We experimentally study the statistics of non-local speckle patterns, obtained when spatially entangled photon pairs are scattered through a random medium. Striking differences arise between the scattering of highly entangled states and almost separable states. Both the purity of the field and the Schmidt number, which quantifies the number of entangled modes, can be obtained from the visibility of the speckles. We observe non-exponential statistics for both the intensities and the two-photon correlations.
9. STATISTICAL PROPERTIES OF NONLOCAL SPECKLES

9.1 Introduction

Speckles are the random intensity patterns that appear when a wave is reflected from or transmitted through a random scattering medium [171,172]. At the time of its discovery, speckles were mainly seen as a drawback in coherent imaging systems. Further studies revealed, however, that the speckle pattern carries information both on the coherence properties of the radiation and on the microscopic details of the scattering object. After averaging over many realizations of the disorder, useful information can be retrieved through statistical arguments. The study of wave propagation in random media has revealed many interesting phenomena, such as conductance fluctuations [173,174], enhanced backscattering [175], and Anderson localization [176].

More recently, considerable effort has been devoted to understand how the quantum nature of light manifests after multiple scattering [177–188]. A broad range of subjects have been investigated, such as the degradation of polarization entanglement [178,179,182], the transport of quantum noise [177,180], and the dynamics of photons in disordered lattices [188]. It has also been shown that entanglement can be induced by multiple scattering of squeezed states and that quantum interference can survive ensemble average [185,186].

The special features of scattering of quantum light are best appreciated in the spatial domain. When a pure two-photon state is scattered by a random medium, it will produce so-called “two-photon speckles” [187]. These patterns are remarkable because they exist in the more abstract space of fourth-order correlations. They show up in the coincidence count rate of two (scanning) detectors and are a function of two position coordinates.

In this Chapter we present the first experimental investigation of the statistics of two-photon speckles. We show that either non-local or separable speckle patterns can be observed, depending on the degree of spatial entanglement of the initial state. By averaging over many realizations of the disorder, important properties of the source can be retrieved. In this way, the two-photon state can be proven to be pure and the Schmidt number \( K \), which quantifies the number of entangled modes, can be obtained. Experimentally quantifying multi-dimensional entanglement is an important, but very demanding task [39]. Our approach provides a feasible and theoretically sound solution for this problem. Finally, we recover the probability distributions of single-photon intensities and two-photon coincidences, which, contrary to most classical speckles, are in general non exponential.

9.2 Theory

Statistical distributions of two-photon speckles were first theoretically discussed by Beenakker et al. [184]. In this Chapter we will greatly benefit from their results. But in order to extend their conclusions to more realistic experimental conditions, our analysis will deviate at several points. First of all, we do not use a random matrix description, with its discrete number of input and output
channels, but instead we use a continuous description. The only requirement we will impose on the scattering is that it is sufficiently random and unitary (i.e. energy conserving). Second, we will start from the most general pure input state, whose Schmidt coefficients are not necessarily equal, i.e., we show how the Schmidt number $K$ can be measured, instead of the Schmidt rank. Finally, we do not separate the two-photon phase space into half-spaces, $q > 0$ and $q < 0$. This allows us to investigate separable states ($K = 1$) as well.

We begin by reviewing some properties of classical speckles. When a field $f(x)$ is scattered by a random medium, all possible light paths will acquire arbitrary phases. When these components are added together, they will form a complex interference pattern known as speckle. For unitary scattering, we can describe this pattern by the transformation $F(x) = U[f(x)]$. If the number of scattering centers is very large, the Central Limit Theorem assures that the probability density function for both the real and imaginary components of $F(x)$ is asymptotically Gaussian. The intensity $I = |F|^2$ has then an exponential distribution $P(I) \propto \exp(-I/\langle I \rangle)$, with average $\langle I \rangle$. The visibility or contrast of any speckle pattern is defined by $V = \langle I^2 \rangle / \langle I \rangle^2 - 1$, where the brackets denote ensemble average. The exponential distribution has unity visibility.

These results can be generalized to a two-photon field $A(x_1, x_2)$, which describes the probability amplitude of finding one photon at transverse position $x_1$ and the other at $x_2$. The quantum-entangled nature of the state is best appreciated in the so-called Schmidt decomposition [13], where the two-photon state is expressed as a discrete sum over factorizable two-photon states of the form $A_k(x_1, x_2) = f_k(x_1)g_k(x_2)$. Part of the beauty of the Schmidt decomposition is that it remains intact upon unitary scattering, as orthogonal states remain orthogonal under this scattering; only the eigenstates are modified. If we write the input state in the Schmidt form, the output state will be

$$A_{out}(x_1, x_2) = \sum_k \sqrt{\lambda_k} F_k(x_1)G_k(x_2),$$

(9.1)

where $\lambda_k$ are the Schmidt coefficients and, analogously to the classical case, $F_k(x) = U[f_k(x)]$ and $G_k(x) = U[g_k(x)]$ are the speckle fields corresponding to the transformation of the Schmidt modes $f_k$ and $g_k$. The coincidence rate measured by two photon counters at positions $x_1$ and $x_2$ is $R_{cc}(x_1, x_2) \propto |A_{out}(x_1, x_2)|^2$. If the initial state is separable, i.e., just one term in the Schmidt decomposition, the two-photon speckles observed in $R_{cc}$ will also be separable. On the other hand, if the input state is highly entangled, $R_{cc}$ will reveal a nonlocal speckle pattern. The effective number of entangled modes in the decomposition (9.1) is usually quantified by the Schmidt number $K = 1/\sum_k |\lambda_k|^2$. The Schmidt coefficients are normalized such that $\sum_k |\lambda_k| = 1$.

We will now study the statistics of the intensities $I$ (single photon rate) and
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of the coincidences \( R_{cc} \). From Eq. (9.1) we immediately obtain

\[
R_{cc} = \alpha \sum_{j,k} \sqrt{\lambda^*_{j} \lambda_{k}} F_{j}^* F_{k} G_{j}^* G_{k}, \tag{9.2}
\]

\[
R^2_{cc} = \alpha^2 \sum_{i,j,k,m} \sqrt{\lambda^*_{i} \lambda^*_{j} \lambda_{k} \lambda_{m}} F_{i}^* F_{j}^* F_{k} F_{m} G_{i}^* G_{j}^* G_{k} G_{m}, \tag{9.3}
\]

where we omit the coordinates \( x_1 \) and \( x_2 \). The proportionality constant \( \alpha \) incorporates the experimental factors that relate the theory to the measured coincidences rate. Let's first assume that \( x_1 \neq x_2 \). In this case, the speckle fields \( F \) and \( G \) are statistically independent. Furthermore, the fields \( F_i \) and \( F_j \) are also statistically independent, unless \( i = j \). The same holds for \( G \). Due to the Gaussian-random nature of the scattered fields, \( \langle |F_i|^2 n \rangle = \langle I^n \rangle = n! \), where \( I \) is the exponentially distributed intensity, but \( \langle F_i^{2n} \rangle = 0 \). With these ingredients, it is straightforward to show that

\[
\begin{align*}
\langle R_{cc} \rangle &= \alpha, \\
V_c &= 1 + 2 K, \\
\end{align*}
\tag{9.4}
\]

where \( K \) is the Schmidt number and \( V_c \) is the visibility of the two-photon speckle pattern. We see that \( V_c \) varies from 3 (separable state) to 1 (maximally entangled state).

The single photon intensities can be obtained by a partial trace of the two-photon state as

\[
I_{\text{out}}(x) = \sum_{k} |\lambda_{k}|^2 |F_{k}(x)|^2 = \sum_{k} |\lambda_{k}|^2 |G_{k}(x)|^2, \tag{9.5}
\]

which is an incoherent sum of many speckle patterns. The more terms in the distribution, the more uniform the intensity becomes. The visibility of the one-photon speckle reduces with the number of modes as

\[
V_I = \frac{1}{K}. \tag{9.6}
\]

When the number of terms in (9.5) is very large, the Central Limit Theorem can be used to show that \( P(I) \) is normally distributed with mean \( \langle I \rangle \) and standard deviation \( 1/\sqrt{K} \).

We have assumed so far that the input state is pure. In a more general sense, the purity \( P \) of the two-photon state can be calculated from the visibilities \( V_I \) and \( V_c \) of the single-photon and two-photon speckles as

\[
P = V_c - 2V_I. \tag{9.7}
\]

This crucial result, which was first derived in [184], can also be proven using our formalism.
9.2. THEORY

Figure 9.1: Entangled photon pairs are obtained via type-I SPDC by pumping a 5-mm thick periodically poled KTP crystal with a laser beam ($\lambda_p = 413.1$ nm and 200 mW power). The crystal center is imaged onto the incident plane of the scatterer with two $f_1 = 200$ mm lenses. The far field of the scattering medium is imaged with a $f_d = 250$ mm lens onto the detection plane. Detection occurs via projection onto two single-mode fibers. The size of the detection modes ($w_{det} = 140$ $\mu$m) determines the spatial resolution in the far-field plane. Narrow band spectral filters (5 nm at 826.2 nm) are used to select down-converted light close to frequency degeneracy. The inset shows our scattering medium, which comprises two light shaping diffusers positioned in each other’s far field (using a lens $f_c = 10$ mm). This configuration mimics a volume scatter [187], but it also allows sufficient counts to be measured. (a) Generation of a state with $K_{th} = 80$. The pump is weakly focused to a waist $w_p = 160$ $\mu$m. (b) Generation of a state with $K_{th} = 1.4$. The pump lens (not shown) is removed and a $f_2 = 100$ mm lens focuses the beam to a spot $w_p = 11.5$ $\mu$m at the center of the crystal. The two $f_1$ lenses are removed and a single $f_3 = 59$ mm lens images the center of the crystal on the scatter.

The probability distributions for $R_{cc}$ and $I$ can be deduced from Eqs. (9.2) and (9.5), which are weighted sums of products of random Gaussian variables. For the special case of $K$ equally weighted Schmidt modes, with $\lambda_k = 1/K$, we recover the closed expressions of Ref. [184]

$$P_1(\tilde{I}) = iK^{-1}e^{-\frac{\lambda_k K K}{\Gamma(K)}},$$  \hfill (9.8)

$$P_2(\tilde{R}_{cc}) = \frac{K}{\Gamma(K)} (K \tilde{R}_{cc})^K K_{K-1} \left[ 2 \sqrt{K \tilde{R}_{cc}} \right].$$  \hfill (9.9)

where $\tilde{I} = I/\langle I \rangle$ and $\tilde{R}_{cc} = R_{cc}/\langle R_{cc} \rangle$, $\Gamma$ is the Gamma function and $K_{K-1}$ is a modified Bessel function of the second kind. The single-photon probability density $P_1(\tilde{I})$ is a Gamma distribution while, the two-photon probability $P_2(\tilde{R}_{cc})$ is known as the “$K$”-distribution [189].

We finally consider the case $x_1 = x_2 = x$. Because of the symmetry $A(x_1, x_2) = A(x_2, x_1)$ of the two-photon field, the Schmidt modes $\tilde{F}_i(x)$ and
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Figure 9.2: Coincidence counts measured by two scanning detectors for a single realization of the scattering medium. (a) A non-local speckle pattern, corresponding to the scattering of a highly entangled state with $K_{th} = 80$. (b) Separable speckle pattern, corresponding to the scattering of a state with a small number of modes, $K_{th} = 1.4$.

$G_j(x)$ are not all statistically independent. Taking this into consideration, we can repeat the steps above and show that Eqs. (9.4) and (9.9) retain their form, but with the substitution $K \rightarrow K/2$, which implies that $V_c = 1 + \frac{4}{K}$. The average $\langle R_{cc} \rangle = 2\alpha$ is twice as large. This photon bunching effect survives averaging over many realizations of the disorder.

9.3 Experimental results

Figure 9.1 shows the experimental setup used to generate entangled photon pairs and to measure the statistics of the speckles. We investigate two different regimes, namely, a highly entangled state with theoretical Schmidt number $K_{th} = 80$ and an almost separable state with $K_{th} = 1.4$.

Figure 9.2 shows measurements of two-photon speckle patterns for a fixed realization of the scattering medium. These figures are obtained by scanning both detectors horizontally, keeping $y_1 = y_2$ fixed, and recording the coincidences count rate. The results are corrected for accidental counts. Figure 9.2(a) corresponds to a highly entangled state, while Fig. 9.2(b) shows the results for an almost separable state. The differences are striking. When operating under reduced number of modes, the coincidences rate is practically separable in the product of single-photon intensities, $R_{cc}(x_1, x_2) \approx I(x_1)I(x_2)$. On the other hand, when the number of modes is very large the pattern is clearly non-separable. By measuring photon 1 at a certain position, photon 2 is “nonlocally” projected into a speckle pattern that depends on the position of detector 1. Notice also that
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Table 9.1: Overview of the measured statistics, obtained for $x_1 \neq x_2$. The theoretical Schmidt number $K_{th}$ is calculated via the procedure in [13]; $V_I$ and $V_c$ are the visibilities of the intensities and coincidences respectively, $\mathcal{P}$ is the purity, and $K_{ex}$ is the measured Schmidt number.

<table>
<thead>
<tr>
<th>$K_{th}$</th>
<th>$V_I$</th>
<th>$V_c$</th>
<th>$\mathcal{P}$</th>
<th>$K_{ex}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.83 ± 0.02</td>
<td>2.65 ± 0.15</td>
<td>0.98 ± 0.15</td>
<td>1.20 ± 0.03</td>
</tr>
<tr>
<td>80</td>
<td>0.014 ± 0.002</td>
<td>1.04 ± 0.04</td>
<td>1.01 ± 0.04</td>
<td>70 ± 9</td>
</tr>
</tbody>
</table>

both patterns are symmetric with respect to the $x_1 = x_2$ diagonal; this reflects the symmetry of the field $A(x_1, x_2) = A(x_2, x_1)$.

We will next discuss the statistical distributions of the coincidences $R_{cc}$ and intensities $I$ under various conditions. To this end, the detectors are placed at fixed positions, either $x_1 = x_2$ or $x_1 \neq x_2$. Different realizations of disorder are obtained by rotating the first or the second diffuser in steps of $3^\circ$. In this way, we can measure 14,400 realizations of the scattering medium. The acquisition time for each measurement was 5 seconds. An overview of the results for $x_1 \neq x_2$ is shown in table 9.1.

Figure 9.3 shows the probability distributions measured for $K_{th} = 1.4$, i.e., for an almost separable state. The dashed red curves are the theoretical curves, obtained using Eqs. (9.8) and (9.9) with the measured Schmidt number $K$. The dashed black lines correspond to an exponential distribution and confirm that all three distributions are non-exponential. When $K_{th} = 1.4$, the field is not only coherent in fourth order, but it is also almost coherent in second order. The single photon speckles exhibit high visibility, $V_I = 0.83 \pm 0.02$, not very far from unity visibility, which holds for completely coherent light. This visibility allows us to estimate the experimental Schmidt number to be $K = 1.20 \pm 0.03$, confirming that our state is practically separable. The associated probability distribution $P_1$ is shown in Fig. 9.3(a). The distribution is slightly concave on a semi-log scale and is theoretically described by a Gamma distribution. This distribution would have been exactly exponential in the limit $K = 1$.

The results for the coincidence counts are more interesting. Figure 9.3(b) shows the probability distribution $P_2$ of two-photon speckles for $x_1 \neq x_2$. The associated visibility $V_c = 2.65 \pm 0.15$ has a relatively large error margin. The main reason for this error is the occurrence of a few very large fluctuations, associated with the extreme tail of the $P_2$ distribution. As we can see, the distribution has a convex shape on a semi-log scale, such that the probability of very small and very high fluctuations are higher than for an exponential distribution. The error in $V_c$ will propagate to the purity $\mathcal{P}$. We obtain an average value $\mathcal{P} = 0.98 \pm 0.15$ for the purity of the two-photon state.

Figure 9.3(c) shows the probability distribution of $R_{cc}$ for $x_1 = x_2$. The convexity is even more pronounced and the peak close to $R_{cc} \approx 0$ is twice as large,
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Figure 9.3: Probability distributions measured for $K_{th} = 1.4$. The (red) dashed curves are the theoretical distributions and the (black) dashed lines correspond to an exponential distribution. The insets show the results on a linear scale. (a) Distribution $P_1$ of intensities. (b) Distribution $P_2$ of coincidences for $x_1 \neq x_2$. (c) Distribution $P_2$ of coincidences for $x_1 = x_2$.

as can be seen in the insets. Theoretically, the peak at $R_{cc} = 0$ should be much higher. The measured shape and peak around $R_{cc} = 0$, however, are limited by the experimental noise that inevitably dominates at the smallest count rates. The two-photon speckle contrast is $V_c = 4.45 \pm 0.30$, which reflects the almost classical, i.e local, nature of the fluctuations. For a fully factorizable speckle, the visibility of $R_{cc} \approx I^2$ is $V_c = 5$.

Figure 9.4 shows the probability distributions for a highly entangled state with $K_{th} = 80$. In the limit of high $K$, the two-photon speckles are a genuine two-coordinate function and can be considered as a more authentic generalization of classical speckles to fourth-order optics. The fluctuations at $x_1 = x_2$ are now not more special than those at $x_1 \neq x_2$; only the average level will be different due to the photon bunching effect. The probability distribution $P_2$ for $x_1 \neq x_2$, shown in Fig. 9.4(a), is practically exponential and the visibility $V_c = 1.04 \pm 0.04$ is close to unity. Extreme fluctuations do not occur very often for this distribution. As a result, the error in $V_c$ is smaller. The distributions for $x_1 = x_2$ and $x_1 \neq x_2$ have approximately the same shape, but since the number of modes is still finite, the visibility for $x_1 = x_2$ is slightly higher, namely, $V_c = 1.10 \pm 0.05$.

Figure 9.4(b) shows the probability distribution $P_1$ on a linear scale. Because the reduced one-photon state is now practically incoherent, the intensity will exhibit only limited fluctuations around the average $\langle I \rangle$. As expected from the Central Limit Theorem, the curve is approximately Gaussian. The associated one-photon visibility is only $V_I = 0.014 \pm 0.002$. To measure this value, we had to correct for a small wedge effect in the diffusers. This correction introduces a relative larger error, which propagates when calculating the Schmidt number. Nonetheless, the obtained value $K_{ex} = 70 \pm 9$ agrees reasonably well with the large number of modes expected. Finally, we obtain $\mathcal{P} = 1.01 \pm 0.04$ for the purity of the state.

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9.4 Conclusion

We have studied the statistics of two-photon speckle patterns. These patterns are a generalization of classical speckles to fourth-order optics. Depending on the degree of spatial entanglement of the input state, the scattered field can exhibit very different structures and statistics. We have measured the Schmidt number of both an almost separable state and a highly entangled state. We have also proven that both generated states are pure to a good degree of accuracy. These results provide new insights into the role of spatial entanglement to the scattering of light and opens the door to new developments in the field of quantum optics in random media.

Figure 9.4: Probability distributions measured for $K_{th} = 80$. The (red) curves are the theoretical distributions and the (black) dashed lines correspond to an exponential distribution. (a) Distribution $P_2$ of coincidences for $x_1 \neq x_2$. (b) Distribution $P_1$ of intensities.
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