with respect to A_ν,

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathscr{L}}{\partial A_\nu} = -\frac{1}{\mu_0}\partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{\hbar^2}{m^*}\rho_\mathrm{s}\frac{e^*}{\hbar}(\partial_\nu\varphi - \frac{e^*}{\hbar}A_\nu) = 0. \tag{4.3}$$

Now we act with $\epsilon_{\kappa\lambda\rho\nu}\partial_\rho$ on this equation, which leads to,

$$-\lambda^2(\epsilon_{\kappa\lambda\rho\nu}\partial_\mu^2\partial_\rho A_\nu - \cancel{\epsilon_{\kappa\lambda\rho\nu}\partial_\rho\partial_\nu}\partial_\mu A_\mu) + \epsilon_{\kappa\lambda\rho\nu}\partial_\rho A_\nu = \frac{\hbar}{e^*}\epsilon_{\kappa\lambda\rho\nu}\partial_\rho\partial_\nu\varphi = \frac{\hbar}{e^*}J^\mathrm{V}_{\kappa\lambda}. \tag{4.4}$$

Here we have defined the London penetration depth $\lambda = \sqrt{\frac{m^*}{\mu_0 e^{*2}\rho_\mathrm{s}}}$; the second term vanishes because the antisymmetric contraction of two derivatives; and on the right-hand side we recognize from Eq. (2.17) the definition of the vortex current $J^{\mathrm{V}}_{\kappa\lambda}$. Let us consider the special case $\kappa = t$, and use the definition of the magnetic field $B_l = \epsilon_{lrn}\partial_r A_n$,

$$-\lambda^2\partial_\mu^2 B_l + B_l = \frac{\hbar}{e^*}J^{\mathrm{V}}_{tl} = \frac{\hbar}{e^*}2\pi N\delta_l^{(2)}(\mathbf{x}). \tag{4.5}$$

Here we have used Eq. (2.17) in the last equality. This is precisely the textbook equation for the Meissner screening of a vortex source of strength N, with flux quantum $\Phi_0 = 2\pi\hbar/e^*$ Eq. (2.6), [51, eq.(5.10)]. But instead of ad hoc inserting the delta-function source, we actually derived it from the singular phase field. The only difference is that here also the dynamics is taken into account via the double time derivative contained in ∂_μ^2. The true power of the vortex world sheet shows itself when considering the electric field $\mathbf{E} = -\nabla A_0 - \partial_t\mathbf{A}$ and the spatial components J^{V}_{kl} of the vortex field. This will be further elaborated on in §4.4. But let us first analyze how two-form sources couple to electromagnetism in general, followed by a more general derivation of the above relations invoking a duality mapping, by which we can treat the vortex fields in the action itself, rather than only in the equations of motion. This can be regarded as revealing the more fundamental structure of the problem. The reader who is less interested in these theoretical matters may skip ahead directly to §4.4.

4.2 Electrodynamics of two-form sources

We will formulate here the generalization of the standard Maxwell action and equations of motion when the sources are not monopoles with charge density ρ and current J_m, collected in a vector field $J_\mu = (c\rho, J_m)$, but instead (vortex) lines with line densities J_{tl} and line currents J_{kl} (which denote the current in direction k of a line that extends in direction l), collected in a two-form field $J_{\kappa\lambda} = (J_{tl}, J_{kl})$. Let us first recall the established knowledge for ordinary electromagnetism, in terms suited for this generalization. For clarity we again use a shorthand notation where we are intentionally sloppy with contra- and covariant indices, leaving out dimensionful parameters in order to maximally expose the principles. In the next section we will present the final results that are accurate in this regard.

4.2.1 Maxwell action with monopole sources

Let us start by considering a set of electrical monopole sources collected in a source field J_μ as in the above, satisfying a continuity equation/conservation law $\partial_\mu J_\mu = 0$. These sources interact via the exchange of gauge particles, as gauge fields A_μ that couple locally to the source fields, by an interaction term in the Lagrangian of the form $A_\mu J_\mu$. Because of current conservation, any transformation of the gauge field $A_\mu \to A_\mu + \partial_\mu \varepsilon$, where ε is any smooth scalar field, will leave the coupling term invariant. Indeed,

$$A_\mu J_\mu \to A_\mu J_\mu + (\partial_\mu \varepsilon) J_\mu = A_\mu J_\mu - \varepsilon \partial_\mu J_\mu = A_\mu J_\mu. \tag{4.6}$$

Here we performed partial integration in the second step. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is also invariant under the same gauge transformation. An immediate consequence of this definition are the Bianchi identities or homogeneous Maxwell equations,

$$\epsilon_{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu} \partial_\beta \partial_\mu A_\nu = 0, \tag{4.7}$$

because the derivatives commute. These equations comprise $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$. This suggests a Lagrangian of gauge invariant terms,

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + A_\mu J_\mu, \tag{4.8}$$

accompanied by the Euler–Lagrange equations of motion obtained by variation with respect to A_ν,

$$\partial_\mu F_{\mu\nu} = -J_\nu. \tag{4.9}$$

These are the inhomogeneous Maxwell equations comprising $\nabla \cdot \mathbf{E} = \rho$ and $\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}$. In a superconductor, one must also add a Meissner term, which in the unitary gauge fix turns into a mass term for the gauge field A_μ,

$$\mathscr{L}_{\text{Maxwell + Meissner}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} A_\mu A_\mu + A_\mu J_\mu, \tag{4.10}$$

In this form, the Meissner term breaks the gauge invariance of the Lagrangian. This corresponds to releasing the longitudinal degrees of freedom of the photon field. A gauge equivalent perspective is that this degree of freedom represents the phase mode of the superconducting condensate (see §3.2.2). The equation of motion is modified to,

$$\partial_\mu F_{\mu\nu} - A_\nu = -J_\nu. \tag{4.11}$$

4.2.2 General two-form sources

Let us now repeat this procedure for antisymmetric two-form sources $J_{\kappa\lambda} = (J_{tl}, J_{kl})$. These must obey the continuity equations (i.e. conservation laws) $\partial_\kappa J_{\kappa\lambda} = 0$, reflecting that the density of the source can only increase (decrease) when it flows into (out of) the region under consideration, and that vortex lines cannot end within in the system (no monopoles). Consider now that these sources interact by exchanging two-form gauge fields, that we will tentatively denote by $G_{\kappa\lambda}$. Then these gauge fields couple locally to the sources as $G_{\kappa\lambda}J_{\kappa\lambda}$. These fields have to transform under gauge transformations as,

$$G_{\kappa\lambda} \to G_{\kappa\lambda} + \frac{1}{2}(\partial_\kappa \varepsilon_\lambda - \partial_\lambda \varepsilon_\kappa), \tag{4.12}$$

where ε_λ is any smooth vector field, in order to leave the coupling term invariant as required by the current conservation. Indeed,

$$G_{\kappa\lambda}J_{\kappa\lambda} \to G_{\kappa\lambda}J_{\kappa\lambda} + (\partial_\kappa \varepsilon_\lambda)J_{\kappa\lambda} = G_{\kappa\lambda}J_{\kappa\lambda} - \varepsilon_\lambda \partial_\kappa J_{\kappa\lambda} = G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.13}$$

Here we have used the antisymmetry of $J_{\kappa\lambda}$ in the first step, and partial integration in the second. The field strength $H_{\mu\kappa\lambda} = \partial_{[\mu}G_{\kappa\lambda]} = \partial_\mu G_{\kappa\lambda} + \partial_\lambda G_{\mu\kappa} + \partial_\kappa G_{\lambda\mu}$ is also invariant under these gauge transformations. An immediate consequence of this definition is the Bianchi identity,

$$\epsilon_{\nu\mu\kappa\lambda}\partial_\nu H_{\mu\kappa\lambda} = \partial_{[\nu}\partial_\mu G_{\kappa\lambda]} = 0, \tag{4.14}$$

because the derivatives commute. With these definitions, we can write down a gauge invariant Lagrangian,

$$\mathscr{L} = -\frac{1}{12}H^2_{\mu\kappa\lambda} + G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.15}$$

Note that this Lagrangian is in terms of the dynamic variables $G_{\kappa\lambda}$, which we will see later is the dual of the electromagnetic field strength $F_{\mu\nu}$. In other words, this Lagrangian is in terms of the electric and magnetic fields themselves, rather than the gauge potential A_μ. The equations of motion follow after variation with respect to $G_{\kappa\lambda}$,

$$\partial_\mu H_{\mu\kappa\lambda} = -J_{\kappa\lambda}. \tag{4.16}$$

Now, in a gauge-invariance breaking medium such as a superconductor, one must add a Higgs or Meissner term to the Lagrangian as,

$$\mathscr{L} = -\frac{1}{12}H^2_{\mu\kappa\lambda} - \frac{1}{4}G^2_{\kappa\lambda} + G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.17}$$

4.2.3 Abrikosov vortex sources

Up to now we have just reviewed the standard derivation of non-compact $U(1)$ two-form gauge theory. Let us now specialize to the case of a vortex line in a superconductor. For such an Abrikosov vortex, we know that the density J^{V}_{tl} is proportional to the magnetic field, and that the magnetic field is parallel to the spatial orientation of the vortex line. In fact, when the magnetic field intensity coincides with the lower critical field H_{c1}, the dimensionful vortex density may be denoted as before, Eq. (4.5),

$$J^{\mathrm{V}}_{tl} = \Phi_0 \delta^{(2)}_l(\mathbf{r}), \tag{4.18}$$

where Φ_0 is the flux quantum $\frac{h}{e^*}$. Because of these considerations, the vortex line density should couple to the magnetic field B_l. The definition of the Maxwell field strength is,

$$F_{tn} = E_n \qquad\qquad F_{mn} = \epsilon_{mnl} B_l, \tag{4.19}$$

If we contract the last definition with $\sum_{mn} \epsilon_{tbmn}$, one finds $B_l = \epsilon_{tlmn} F_{mn} \equiv G_{tl}$. Here we introduce the Hodge dual of the Maxwell field strength $G_{\alpha\beta} \equiv \frac{1}{2}\epsilon_{\alpha\beta\mu\nu} F_{\mu\nu}$. Then the coupling of the vortex line density J^{V}_{tl} to the magnetic field B_l is written as $G_{tl} J^{\mathrm{V}}_{tl}$ and generalizes to $G_{\kappa\lambda} J^{\mathrm{V}}_{\kappa\lambda}$. Therefore, the general two-form gauge field in Eq. (4.15) is now identified as the dual Maxwell field strength $G_{\kappa\lambda}$.

4.2.4 Gauge freedom of the field strength

This leads immediately to an astonishing consequence: the Maxwell field strength $F_{\mu\nu}$ itself *has now become a gauge field* ! The gauge transformations Eq. (4.12) correspond to,

$$F_{\mu\nu} \to F_{\mu\nu} + \epsilon_{\mu\nu\kappa\lambda} \partial_\kappa \varepsilon_\lambda. \tag{4.20}$$

How does it come about that these all too physical $F_{\mu\nu}$'s have suddenly turned into gauge variant quantities? The reason is simple although perhaps defeating the physical intuition: in normal matter we always have electric monopole sources J_ν with the associated equations of motion $\partial_\mu F_{\mu\nu} = -J_\nu$. In the absence of any such sources, these equations reduce to $\partial_\mu F_{\mu\nu} = 0$. Together with the inhomogeneous Maxwell equations $\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0$, these imply that the field strength cannot be measured at all. It amounts to the

Schwinger wisdom that fields which cannot be sourced do not have physical reality [62]. The formal expression of this fact is that *the field strength becomes pure gauge in the absence of monopole sources*.

Another insight is obtained by taking a closer look at the gauge transformations Eq. (4.20). For the Bianchi identities in Eq. (4.7) these imply,

$$\begin{aligned}\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} &\to \epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} + \epsilon_{\alpha\beta\mu\nu}\partial_\beta\epsilon_{\mu\nu\kappa\lambda}\partial_\kappa\varepsilon_\lambda \\ &= \epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} + (\partial_\alpha\partial_\lambda - \partial^2\delta_{\alpha\lambda})\varepsilon_\lambda. \end{aligned} \tag{4.21}$$

In other words, the Bianchi identities are not invariant under these gauge transformations! This makes sense: these identities are a direct result of expressing the field strength in terms of a gauge potential A_ν, which of itself has three degrees of freedom (four minus one gauge freedom). The Bianchi identities serve to restrict the six degrees of freedom contained in $F_{\mu\nu}$ to the proper number of three[1]. Conversely, in the derivation of the two-form action Eq. (4.15), we have not assumed anything about the origin of the two-form field. Next to three physical degrees of freedom, there are three gauge degrees of freedom. Therefore the constraints $\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0$ are not strictly enforced, but can always be obtained by a suitable gauge transformation.

We never observe the gauge character of the fields $F_{\mu\nu}$ themselves because the only two-form sources to which this action applies that we know of are Abrikosov vortices in a superconductor. The superconducting matter causes a finite penetration depth λ for the fields, which is reflected by the addition of a Meissner term to the Lagrangian. The gauge-invariant form of this term is known to be,

$$H_{\kappa\lambda\mu}\frac{1}{\partial^2}H_{\kappa\lambda\mu} = -G_{\kappa\lambda}\frac{\delta_{\kappa\mu}\partial^2 - \partial_\kappa\partial_\mu}{\partial^2}G_{\kappa\lambda}, \tag{4.22}$$

in the same way as one can formally write the Meissner term in Eq. (4.10) as $F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$ [cf. Eq. (3.23)]. However, since the longitudinal components of $G_{\kappa\lambda}$ are not sourced by the conserved Abrikosov vortices, we are naturally led to the Lorenz gauge condition $\partial_\kappa G_{\kappa\lambda} = 0$, and the gauge freedom has been removed. With this gauge condition Eq. (4.22) reduces to $G_{\kappa\lambda}G_{\kappa\lambda}$, that appears in Eq. (4.17). In other words, the superconducting medium forces us to the fixed frame action Eq. (4.17).

[1] In light of the discussion in §3.A, we refer here to the general case for the field strength, without restricting to a particular action. Surely a massless photon field has only two propagating degrees of freedom, but that follows only after ascertaining the Maxwell action.

4.2.5 Vortex equation of motion

We end up with the action Eq. (4.17), and we now put in dimensionful parameters. Please note that this action is equivalent to the regular action as (4.2), but with the important difference that here we work with the dual field strength $G_{\kappa\lambda}$ as the dynamic variable instead of the gauge potential A_μ .The equations of motion ("Maxwell equations for relativistic vortices") are now obtained straightforwardly by varying with respect to $G_{\kappa\lambda}$ as,

$$\lambda^2(\partial^2 G_{\mu\nu} - \partial_\mu\partial_\kappa G_{\kappa\nu} + \partial_\nu\partial_\kappa G_{\kappa\mu}) - G_{\mu\nu} = -\frac{\hbar}{e^*} J^{\mathrm{V}}_{\mu\nu}. \tag{4.23}$$

This is to be compared with Eq. (4.11) and Eq. (4.5). The second and third term can be set to zero by a gauge transformation Eq. (4.20) or alternatively by invoking the Bianchi identities Eq. (4.7). The meaning of these equations is that the two-form source $J^{\mathrm{V}}_{\kappa\lambda}$, causes an electromagnetic field $G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda}$ that is now Meissner screened over a length scale λ. The case $\mu = t$, together with the definition $B_n = \frac{1}{2}\epsilon_{nab}F_{ab}$ reduces to Eq. (4.5).

4.2.6 Summary

Summarizing, we have shown here that the action Eq. (4.17) can be postulated, from which the correct equations of motion as introduced in §4.1 directly follow, without ever mentioning the gauge potential A_μ. One trades in the Bianchi identities Eq. (4.7) for a set of gauge transformations Eq. (4.20). This action is only meaningful in the absence of monopole sources, but is very appropriate when considering two-form sources such as Abrikosov vortices. In the case that the penetration depth λ becomes infinitely large, the field strength $F_{\mu\nu}$ recovers its status as a gauge field. This would correspond to the Coulomb phase of two-form sources, as opposed to the Higgs phase that is always realized in superconductors.

As a final note it should be stressed, that although the vortex source is intrinsically dipolar in nature, the equations stated above are not generally valid for any dipole source. Here, the direction of the vortex line is always parallel to the dipole moment. If one should instead consider for instance a string of ferromagnetic material with moments not along the string, one must revert to the omnipotent regular Maxwell equations.

For the reader familiar with differential forms, I have included appendix 4.A repeating these considerations in metric-independent language, valid in any spatial dimension higher than 2.

4.3 Vortex duality in charged superfluids

We shall now rigorously derive the coupling of Abrikosov vortex sources to the electromagnetic fields, starting from the action describing a superconductor in 3+1 dimensions. This follows the same pattern as the uncharged superfluid of chapter 3, extended by minimally coupling in the electromagnetic field. For 2+1 dimensions this was done in §2.4.7. We end up with an effective action describing the electrodynamics of vortices.

4.3.1 Dual Ginzburg–Landau action

Our starting point is the partition function Eq. (4.1). To keep the equations readable, we will transform to dimensionless units denoted by a prime (which we suppress when matters are unambiguous),

$$S' = \frac{1}{\hbar}S,\ x'_m = \frac{1}{a}x_m,\ t' = \frac{c}{a}t,\ A'_\mu = \frac{ae^*}{\hbar}A_\mu,\ \rho' = \frac{\hbar a^2}{m^* c}\rho_{\mathrm{s}},\ \frac{1}{\mu'} = \frac{\hbar}{\mu_0 c e^{*2}}. \tag{4.24}$$

Here a is a length scale relevant in the system, for instance the lattice constant. We will assume the relativistic limit $c_{\mathrm{ph}} = c$; later we shall return to dimensionful quantities and it will become clear that the phase velocity is playing an essential role for the description of the non-relativistic vortices. The partition function in these dimensionless units reads,

$$Z = \int \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\ \mathscr{L}}, \tag{4.25}$$

$$\mathscr{L} = -\frac{1}{4\mu}F_{\mu\nu}^2 - \frac{1}{2}\rho(\partial_\mu\varphi - A_\mu)^2. \tag{4.26}$$

Now we perform the dualization procedure. A Hubbard–Stratonovich transformation of Eq. (4.25) leads to,

$$Z = \int \mathscr{D}w_\mu \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.27}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu}F_{\mu\nu}^2 + \frac{1}{2\rho}w^\mu w_\mu - w^\mu(\partial_\mu\varphi - A_\mu). \tag{4.28}$$

Here w_μ is the auxiliary variable in the transformation, but it is actually the canonical momentum related to the velocity $\partial_\mu\varphi$, which can be found as,

$$w_\mu = -\frac{\partial\mathscr{L}}{\partial(\partial^\mu\varphi)} = \rho(\partial_\mu\varphi - A_\mu), \tag{4.29}$$

and is related to the supercurrent as $w_\mu = \frac{e^*}{\hbar} J_\mu$. If one integrates out the field w_μ from Eq. (4.27), one retrieves Eq. (4.25). In the presence of Abrikosov vortices, the superconductor phase φ is no longer everywhere single-valued. Therefore it is separated into smooth and multivalued parts $\varphi = \varphi_{\mathrm{smooth}} + \varphi_{\mathrm{MV}}$. The smooth part can be partially integrated yielding,

$$Z = \int \mathscr{D} w_\mu \mathscr{D}\varphi_{\mathrm{smooth}} \mathscr{D}\varphi_{\mathrm{MV}} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.30}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} w^\mu w_\mu + \varphi_{\mathrm{smooth}} \partial_\mu w^\mu - w^\mu \partial_\mu \varphi_{\mathrm{MV}} + w^\mu A_\mu. \tag{4.31}$$

Notice that the photon field is wired in just by coupling to the supercurrent. The smooth part can now be integrated out as a Lagrange multiplier turning into the constraint $\partial_\mu w^\mu = 0$, the supercurrent continuity equation. This constraint can be explicitly enforced by expressing w^μ as the curl of a gauge field,

$$w^\mu = \epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda}. \tag{4.32}$$

4.3.2 Abrikosov vortex world sheets

We can now substitute this expression in the partition function; the integral over the fields w_μ is replaced by one over $b_{\kappa\lambda}$, as long as we apply a gauge fixing term $\mathscr{F}(b_{\kappa\lambda})$ to take care of the redundant degrees of freedom. Since the gauge field is smooth it can be partially integrated to give,

$$Z = \int \mathscr{D}\varphi_{\mathrm{MV}} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda}) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.33}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} (\epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda})^2 - b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu \partial_\mu \varphi_{\mathrm{MV}} + b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu A_\mu. \tag{4.34}$$

Here we recognize the definition Eq. (2.17) of the vortex source,

$$J^{\mathrm{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\nu\mu} \partial^\nu \partial^\mu \varphi_{\mathrm{MV}}, \tag{4.35}$$

and we have derived the dual partition function,

$$Z = \int \mathscr{D} J^{\mathrm{V}}_{\kappa\lambda} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda}) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.36}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} (\epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda})^2 - b^{\kappa\lambda} J^{\mathrm{V}}_{\kappa\lambda} + b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu A_\mu. \tag{4.37}$$

The interpretation is as follows. The vortex sources $J^{\mathrm{V}}_{\kappa\lambda}$ interact through the exchange of dual gauge particles $b_{\kappa\lambda}$ coding for the long range vortex–vortex interactions mediated by the condensate. The gauge field $b_{\kappa\lambda}$ couples as well to the electromagnetic field A_μ. Integrating out the electromagnetic field will lead to a Meissner/Higgs term $\sim b^2_{\kappa\lambda}$, showing that the interaction between vortices is actually short-ranged in the superconductor. However, we are instead interested in how the electromagnetic field couples to the vortices themselves. Therefore, we shall integrate out the dual gauge field $b_{\kappa\lambda}$.

The first step is to complete the square in $b_{\kappa\lambda}$. The kinetic term for $b_{\kappa\lambda}$ is proportional to,

$$-b_{\kappa\lambda}\epsilon^{\kappa\lambda\mu\nu}\partial_\nu\epsilon_{\rho\sigma\alpha\mu}\partial^\alpha b^{\rho\sigma} = -b_{\kappa\lambda}(\delta^{\kappa\mu}\partial^2 - \partial^\kappa\partial^\mu)b_{\mu\lambda} \equiv -b_{\kappa\lambda}\mathscr{G}_0^{-1\kappa\mu}b_{\mu\lambda}. \tag{4.38}$$

Here $\mathscr{G}_0^{-1\kappa\mu}$ is the inverse propagator. However, this expression cannot be inverted (the same problem arises in the quantization of the photon field). We can solve this by imposing the Lorenz gauge condition $\partial^\kappa b_{\kappa\lambda} = 0$. Then the inverse propagator is simply $\mathscr{G}_0^{-1\kappa\mu} = \delta^{\kappa\mu}\partial^2$, and its inverse is $\mathscr{G}_{0\kappa\mu} = \delta_{\kappa\mu}\frac{1}{\partial^2}$. Now we can complete the square,

$$\begin{aligned}\mathscr{L}_{\text{dual}} = {} & \frac{1}{2}\big(b_{\kappa\lambda} - \frac{\rho}{\partial^2}J^{\text{V}}_{\kappa\lambda} + \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\big)\big(-\frac{\partial^2}{\rho}\big)\big(b^{\kappa\lambda} - \frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} + \epsilon^{\kappa\lambda\rho\sigma}\partial_\rho A_\sigma\big)\\ & - \frac{1}{2}\big(-J^{\text{V}}_{\kappa\lambda} + \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\big)\big(-\frac{\rho}{\partial^2}\big)\big(-J^{\text{V}\kappa\lambda} + \epsilon^{\kappa\lambda\nu\mu}\partial_\nu A_\mu\big) - \frac{1}{4\mu}F^2_{\mu\nu}.\end{aligned} \tag{4.39}$$

Then we shift the field $b_{\kappa\lambda} \to b_{\kappa\lambda} + \frac{\rho}{\partial^2}J^{\text{V}}_{\kappa\lambda} - \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu$ and integrate it out in the path integral to leave an unimportant constant factor. Expanding the remaining terms leads to,

$$\begin{aligned}\mathscr{L}_{\text{dual}} &= \frac{1}{2}J^{\text{V}}_{\kappa\lambda}\frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} + \frac{1}{2}\epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\frac{\rho}{\partial^2}\epsilon^{\kappa\lambda\rho\sigma}\partial_\rho A_\sigma - \rho J^{\text{V}}_{\kappa\lambda}\epsilon^{\kappa\lambda\nu\mu}\frac{\partial_\nu}{\partial^2}A_\mu - \frac{1}{4\mu}F^2_{\mu\nu}\\ &= \frac{1}{2}J^{\text{V}}_{\kappa\lambda}\frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} - \frac{1}{2}\rho A^\mu A_\mu - \rho J^{\text{V}}_{\kappa\lambda}\epsilon^{\kappa\lambda\nu\mu}\frac{\partial_\nu}{\partial^2}A_\mu - \frac{1}{4\mu}F^2_{\mu\nu}.\end{aligned} \tag{4.40}$$

In going to the second line we have performed partial integration on the second term and invoked the Lorenz gauge condition $\partial^\mu A_\mu = 0$. We can immediately read off the physics encoded in this action: the first term describes the core energy of the vortices and we shall not need it in this work; the second term is the Higgs mass (including Meissner) for the electromagnetic field; the third term is the coupling term between the electromagnetic field and the vortex source. This term looks rather awkward given the derivatives in the denominator. This could signal that the coupling is non-local but that is not the case here. The origin of this coupling follows from the notions presented in section 4.2: it is not the gauge potential A_μ but rather the field strength $F_{\mu\nu}$ itself that couples to the vortex source.

4.3.3 Equations of motion

We can confirm this expectation by computing the equations of motion,

$$\frac{1}{\mu}\partial_\mu F^{\mu\nu} + \rho\epsilon^{\mu\nu\kappa\lambda}\frac{\partial_\mu}{\partial^2}J^{\mathrm{V}}_{\kappa\lambda} - \rho A^\nu = 0. \tag{4.41}$$

Acting with $\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma$ on this equation, one obtains,

$$\frac{1}{\mu\rho}\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma\partial_\mu F^{\mu\nu} + \epsilon_{\alpha\beta\gamma\nu}\epsilon^{\mu\nu\kappa\lambda}\frac{\partial^\gamma\partial_\mu}{\partial^2}J^{\mathrm{V}}_{\kappa\lambda} - \epsilon_{\alpha\beta\gamma\nu}\partial^\gamma A^\nu = 0 \tag{4.42}$$

Using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ one can see that from the first term only $\epsilon_{\alpha\beta\mu\nu}\partial^2 F^{\mu\nu}$ survives. Also, using $\partial^\kappa J^{\mathrm{V}}_{\kappa\lambda} = 0$ one can see that $\epsilon_{\alpha\beta\gamma\nu}\epsilon^{\mu\nu\kappa\lambda}\partial^\gamma\partial_\mu J^{\mathrm{V}}_{\kappa\lambda} = \partial^2 J^{\mathrm{V}}_{\alpha\beta}$, cancelling the derivatives in the denominator. Altogether we find,

$$\frac{1}{2\mu\rho}\epsilon_{\alpha\beta\mu\nu}\partial^2 F^{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu} = -J^{\mathrm{V}}_{\alpha\beta}. \tag{4.43}$$

This is the same result as Eq. (4.23). Notice that it is a completely local expression. As we announced earlier, we have derived here with a completely controlled procedure the dimensionless version of Eq. (4.5), describing the interactions between the vortices and electromagnetic fields inside a relativistic superconductor. Departing from this result we will derive in the next section various physical consequences. Summarizing this section, by dualizing the Ginzburg–Landau action for the superconductor, Eq. (4.25) was reformulated in terms of the vortex currents Eq. (4.35) as the active degrees of freedom, that interact via the effective gauge fields parametrizing the rigidity of the superconductor. The latter were integrated out to obtain the direct coupling of the vortices to the electromagnetic field, leading eventually to the concise equations of motion Eq. (4.43). Although this strategy is well known dealing with vortex 'particles' in 2+1 dimensions we are not aware that it was ever explored in the context of the electrodynamics of vortices in 3+1d. Surely, the derivation presented in the above is in regard with its rigour and completeness strongly contrasting with the rather ad hoc way that the problem is addressed in the standard textbooks [51, eq.(5.13)].

4.4 Vortex electrodynamics

In order to establish contact with the physics in the laboratory all that remains to be done is to break the Lorentz invariance, doing justice to the fact that the phase velocity of the superconductor as introduced in the first paragraphs of Section 4.1 is of order of the Fermi velocity of the metal and thereby a tiny fraction of the speed of light. Subsequently we will analyze what the physical ramifications are of our "Maxwell equations for vortices".

4.4.1 Non-relativistic dual action

The non-relativistic version of the vortex action Eq. (4.40) is,

$$\begin{aligned}\mathscr{L} = & \frac{\hbar^2}{2m^*}\rho_s J^{\mathrm{V}}_{tl}\frac{1}{-1/c^2_{\mathrm{ph}}\partial_t^2+\partial_k^2}J^{\mathrm{V}}_{tl} - \frac{\hbar^2}{2m^*}\rho_s J^{\mathrm{V}}_{kl}\frac{c^2_{\mathrm{ph}}}{-1/c^2_{\mathrm{ph}}\partial_t^2+\partial_k^2}J^{\mathrm{V}}_{kl} \\ & - \frac{e^{*2}}{2m^* c^2_{\mathrm{ph}}}\rho_s V^2 - \frac{e^{*2}}{2m^*}\rho_s A_m^2 \\ & - \frac{e^*\hbar}{m^*}\rho_s\frac{1}{-\frac{1}{c^2_{\mathrm{ph}}}\partial_t^2+\partial_k^2}\Big[\frac{1}{c_{\mathrm{ph}}}J^{\mathrm{V}}_{ab}\epsilon_{abtm}(\partial_t A_m+\partial_m V)+\frac{1}{2}J^{\mathrm{V}}_{ta}\epsilon_{tamn}\partial_m A_n\Big] \\ & + \frac{1}{2\mu_0 c^2}(\partial_t A_n+\partial_n V)^2 - \frac{1}{4\mu_0}(\partial_m A_n - \partial_n A_m)^2. \end{aligned} \tag{4.44}$$

4.4.2 Non-relativistic equations of motion

Varying with respect to A_ν, acting with $\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma$ and imposing current conservation $\partial^\kappa J^{\mathrm{V}}_{\kappa\lambda}=0$ will lead to the correct non-relativistic form of the equations of motion Eq. (4.4). However the easiest way to obtain these is to vary Eq. (4.2) directly with respect to V and A_n respectively,

$$-\frac{c^2_{\mathrm{ph}}}{c^2}\lambda^2\partial_n E_n - V = \frac{\hbar}{e^*}\partial_t\varphi, \tag{4.45}$$

$$-\lambda^2\frac{1}{c^2}\partial_t E_n + \lambda^2\epsilon_{nmk}\partial_m B_k + A_n = \frac{\hbar}{e^*}\partial_n\varphi. \tag{4.46}$$

Here $\lambda = \sqrt{\frac{m^*}{\mu_0 e^{*2}\rho_s}}$ is the London penetration depth. Now we operate on the first equation by $\partial_m = \frac{1}{2}\epsilon_{mtab}\epsilon_{abrt}\partial_r$, and on the second by $\delta_{mn}\partial_t = \frac{1}{2}\epsilon_{tmab}\epsilon_{abtn}\partial_t$ and $\epsilon_{tamn}\partial_m$ respectively to obtain,

$$-\frac{c^2_{\mathrm{ph}}}{c^2}\lambda^2\partial_m\partial_n E_n - \partial_m V = \frac{\hbar}{e^*}c_{\mathrm{ph}}\frac{1}{2}\epsilon_{mtab}J^{\mathrm{V}}_{ab}, \tag{4.47}$$

$$-\lambda^2\frac{1}{c^2}\partial_t^2 E_m - \lambda^2\partial_n^2\partial_t A_m + \partial_t A_m = \frac{\hbar}{e^*}c_{\mathrm{ph}}\frac{1}{2}\epsilon_{tmab}J^{\mathrm{V}}_{ab}, \tag{4.48}$$

$$\lambda^2(\nabla^2 - \frac{1}{c^2}\partial_t^2)B_a - B_a = -\frac{\hbar}{e^*}J^{\mathrm{V}}_{ta}. \tag{4.49}$$

For the last equation we used the Maxwell equations $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}$ and $\nabla\cdot\mathbf{B} = 0$. This one is equal to the one we found before in Eq. (4.5), obviously, since there the temporal terms do not come into play.

For the equations for the electric field is it useful to choose the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, and separate the electric field in longitudinal and transversal parts: $\mathbf{E} = \mathbf{E}^{\mathrm{L}} + \mathbf{E}^{\mathrm{T}}$, where $\nabla\times\mathbf{E}^{\mathrm{L}} = 0$ and $\nabla\cdot\mathbf{E}^{\mathrm{T}} = 0$. In the Coulomb gauge we see from the definition $\mathbf{E} = -\nabla V - \partial_t\mathbf{A}$ that $\mathbf{E}^{\mathrm{L}} = -\nabla V$ and $\mathbf{E}^{\mathrm{T}} = -\partial_t\mathbf{A}$. We can subtract the first equation above from the second to obtain,

$$\lambda^2\Big(-\frac{1}{c^2}\partial_t^2 E_m + \nabla^2 E_m^{\mathrm{T}} + \frac{c_{\mathrm{ph}}^2}{c^2}\nabla^2 E_m^{\mathrm{L}}\Big) - E_m = \frac{\hbar}{e^*} c_{\mathrm{ph}}\epsilon_{tmab} J_{ab}^{\mathrm{V}}. \tag{4.50}$$

Hence, as in the case of the Maxwell theory for non-relativistic matter one finds instead of the highly symmetric relativistic result Eq. (4.4) two equations of motion that are representing the spatial (magnetic) and temporal (electrical) sides of the physics, Eq. (4.49) and Eq. (4.50). One notices that the first 'magnetic' equation is quite like the relativistic one while the 'electrical' equation is now more complicated for reasons that will become clear in a moment.

The factor c_{ph} on the right-hand side of the electric equation is due to our convention of rescaling the time derivative to having units of 1/length in the definition of $J_{\kappa\lambda}^{\mathrm{V}}$. Thus all components of $J_{\kappa\lambda}^{\mathrm{V}}$ have dimensions of a surface density, and multiplying by a velocity is necessary to end up with a current density. The sign difference on the right-hand side between the electric and magnetic equations is related to the continuity equation $\frac{1}{c_{\mathrm{ph}}}\partial_t J_{tn}^{\mathrm{V}} = -\partial_m J_{mb}^{\mathrm{V}}$.

To grasp the content of these equations, one should compare the magnetic equation Eq. (4.49) with the standard form [51, eq.(5.13)],

$$\lambda^2\nabla^2 B_a - B_a = -\Phi_0\delta_a^{(2)}(\mathbf{r}), \tag{4.51}$$

Here $\Phi_0 = 2\pi\hbar/e^*$ is the flux quantum. The factor of 2π is associated with the definition of J^{V} as in Eq. (2.17). Our treatment automatically takes dynamics into account in the form of temporal derivatives. Otherwise, the correspondence is complete. We have indeed exactly recovered the well-established vortex equation of motion.

The equation for the electric field (4.50) looks more involved, but this can be made more insightful by writing the equations for the longitudinal and transversal parts separately,

$$\lambda^2(\frac{c_{\rm ph}^2}{c^2}\nabla^2 - \tfrac{1}{c^2}\partial_t^2)E_m^{\rm L} - E_m^{\rm L} = \frac{\hbar}{e^*} c_{\rm ph}\epsilon_{tmab}^{\rm L} J_{ab}^{\rm V}, \tag{4.52}$$

$$\lambda^2(\nabla^2 - \tfrac{1}{c^2}\partial_t^2)E_m^{\rm T} - E_m^{\rm T} = \frac{\hbar}{e^*} c_{\rm ph}\epsilon_{tmab}^{\rm T} J_{ab}^{\rm V}. \tag{4.53}$$

The labels on the ϵ-symbol denote that they include a longitudinal or transversal projection.

We want to point out for future reference that, applying the curl operator to Eq. (4.49), in the absence of vortex sources, and using $\nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$ (the Ampère–Maxwell equation in the static limit), one finds,

$$\lambda^2 \nabla^2 \mathbf{J} - \mathbf{J} = 0. \tag{4.54}$$

This denotes the perhaps counterintuitive result that the current is screened inside the superconductor. The reason is that a current induces a magnetic field locally, and the superconductor wants to expel the magnetic field. As such, all current through a superconductor flows through a thin layer near the boundary of typical size λ.

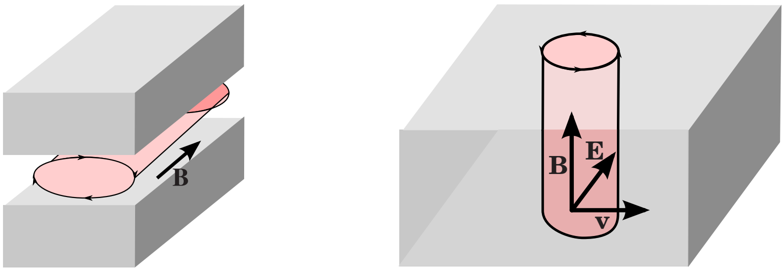
4.4.3 Vortex phenomenology

We can now read off the following physical relations:

1. *Meissner screening*: from Eq. (4.49) in the static limit $\partial_t \to 0$, a vortex line sources a magnetic field, that falls off in the superconductor with a length scale λ, the familiar Meissner effect.

2. *Thomas–Fermi screening*: from Eq. (4.52) one infers that the longitudinal (electrostatic) electric field penetrates up to a much smaller length $\frac{c_{\rm ph}}{c}\lambda$, which is the Thomas–Fermi length ($c \approx 300 c_{\rm ph}$). This just amounts to the well-known fact that the electrical screening is the same in the metal as in the superconductor. Notice that this length scale is obtained without referral to the electrons in the normal metal state as in the textbook derivation.

3. *Dynamic Meissner screening or the Higgs mass*: taking into account the time-dependence, Eq. (4.49) and Eq. (4.53) show that the transversal photon parts of the fields are screened not only in space, but also in time with characteristic time scale $\frac{\lambda}{c}$. This is just the familiar statement that the two propagating photon polarizations in 3+1 dimensions acquire a “Higgs mass” $\sim \frac{\hbar}{\lambda c}$ inside the superconductor.



(a) A vortex in a Josephson junction between two superconductors (grey); it has no normal core. The magnetic field **B** is along the vortex; any electric field across the junction causes the vortex to move in the perpendicular direction. Such motion induces electromagnetic radiation that may escape to the outside world.

(b) Geometry of the electric field **E** generated by a vortex line parallel to the magnetic field **B** and moving with a speed **v**. This phenomenon related to the Lorentz force follows directly from the vortex equations of motion.

Figure 4.1: Additional vortex configurations

4. *Electrical field of a moving vortex and the Nernst effect*: disregarding the dynamical term in Eq. (4.50), one is left with

$$E_m = -\frac{\hbar}{e^*} c_{\text{ph}} \epsilon_{tmkl} J^{\text{V}}_{kl}. \tag{4.55}$$

Recall from section 2.2.4 that we had interpreted J^{V}_{kl} as the flow or velocity in the k-direction of a vortex line in the l direction. Since we know that one vortex line carries a magnetic flux of $\Phi_0 = 2\pi \frac{\hbar}{e^*}$, we can write $\frac{\hbar}{e^*} c_{\text{ph}} J^{\text{V}}_{kl} = v_l B^0_k$, where B^0 denotes the field associated with one quantum of flux, and $v_l = c_{\text{ph}} \hat{e}_l$ is the velocity. In practice there is always a drag force that greatly slows down the vortices. Still, Josephson vortices that do not have a normal core (Fig. 4.1(a)) may achieve this large speed. With this interpretation, (4.50) reads,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}^0, \tag{4.56}$$

which is precisely the known result [74] for the electric field generated by a vortex moving in a magnetic field B^0 (Fig. 4.1(b)). When the motion is caused by a temperature gradient this is responsible for the large Nernst effect of the vortex fluid.

5. *AC Josephson relation*: another interpretation of Eq. (4.52) is found by inserting the definition of the vortex current, $J^{\text{V}}_{ab} = \epsilon_{abtn} \frac{1}{c_{\text{ph}}} \partial_t \partial_n \varphi$, taking

m as the longitudinal direction and neglecting the higher derivative terms. In this case,

$$\partial_m V = \frac{\hbar}{e^*}\partial_t(\partial_m \varphi). \tag{4.57}$$

Here the left-hand side is the potential difference, and the right-hand side is the time derivative of the superconducting phase difference. This is exactly the AC Josephson relation. The full equations Eq. (4.50) reveal also that the induced electric field is screened inside the superconductor.

6. *Moving vortices as radiation sources*: in the same spirit, the moving vortex is also inducing dynamic transversal fields according to Eq. (4.53). In other words: moving vortices radiate [71]. But since the field is Meissner screened, it is very hard to detect this radiation. All our results also apply to Josephson vortices (line vortex solutions in a Josephson junction between two superconductors parallel to the interface, Fig. 4.1(a)), which differ only in the regard that they do not have a normal core. There is much recent interest in radiation from (arrays of) Josephson junctions, see e.g. [75]. Since inside the junction the field is not expelled by Meissner and metallic screening, the radiation may escape to the outside world. In this literature one finds the following result [76, eq.(13)],

$$-\hat{\lambda}^2\nabla^2\mathbf{A} + \mathbf{A} = \frac{\hbar}{e^*}\nabla\phi. \tag{4.58}$$

Here $\hat{\lambda}$ differs from λ because of a special geometry. Compare this with a result that follows from Eq. (4.53),

$$\partial_t\big[-\lambda^2\big((\nabla^2 - \tfrac{1}{c^2}\partial_t^2)\mathbf{A} + \mathbf{A}\big] = \partial_t\big[\frac{\hbar}{e^*}\nabla\varphi\big], \tag{4.59}$$

confirming Eq. (4.58) but showing in addition how to take care of a possible time dependence of the photon field.

Summarizing, to the best of my knowledge we have addressed all known electrodynamical properties of vortex matter departing from a single action principle.

4.5 Outlook

I am of the opinion that our action principle for vortex electrodynamics Eq. (4.40) resp. (4.44) and the associated "vortex-Maxwell" equations Eq. (4.43), (4.49) and (4.50) deserve a place in the textbooks on the subject. In contrast with the clever but improvising discussions one usually finds, our formulation has the same 'mechanical' quality as for instance the Landau–Lifshitz treatise of electromagnetism. One just departs from the fundamentals, to expose the consequences by unambiguous and straightforward algebraic manipulations that are worshipped by any student of physics. A potential hurdle is that one has to get familiar with the two-form gauge field formalism, but then again this belongs to the kindergarten of differential geometry and string theory.

The analysis also reveals the origin of the peculiar nature of this vortex electrodynamics. The realization that it is in fact governed by a two-form gauge structure amounts to an entertaining excursion in the fundamentals of gauge theory itself, nota bene associated with the superficially rather mundane and technology-oriented vortex physics, at least when viewed from the perspective of fundamental physics. In the next chapter we will encounter more surprises when we investigate the electrodynamics of vortices in Bose-Mott insulators

On the practical side, as we implicitly emphasized in the last section our approach offers a unified description of the electrodynamics of vortices. Although we got as far as recovering the known physical effects in terms of special limits of our equations, there is potential to use them to identify hitherto unknown effects and perhaps to arrive at a more complete description of the electrodynamics vortex matter. As we are well aware of the large body of knowledge of this large field in physics, this is left as an open question to the real experts.

4.A Electrodynamics with differential forms

For the reader familiar with the mathematical language of differential forms, we present the electrodynamics of vortex sources for any dimension $d = D+1$ higher than 2. For our purposes, a differential form can be thought of as something that can be integrated over; in other words: it is a density function combined with the integrand. For instance, the electric field is a 1-form $\mathsf{E} = E_i \mathrm{d}x_i = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z$. Higher forms are always obtained through the wedge product $\mathsf{a} \wedge \mathsf{b}$, which is the antisymmetrization of the tensor product of a and b. Another common operation is the Hodge dual $*\mathsf{a}$ of a, which turns an n-form into a $(d-n)$-form. For instance in three spatial dimensions

name	field	?-form	2+1d	3+1d	representative in d=3+1
electric field	E	1	1	1	$E_x\,\mathrm{d}x$
dielectric current	$\mathsf{D} = \varepsilon *_s \mathsf{E}$	d-2	1	2	$D_x\,\mathrm{d}y \wedge \mathrm{d}z$
magnetic field	B	2	2	2	$B_x\,\mathrm{d}y \wedge \mathrm{d}z$
magnetic intensity	$\mathsf{H} = \mu *_s \mathsf{B}$	d-3	0	1	$H_x\,\mathrm{d}x$
charge density	ρ	d-1	2	3	$\rho\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
current density	J	d-2	1	2	$J_x\,\mathrm{d}y \wedge \mathrm{d}z$
covariant current	$\mathsf{j} = \rho + \mathsf{J} \wedge \mathrm{d}t$	d-1	2	3	$j_x\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t$
field strength	$\mathsf{F} = \mathsf{B} + \mathsf{E} \wedge \mathrm{d}t$	2	2	2	$F_{xy}\,\mathrm{d}x \wedge \mathrm{d}y$
gauge potential	A	1	1	1	$A_x\,\mathrm{d}x$
vortex source	J^{V}	d-2	1	2	$J^{\mathrm{V}}_{xy}\,\mathrm{d}x \wedge \mathrm{d}y$
Lagrangian density	$\mathscr{L}$	d	3	4	$\mathscr{L}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Table 4.1: Electrodynamical quantities in differential forms. Here ε and μ are the electric permittivity and the magnetic permeability, and $*_s$ is the spatial Hodge dual. Other factors of c are suppressed. Minus signs are subject to convention.

$*\mathsf{E} = E_x \mathrm{d}y \wedge \mathrm{d}z + E_y \mathrm{d}z \wedge \mathrm{d}x + E_z \mathrm{d}x \wedge \mathrm{d}y$. For a pedagogical introduction to differential forms in Maxwell electrodynamics see [77].

In the familiar case of $d = 3+1$, a 1-form is a line density or "field intensity" like the electric field; a 2-form is a surface density or flux density like the magnetic field; a 3-form is a volume density like the charge density. Confusion may arise when it is not immediately clear whether an object is an n-form or a $d-n$-form, which is important for generalization to other dimensions. We distinguish the regular Hodge dual $*$ from the spatial Hodge dual $*_s$, where the latter does not involve the temporal dimension. The exterior derivative operator is $\mathsf{d} = \frac{\partial}{\partial t}\,\mathrm{d}t \wedge + \sum_i \frac{\partial}{\partial x_i}\,\mathrm{d}x_i \wedge$, and the one with only spatial components is $\mathsf{d}_s = \sum_i \frac{\partial}{\partial x_i}\,\mathrm{d}x_i \wedge$. The Leibniz rule is $\mathsf{d}(\mathsf{a} \wedge \mathsf{b}) = \mathsf{d}\mathsf{a} \wedge \mathsf{b} + (-1)^r \mathsf{a} \wedge \mathsf{d}\mathsf{b}$, where a is an r-form. This can be used for partial integration.

In table 4.1 we have listed the differential forms of the relevant fields. Some of these definitions seem perhaps unfamiliar. In particular, we are used to thinking of the magnetic field as a vector field; however, its solenoidal nature is typical of a two-form. This becomes even more clear when it is expressed as the curl of the vector potential $\mathsf{B} = \mathsf{d}_s\mathsf{A}$, which holds in $3+0$

dimensions. Also the current density J is naturally a flux or a 2-form, but its generalization is as a $d-2$-form. One way to see that this must be so, is to write down the continuity equation in differential forms,

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \qquad \rightarrow \qquad (\partial_t \rho + \mathsf{d}_s \mathsf{J}) \wedge \mathsf{d}t = \mathsf{dj} = 0. \tag{4.60}$$

The current density appearing as a vector field for instance in Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$ is actually the spatial Hodge dual of J.

We shall now write down the familiar expressions of Maxwell electrodynamics. The Lagrangian density is a spacetime volume density. All terms must therefore combine into d-forms. The field strength is $\mathsf{F} = \mathsf{dA}$. From this definition it is clear that the gauge transformations $\mathsf{A} \rightarrow \mathsf{A} + \mathsf{d}\xi$, with ξ any 0-form, leave the field strength unchanged, since $\mathsf{d}^2 = 0$. The field strength is contracted with its dual to obtain a d-form in the Lagrangian. The sources couple to the gauge potential (this is another reason why the source is a $d-1$ form). The Maxwell action is then,

$$S = \int -\mathsf{F} \wedge *\mathsf{F} + \mathsf{A} \wedge \mathsf{j}. \tag{4.61}$$

The second term is also invariant under the same gauge transformations, provided that $\mathsf{dj} = 0$, the continuity equation. The Euler–Lagrange equations are,

$$\mathsf{d}\frac{\partial \mathsf{L}}{\partial \mathsf{dA}} - \frac{\partial \mathsf{L}}{\partial \mathsf{A}} = 0, \tag{4.62}$$

resulting in the inhomogeneous Maxwell equations,

$$\mathsf{d} * \mathsf{dA} = \mathsf{d} * \mathsf{F} = -\mathsf{j}, \qquad (\partial_\mu F_{\mu\nu} = -J_\nu). \tag{4.63}$$

Applying the exterior derivative on this equation directly leads to the continuity equation, since $\mathsf{d}^2 = 0$. Similarly, from the definition $\mathsf{F} = \mathsf{dA}$ it immediately follows that,

$$\mathsf{dF} = 0, \tag{4.64}$$

which are the homogeneous Maxwell equations, or in this context rather the Bianchi identities.

Now let us repeat the reasoning of section 4.2. In the absence of monopole sources J, we have both $\mathsf{d} * \mathsf{F} = 0$ and $\mathsf{dF} = 0$. This implies that the field strength has become "pure gauge". The first of these equations still holds when we add any 1-form ξ as $*\mathsf{F} \rightarrow *\mathsf{F} + \mathsf{d}\xi$. The original Bianchi identities are not invariant under these transformations. The dual field strength $*\mathsf{F}$ turns into a gauge potential, and is accompanied by its own field strength $\mathsf{K} = \mathsf{d} * \mathsf{F}$, which contracts with its dual in the Lagrangian. The field strength can couple to a $d-2$-form source, which we anticipatively denote by J^{V}, provided that $\mathsf{d} * \mathsf{J}^{\mathrm{V}} = 0$. Indeed,

$$\mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} \to \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} + *\mathsf{d}\xi \wedge \mathsf{J}^{\mathrm{V}} = \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} - \xi \wedge \mathsf{d} * \mathsf{J}^{\mathrm{V}} = \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}}. \tag{4.65}$$

The second step is achieved by partial integration, and the last equality holds if the vortex current is conserved, $\mathsf{d} * \mathsf{J}^{\mathrm{V}} = 0$. The action for vortices directly sourcing the field tensor is, (with $\mathsf{G} = *\mathsf{F}$),

$$S = \int -\mathsf{K} \wedge *\mathsf{K} + \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} = \int -\mathsf{K} \wedge *\mathsf{K} + \mathsf{G} \wedge *\mathsf{J}^{\mathrm{V}}. \tag{4.66}$$

Variation with respect to G leads to,

$$* \mathsf{d} * \mathsf{d}\mathsf{G} = -\mathsf{J}^{\mathrm{V}}. \tag{4.67}$$

This equation corresponds to $\epsilon_{\kappa\lambda\mu\nu}\partial^2 F_{\mu\nu} = -J^{\mathrm{V}}_{\kappa\lambda}$ as in Eq. (4.43), but is valid in any dimension. The addition of a Meissner term results in

$$S = \int -\mathsf{K} \wedge *\mathsf{K} - \mathsf{G} \wedge *\mathsf{G} + \mathsf{G} \wedge *\mathsf{J}^{\mathrm{V}}. \tag{4.68}$$

and,

$$* \mathsf{d} * \mathsf{d}\mathsf{G} - \mathsf{G} = -\mathsf{J}^{\mathrm{V}}. \tag{4.69}$$

This is the equation of motion for $d-1$-dimensional superconductors, which have $d-2$-dimensional vortex world branes J^{V}.