CHAPTER 9

Transport of orbital-angular-momentum entanglement through a turbulent atmosphere

We demonstrate experimentally how orbital-angular-momentum entanglement of two photons evolves under the influence of atmospheric turbulence. Experimental results are in excellent agreement with our theoretical model, which combines the formalism of two-photon coincidence detection with a Kolmogorov description of atmospheric turbulence. We express the robustness to turbulence in terms of the dimensionality of the measured correlations. This dimensionality is surprisingly robust: scaling up our system to real-life dimensions, a horizontal propagation distance of 2 km seems viable.

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9.1 Introduction

Quantum communication by means of entangled photon pairs allows for an intrinsically secure transmission of data, by distributing the pairs via a free-space or fiber channel to distant parties [10]. Most popular is polarization entanglement, which has dimensionality 2. Higher dimensionalities can be achieved using orbital-angular-momentum (OAM) entanglement [40, 215] or energy-time entanglement [22, 60]; this route provides for a larger channel capacity and an increased security against eavesdroppers [17, 19]. However, the performance of a real-world high-dimensional quantum channel is an open issue. Here, we address this issue for the case of OAM entanglement distribution via a free-space channel.

For quantum communication to be of practical relevance, it is imperative that the entanglement between the photons carrying the information survives over a reasonably long propagation distance. Entanglement distribution over fiber-based transmission lines has proven to be feasible over distances over a hundred kilometers [53–55]. However, the use of free-space links is needed when considering such purposes as airplane and satellite quantum links or hand-held communication devices [56–58].

The increased quantum-channel capacity that is available when encoding the information in the OAM of the entangled photons was argued to be severely limited in a practical free-space link, due to atmospheric turbulence that causes wavefront distortions. Several studies have addressed this aspect [63–71], but there is no unanimity on exactly how sensitive OAM entanglement is to atmospheric perturbations. So far, no experimental verdict has been obtained to clarify this issue.

In this Chapter, we present the first such experiment. We start with bipartite OAM entanglement of effective dimensionality 6, and demonstrate how the corresponding correlations evolve when one of the photons traverses a turbulent atmosphere, emulated by controlled mixing of cold and hot air. Our experimental results are in excellent agreement with our theoretical model, which combines a Kolmogorov description of atmospheric turbulence with our formalism of bi-photon correlation detection.

9.2 Experimental setting

Our experimental setup is depicted in Fig. 9.1. A PPKTP down-conversion crystal of length 2 mm is pumped by the single transverse mode output of a Kr⁺ laser operating at 413 nm. It emits correlated photon pairs with complementary OAM at 826 nm, in a state of the form [33]

$$|\Psi\rangle = \sum_{m,p} c_{m,p} |m, p\rangle |- m, p\rangle.$$  \hspace{1cm} (9.1)

Here, |m, p\rangle indicates the Schmidt mode containing one photon with orbital angular momentum \(m\hbar\), with \(p\) the radial mode index, and we can write \langle r|m, p\rangle = \langle r|m, p\rangle e^{i m \theta}/\sqrt{2\pi}. \) For our source, the total number of entangled azimuthal modes, the so-called angular Schmidt number \(K\), is of order 30 [34]. The correlated photons are spatially separated by a 50/50 beam splitter.

*Within the mode space our analysers have access to, we can safely approximate \(c_{m,p}\) to be independent of \(m\).
9.3 Turbulence cell

In one of the beam lines, we place a turbulence cell where cold and hot air are mixed to bring about random variations of the refractive index that vary over time (Fig. 9.2(a)). We can tune the strength of the turbulence by varying the heating power and air flow through the cell. Similar cells have been used as a realistic emulation of atmospheric turbulence [203]. Figure 9.2(b) gives an impression of the cell’s functioning: We inject one of the analysers backwards with diode laser light and monitor the beam, which traverses the turbulence cell, in the far field. We do this for two cases; the analyser is equipped with no phase plate (top row), or with the quadrant phase plate (bottom row). We observe that the input beams (left column) become deformed by the refractive index fluctuations, as can be seen when taking a 10 ms snapshot (middle column). Time averaging these fluctuations over 10 s reveals a beam broadening that...
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Figure 9.2: Beam corruption after passage through the turbulence cell. (a) Impression of how an OAM eigenmode, having a helical wavefront, gets distorted when transiting the turbulence cell. The cell consists of a 7 cm long, 26 mm diameter glass tube, containing several resistors that produce up to 60 W of heat. A gentle flow of room temperature air is driven through the tube. (b) Far-field intensity patterns of the analyser, which is fed backwards with diode laser light at 826 nm. The analyser is equipped with no phase plate (top row), or quadrant phase plate (with its sector aligned along the Cartesian axes) (bottom row). The diffraction limited patterns (left column) get perturbed when turbulence is switched on (middle column): for mild turbulence, the dominant effect is a randomly evolving beam deflection; for the more severe turbulence conditions used here ($w_0/r_0 = 0.65$), the beam profile can get significantly distorted. Taking a 10 s time average reveals an isotropic beam broadening (right column). The apparent asymmetry along the diagonal in the bottom left and right windows is due to the 3% discrepancy of the quadrant phase step from the ideal value of $\pi$.

is spatially isotropic (right column).

We describe our cell by the Kolmogorov theory of turbulence [188]. This standard model treats the optical effects of the atmosphere at any moment as a random phase operation $e^{i\phi(r)}$, the time evolution of which follows a Gaussian distribution. It is conveniently described in terms of its coherence function, given by

$$\langle e^{i\phi(r_1) - i\phi(r_2)} \rangle_t = e^{-\frac{2}{\pi^2} \left[ \frac{r_1 - r_2}{r_0} \right]^2},$$

where $\langle \ldots \rangle_t$ denotes averaging over time [189]. The relevant parameter in this model is the Fried parameter $r_0$, being the transverse distance over which the beam profile gets distorted by approximately 1 rad of root-mean-square phase aberration [189]. In the absence of turbulence $r_0 \rightarrow \infty$, but when turbulence becomes stronger, the spatial coherence is reduced and hence

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Figure 9.3: Mode scattering due to turbulence. (a) Time-averaged survival probability of an analyser’s OAM eigenmode \( m = m_0 \) as a function of turbulence strength (blue). The red and green curves denote turbulence-induced coupling probabilities to neighbouring modes for \( \Delta m = \pm 1 \) and \( \Delta m = \pm 2 \), respectively. (b) Time-averaged spreading of the \( m_0 \) OAM eigenmode (blue bar) over its neighbours for \( w_0/r_0 = 0.65 \) (red bars).

\[ \frac{w_0}{r_0} = \sqrt{\left(\frac{w_{el}}{w_{dl}}\right)^2 - 1} \]

with \( w_{dl} \) and \( w_{el} \) the 1/e far-field radius of the diffraction limited beam and long-exposure broadened beam, respectively [199].

We calculated the effect of Kolmogorov turbulence on a single beam with a mode profile described by \( |A(\alpha)| \). The blue curve in Fig. 9.3(a) shows the survival probability of an OAM eigenmode \( m = m_0 \) upon passing through a turbulent atmosphere as described by Eq. (9.2). The survival probability degrades gradually for increasing turbulence strength. We note that this decay depends on the ratio \( w_0/r_0 \) only and not on the specific OAM eigenvalue \( m_0 \), provided that the propagation distance \( L \) is small compared to the diffraction length \( z_R = \pi w_0^2/\lambda \). Furthermore, the turbulence produces a coupling between the orthogonal OAM modes, leading to a non-vanishing mode overlap between the \( m_0 \) eigenmode and its neighbours \( \Delta m = \pm 1 \) (red) and \( \Delta m = \pm 2 \) (green). A different perspective on this mode mixing is presented in histogram Fig. 9.3(b), which shows how an OAM eigenmode (blue bar) spreads out over its neighbouring azimuthal modes for \( w_0/r_0 = 0.65 \) (red bars). We note that normalisation is not preserved, because some intensity is scattered to radial modes that are not sustained by the single-mode fiber. This illustrates the importance of taking into account the radial content of the generated two-photon state and the analysers’ detection states when dealing with OAM modes in the presence of turbulence.

Although turbulence acts as a decohering process when time averages are observed, in real time it simply imprints a phase perturbation on the beam. We note that it is possible to fully undo these perturbations if one could monitor and unwrap the wavefront deformations in real time by means of a phase corrector. This can be done using modern adaptive-optics techniques [216].
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Figure 9.4: Survival of OAM coincidence curves under influence of turbulence. Experimental coincidence rates (data points) and theoretical predictions (curves) obtained with two quadrant-sector phase plates for: no turbulence (blue), \( w_0/r_0 = 0.30 \) (green) and \( w_0/r_0 = 0.65 \) (red). The inset shows a blow-up of the wiggles around \( \alpha - \beta = \pi/2 \).

9.4 Results

In the experiment, the phase plates are rotated around their normals, and the time-averaged photon coincidence probability

\[
P(\alpha - \beta) = \left\langle \left| \langle A(\alpha) | B(\beta) | \hat{S}_A | \Psi \rangle \right|^2 \right\rangle_t
\]

is recorded as a function of their independent orientations. Here, the time-averaged behaviour of the turbulence scattering operator \( \hat{S}_A \) working on channel \( A \) can be described in terms of its coherence function Eq. (9.2),

\[
\left\langle \hat{S}_A | A(\alpha) \rangle \langle A(\alpha) | \hat{S}_A^\dagger \right\rangle_t = \int d\mathbf{r}_1 d\mathbf{r}_2 |\mathbf{r}_1| A(\alpha) |\mathbf{r}_2| e^{i\mathbf{\theta}(\mathbf{r}_1) - i\mathbf{\phi}(\mathbf{r}_2)} \right\rangle_t.
\]

Figure 9.4 shows our main experimental results. In the absence of turbulence, we observe a piecewise-parabolic coincidence curve (blue circles), i.e., the coincidence rate follows a parabolic dependence for \( |\alpha - \beta| \leq \pi/2 \) and is zero elsewhere (see Chapter 4). The coincidence rate depends on the relative orientation of the phase plates only. We have investigated how the coincidence rates evolve for 6 different turbulence strengths, two of them shown in Fig. 9.4: \( w_0/r_0 = 0.30 \) (green triangles) and \( w_0/r_0 = 0.65 \) (red stars). The latter strength was also used for Fig. 9.2(b) and 9.3(b). Note that the 20 s integration time used in the experiment assures isotropic sampling of the wavefront fluctuations (see Fig. 9.2(b)). We observe a partial “smoothening” of the coincidence curve, which is excellently described by our theoretical predictions based on Eqs. (9.2) and (9.4), without any fit parameter. The turbulence-induced wiggles at \( |\alpha - \beta| = \pi/2 \) are reproduced remarkably well (see inset).

We stress that, while the coincidence count rates exhibit this rich behaviour, the single
9.5 Discussion

Figure 9.4 shows that the coherence of the two-photon state is partly conserved even in the presence of rather strong turbulence. However, the turbulence inevitably damages the purity of the quantum state to some degree. Naturally, this damage, or equivalently the mixedness of the state, increases with increasing turbulence strength.

Quantifying entanglement for mixed states is a notoriously hard problem, especially if the entanglement is high dimensional [21, 217, 218]. Nevertheless, for mixed states of two qubits, some mathematical techniques exist to quantify the entanglement [219]. Two recent theoretical studies used this approach to investigate the robustness of entanglement between two spatial (OAM) modes against transmission through a turbulent noise channel [71, 220]. In our experiment, however, we are explicitly in the regime of high-dimensional OAM entanglement. Consequently, it is near impossible to extract a proper entanglement measure from the experimental data at hand. This notwithstanding, the prospect of a high dimensionality is the very motivation to study OAM entanglement in the first place.

In the following, we therefore take a first step towards analysing mixed high-dimensional entanglement, and apply the techniques developed in Chapter 4 for pure entangled states to the partially mixed states we are dealing with here. We attempt to quantify the robustness of the correlations in terms of the Shannon dimensionality $D$, as introduced in Chapter 4. It is an operationally defined measure and gives the effective number of modes the combined analysers have access to when scanning over their possible settings, viz. the phase-plate orientations. For two identical analysers in the absence of turbulence, we can express $D$ in terms of the pure detection state operator $\rho_A$ (or $\rho_B$), where $\rho_A = |A(\alpha)\rangle\langle A(\alpha)|$, as

$$D = \frac{1}{\text{Tr}[(\rho_A)^2]}.$$  \hspace{1cm} (9.6)

Here, $\langle \rho_A \rangle_\alpha$ is the density operator obtained by averaging $\rho_A$ over all phase-plate orientations $\alpha$.

In the presence of turbulent scattering, however, the detection state becomes randomly time dependent: $\rho_A = S_A|A(\alpha)\rangle\langle A(\alpha)|S_A^\dagger = \rho_A(t)$. The relevant detection state operator is therefore not $\rho_A$, but rather $\langle \rho_A \rangle_t$, i.e., the density operator averaged over time. In general, $\langle \rho_A \rangle_t$ is no longer a single-mode projector, but just a positive operator. In other words, when averaging over the random fluctuations, the detection state becomes multimode. We can attach a non-utilisable - dimensionality $D^{-1} = \text{Tr}[(\langle \rho_A \rangle_t)^2]$ to this detection operator, which gives the effective number of modes captured by the analyser for fixed orientation $\alpha$. The mixed nature of the detection operator blurs the analyser's modal resolution when scanning its orientation setting $\alpha$.

In analogy to Eq. (9.6), the total number of modes captured by the analyser when scanning its orientation setting $\alpha$ is given by $\text{Tr}[(\langle \rho_A \rangle_{t,\alpha})^2]$, where $\langle \rho_A \rangle_{t,\alpha}$ denotes the average of the detection state operator $\rho_A$ over time $t$ and orientation $\alpha$. However, we need to compensate for the contribution that arises from the degradation of resolution due to turbulence. Therefore,
in the presence of turbulence, the Shannon dimensionality for mixed detection states is written as

$$\tilde{D} = \frac{\text{Tr}[(\rho_A)_{1,\alpha}^2]}{\text{Tr}[(\rho_A)_{1,\alpha}^2]}$$

(9.7)

This expression can be understood as being the ratio between the dimensionality of the operator $\langle \rho_A \rangle_{1,\alpha}$ that is averaged over $\alpha$ and the dimensionality of $\langle \rho_A \rangle_{1}$, for fixed $\alpha$. Note that in the limit of no turbulence, this result reduces to Eq. (9.6).

It is worth noting that the numerator in Eq. (9.7) is independent of the specific phase plate in use. It can be shown that its evolution under the action of turbulence follows the survival probability discussed in Fig. 9.3(a). The denominator in Eq. (9.7), on the other hand, does depend on the specific phase plate in use.

Continuing with our naive approach, we can extract $\tilde{D}$ straightforwardly from the experimental coincidence curves in Fig. 9.4. Working in the regime $K \gg D$ and using identical phase-plate analysers in both arms, it can be shown that the numerator in Eq. (9.7) is associated to the maximum coincidence probability, and the denominator in Eq. (9.7) is associated to the average coincidence probability. It then follows that $\tilde{D} = 2\pi N_{\text{max}}/A$, where $N_{\text{max}}$ is the maximum coincidence rate and $A$ is the area underneath the curve.

Figure 9.5 shows how $\tilde{D}$ evolves for increasing turbulence strength according to theory and experiment. In the absence of turbulence, we find an experimental value $D = 5.7$ vs. a theoretical prediction $D = 6$ (see Chapter 4). As the turbulence strength increases, the modal

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*We note that in our experiment we have turbulence in one arm only. For this case, the Shannon dimensionality can be generalised as $\tilde{D} = \text{Tr}[(\rho_A)_{1,\alpha}\rho_B]/\text{Tr}[(\rho_A)_{1,\alpha}(\rho_B)_{\theta}]$. It can be shown that this reduces to Eq. (9.7) when one has similar but weaker turbulence in both arms.
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Figure 9.6: Robustness of the Shannon dimensionality for various OAM superposition states. Decay of the dimensionality for two quadrant phase plates (blue solid) and two double-octant plates (blue dashed), both of initial dimensionality $D = 6$. Similarly, decay of the dimensionality for half-sector phase plates (red solid) and half-integer spiral phase plates (red dashed), both of initial dimensionality $D = 3$. The resolution of the analysers degrades, constraining the dimensionality to smaller values, ultimately to $\tilde{D} = 1$. The number of modes is reduced by $\sim 50\%$ to $\tilde{D} = 3.1$ when $w_0/r_0 = 0.65$. Considering the severity of the wavefront distortions (see Fig. 9.2(b)), we conclude that our dimensionality $\tilde{D}$ is surprisingly robust. For comparison, we also plotted our experimental results obtained with two half-sector phase plates, having one semicircle phase shifted by $\pi$ (see inset Fig. 9.5). In this case we observe that the dimensionality, initially at a value $D = 3$, decays considerably more slowly. This indicates that the resilience to atmospheric turbulence is quite sensitive to the nature of the OAM superposition state, an aspect also noted in Ref. [68].

To further substantiate this observation, we consider two distinct OAM superposition states that have the same zero-turbulence dimensionality, and compute how their dimensionality decays for increasing turbulence strength (see Fig. 9.6). For instance, we compare the decay of the dimensionality for the case $D = 6$, using analysers equipped with (i) quadrant phase plates and (ii) phase plates having two opposing octants of optical thickness $\lambda/2$ (see inset Fig. 9.6 for the plate profile). These two types of phase plates bear a large similarity; the OAM superposition state corresponding to the double-octant phase plate has an identical OAM eigenvalue distribution as the superposition of the quadrant phase plate, albeit with twice the mode spacing. This double mode spacing reflects the two-fold symmetry of the double-octant plate as compared to the quadrant phase plate.

Fig. 9.6 shows that the dimensionality $\tilde{D}$ decays considerably more slowly for the double-octant plates (dashed blue curve) as compared to the quadrant phase plates (solid blue curve). A qualitative understanding of this difference in robustness can be obtained from the data

*In this limit for extreme turbulence, the azimuthal fingerprint of the analyser mode is fully wiped out. The detection state thus becomes circularly isotropic, leading to $D = 1$. 

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represented in Fig. 9.3(b). First of all, the dominant effect of turbulence is a loss of modal strength ($\Delta m = 0$ scattering). The non-conserving OAM scattering probability decays rapidly as a function of $|\Delta m|$. Therefore, the spreading of a mode applies to any initial value of $m$. Therefore, neighbouring modes in the OAM superposition associated with a certain phase-plate analyser affect each other strongly when separated by $|\Delta m| = 1$ (as for the quadrant phase plates) and much weaker when separated by $|\Delta m| = 2$ (as for the double-octant phase plates). This explains the robustness of the latter phase plates as compared to the standard quadrant phase plates.

As a second example, we compare OAM superposition states for the case $D = 3$, associated to (i) our half-sector phase plates (solid red curve) and (ii) half-integer spiral phase plates having a helical phase ramp of optical height $\lambda/2$ (dashed red curve). Since the OAM eigenvalue spectrum of the half-sector phase plate has twice the modal spacing of the half-integer spiral phase plate, the argumentation given above applies also here. Naturally, the argument can be extended, suggesting that OAM superposition states can be designed that have an optimal robustness against atmospheric perturbations.

So far, we have expressed the turbulence strength in terms of the ratio $w_0/r_0$. However, this quantity allows us to estimate the propagation distance $L$ that can be reached outside the laboratory, since the Kolmogorov theory (see Eq. (9.2)) used to describe our data is also a fair description of a real-life atmosphere. For horizontal propagation, the Fried parameter can be expressed as $r_0 = 3.02(k^2LC_n^2)^{-3/5}$, with $k = 2\pi/\lambda$ the wavenumber of the light, $L$ the propagation length and $C_n^2$ the structure constant quantifying the phase perturbations. To put this correspondence in perspective, we consider a wavelength $\lambda = 1550$ nm in the transmission window of the atmosphere and assume moderate ground-level perturbations ($C_n^2 = 10^{-14}$ m$^{-2/3}$) and a beam size $w_0 = 6$ cm. At $w_0/r_0 = 0.65$, where $D$ has decayed to 50% of its initial level, we find a propagation length of 2 km (satisfying the requirement $L < z_R$). This distance would suffice for use in a metropolitan environment. For vertical propagation from ground level through the entire column of the atmosphere, the Fried parameter is typically of the order of 5-15 cm, depending on the elevation and weather conditions. For the turbulence strength and beam size mentioned above, the Fried parameter equals 9 cm, suggesting that also a satellite communication link may be viable.

9.6 Conclusions

We have presented the first experimental data on the transmission of OAM-entangled photons through a turbulent atmosphere. We have found that the shape of the coincidence curve is quite robust under the action of the turbulence, and that the robustness can be enhanced by judiciously designing the OAM superposition that acts as an information carrier. Present-day adaptive-optical techniques are sufficiently developed that OAM entanglement for free-space distribution is viable.

9.7 Appendix

In this Appendix, we show the full measurement set of the coincidence count rates used for Figs. 9.4 and 9.5. Figures 9.7 and 9.8 show data obtained with half-sector phase plates and quadrant-sector phase plates, respectively. The agreement between experimental data (open
circles) and theoretical predictions (solid curves) is seen to be excellent. Note that the theoretical curves require just a single fit parameter; a trivial vertical scaling factor, which is determined for the initial case of no turbulence (see Figs. 9.7(a) and 9.8(a)) and kept fixed for increasing turbulence strengths (see Figs. 9.7(b)-(g) and 9.8(b)-(g)).

Figure 9.7: Coincidence curves for two half-sector phase plates. Circles denote experimental data points measured during a 20 s collection time. Curves denote theoretical predictions, using no fit parameter other than a trivial scaling factor that is determined in the absence of turbulence and kept fixed for all other graphs. (a) $w_0/r_0 = 0$ (no turbulence), (b) $w_0/r_0 = 0.24$, (c) $w_0/r_0 = 0.30$, (d) $w_0/r_0 = 0.36$, (e) $w_0/r_0 = 0.42$, (f) $w_0/r_0 = 0.53$, (g) $w_0/r_0 = 0.65$. 

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Figure 9.8: Coincidence curves for two quadrant-sector phase plates. Circles denote experimental data points measured during a 20 s collection time. Curves denote theoretical predictions, using no fit parameter other than a trivial scaling factor that is determined in the absence of turbulence and kept fixed for all other graphs. (a) \( \omega_0/r_0 = 0 \) (no turbulence), (b) \( \omega_0/r_0 = 0.24 \), (c) \( \omega_0/r_0 = 0.30 \), (d) \( \omega_0/r_0 = 0.36 \), (e) \( \omega_0/r_0 = 0.42 \), (f) \( \omega_0/r_0 = 0.53 \), (g) \( \omega_0/r_0 = 0.65 \).