CHAPTER 8

Atmospheric turbulence and optical propagation

8.1 Introduction

In this Chapter, we present an outline of the theory of light propagation through a turbulent atmosphere. Historically, this topic has received much attention from the astronomy community, as the quality of night-sky observations is degraded by atmospheric perturbations. As early as the 1600s, Christiaan Huygens noted when observing the sun that “that very small inequality on his Surface, which is discovered by the Telescopes […] , which makes Men fancy they see boiling Seas and belching Mountains of Fire, is nothing but the trembling Motion of the Vapours our Atmosphere is full of near the Earth; which is likewise the Cause of the Stars twinkling” [182]. Since the days of Huygens, the field of astronomy has witnessed tremendous advance in observation techniques. Nowadays, large ground-based telescopes are often equipped with adaptive optics in order to unwrap image distortions and retrieve the unperturbed light from distant sources [102, 183, 184].

As of late, the emerging field of (quantum) optical communications has triggered renewed interest in atmospheric turbulence, because it hinders the fidelity of information transfer across free-space channels. It is in this context that we will present a study on the suitability of orbital-angular-momentum entanglement for free-space quantum communication in Chapter 9. In the present Chapter, we set the stage for this work. We present a simple atmospheric model and treat the basics of the Kolmogorov theory of turbulence. Next, we introduce an optical description of the loss of coherence this inflicts on light beams. We will establish a connection to Chapter 9 where applicable. Finally, we compare different techniques to produce controlled and realistic atmospheric turbulence. In particular, we discuss the characteristics of our home-built turbulence generator.

8.2 Kolmogorov turbulence

Whirled up through meteorological processes, the earth’s atmosphere is in constant motion. Large scale wind flows connect meteorological zones of low and high pressure. Under turbulent conditions, such flows turn unstable, as local fluctuations of the mean velocity will cause
8. ATMOSPHERIC TURBULENCE AND OPTICAL PROPAGATION

Figure 8.1: Illustration of the emergence of turbulence in the atmosphere. Large-scale wind flows of size \( L_0 \) turn unstable. The resulting eddies of dimension \( \ell \) sequentially disintegrate, a process that continues down to the damping scale \( \ell_0 \).

the current to break down into a cascade of eddies. Here, we give a short overview of the Kolmogorov model of turbulence \([185, 186]\).

Let us start out with a laminar wind field that we describe macroscopically in terms of its velocity \( v \), the kinematic viscosity \( \eta \) and its typical length scale \( L_0 \). Turbulent motion is a stochastic process in the sense that density and velocity fluctuations emerge randomly. Now suppose a fluctuation of the velocity \( v \) occurs in a domain of size \( \ell \) within this field. The kinetic energy (per unit mass) of this fluctuation is equal to \( v^2 \ell \), and the typical time scale attached to its occurrence is of order \( \tau = \ell / v_\ell \). Thus the energy per unit time transported by this fluctuation can be estimated at

\[
\mathcal{E} \sim \frac{v^2 \ell}{\tau} = \frac{v^2}{\ell}. \tag{8.1}
\]

On the other hand, the occurrence of velocity gradients, given by \( v_\ell / \ell \), cause different domains to shear. This results in viscous damping. We find that the amount of energy per unit time \( \varepsilon \) that is dissipated from this fluctuation is given by

\[
\varepsilon \sim \frac{\eta v_\ell^2}{\ell^2}. \tag{8.2}
\]

Obviously, the fluctuation can only survive if \( \mathcal{E} > \varepsilon \). In a more formal formulation, this is expressed in the condition that the parameter \( \text{Re} = v\ell / \eta \), also known as the Reynolds number, is sufficiently large \([187]\).

If the Reynolds number is sufficiently large (\( \text{Re} \gtrsim 4000 \)), the fluctuation will in turn be unstable and break down into structures of smaller scale. Extrapolating this reasoning, we infer that the energy is transferred along a whole cascade of eddies of progressively decreasing size, down to a length scale \( \ell_0 \) at which energy is dissipated. Hence, the atmosphere shows turbulent behaviour on length scales between \( \ell_0 \) and \( L_0 \), known as the inner and outer scales of turbulence, respectively \([186]\). Typical values for the outer scale amount to 20 m, whereas the inner scale is of order 1 mm. An illustration of the emergence of turbulence in the atmosphere is given in Fig. 8.1.

Because of the stochastic nature of atmospheric turbulence, any description is necessarily statistical. Consider the fluctuations of some physical property \( f(\rho) \) at a point \( \rho = (x, y, z) \). Examples of such properties are the aforementioned velocity, or the density. Its statistical char-
acteristics are conveniently described in terms of the structure function \[ D(p_1, p_2) = \left\{ [f(p_1) - f(p_2)]^2 \right\}, \] (8.3)

where \( \langle \ldots \rangle \) denotes averaging over time. This quantity plays a pivotal role in the theory of turbulence, and is often more expedient than the related correlation function

\[
C(p_1, p_2) = \langle [f(p_1) - \langle f(p_1) \rangle] [f(p_2) - \langle f(p_2) \rangle] \rangle. \tag{8.4}
\]

Under the fair assumption that the turbulence is homogeneous (i.e., the same at all locations) and isotropic (i.e., the same in all directions), one can show that \( D(p_1, p_2) = D(|p_1 - p_2|) \). Roughly speaking, the structure function characterises the strength of fluctuations of \( f(p) \) with a length scale comparable to \( |p_1 - p_2| \) \([188]\).

In order to find an explicit expression for the structure function \( D_r(p_1, p_2) \) for the velocity \( v(p) \), we make the following two observations. First, the dominant contribution to the velocity difference between points \( p_1 \) and \( p_2 \) comes from eddies of size \( \ell \approx |p_1 - p_2| \), due to the stochastic nature of the perturbations. Second, we note that the dissipation of structures of length scales \( \ell > \ell_0 \) is marginal. Therefore, the energy they receive from larger perturbations upstream the cascade must be almost the same as the energy they transfer downwards, and Eq. (8.1) must thus hold at all length scales but for the smallest. Based on these observations, we then find that

\[
D_r(p_1, p_2) \propto (|p_1 - p_2|/\rho_0)^{2/3}, \tag{8.5}
\]

This result is known as "Kolmogorov’s two-thirds law" \([185]\) and can also be obtained through a dimensional analysis.

### 8.3 Propagation of light through a turbulent atmosphere

#### 8.3.1 Phase structure function

We will now consider the propagation of light through such a turbulent medium \([188–190]\). We shall restrict ourselves to the regime of weak turbulence, meaning that we take into account weak phase perturbations only, which occur due to variations of the atmospheric density along the beam trajectory. Consequences of phase perturbations leading to such effects as scintillation are a second-order effect and can be ignored as long as diffraction arising from the phase aberrations is negligible. This model is a good approximation for most applications and is supported by a variety of experimental characterisations of the atmosphere \([191–193]\). It is further corroborated by the fact that adaptive optics works.

In this regime, the cumulative effect of the turbulence along the entire propagation path can be treated as a phase perturbation \( \phi(r) \) at the output plane, where \( r = (r, \theta) \) are transverse polar coordinates (see Fig. 8.2). This constitutes a significant conceptual simplification.

We are interested in the stochastic time-evolution of the phase correlation between two points \( r_1 \) and \( r_2 \) in the output plane. This can be formalised in terms of the time-averaged coherence function

\[
C(r_1, r_2) = \langle e^{i\phi(r_1) - i\phi(r_2)} \rangle, \tag{8.6}
\]
8. ATMOSPHERIC TURBULENCE AND OPTICAL PROPAGATION

Figure 8.2: Beam propagation through turbulent air. In the weak turbulence regime, phase perturbations \( \phi(r) \) are taken into account, but amplitude perturbations, which are of second order, are neglected. (a) The cumulative effect of turbulence along the entire propagation length \( L \) can be reduced to (b) a single phase operation at the output plane \( B \).

where \( \langle \ldots \rangle \) denotes averaging over time. By virtue of the central limit theorem, which states that for Gaussian random processes one can approximate \( \langle e^{ix} \rangle = e^{-\frac{1}{2}(|x|^2)} \), Eq. (8.6) can be rewritten as

\[
\langle e^{i(\phi(r_1)-\phi(r_2))} \rangle = e^{-\frac{1}{2}(|\phi(r_1)-\phi(r_2)|^2)} = e^{-\frac{1}{2}D_\phi(r_1-r_2)},
\]

where \( D_\phi(|r_1 - r_2|) \) is the structure function for the phase (see Eq. (8.3)). The final objective is thus to find an explicit expression for the phase structure function.

The phase difference accumulated over a propagation length \( L \) between two parallel trajectories at a distance \( |r_1 - r_2| \) from each other is simply given by

\[
\phi(r_1) - \phi(r_2) = k \int_0^L \left[ n(r_1, z) - n(r_2, z) \right] dz,
\]

where \( k = 2\pi/\lambda \). This reveals that the structure function for the phase is intimately related to that of the refractive index \( n(r) \). Indeed, it can be shown that the structure functions of both the density and the refractive index follow a similar two-thirds power law as derived for the velocity in Eq. (8.5) [188]. Exploiting this knowledge, one may derive the structure function for the phase, which is conventionally written in the following form [188, 190, 194]

\[
D_\phi(|r_1 - r_2|) = 6.88 \left( \frac{|r_1 - r_2|}{r_0} \right)^{5/3}.
\]

We observe that the phase structure function goes with the five-thirds power of the distance between two points, a corollary of the Kolmogorov assumptions. The distance \( r_0 \), named the
8.3. PROPAGATION OF LIGHT THROUGH A TURBULENT ATMOSPHERE

Figure 8.3: **Propagation of a diffraction-limited beam through the turbulent atmosphere.** Under influence of turbulent scattering, the smooth wavefront of the light beam becomes deformed. The directionality of the beam is reduced, causing a wave-vector spread and a broadening of the beam when averaged over time. The time-averaged phase front is ill-defined, having a phase variance $\Delta \phi$ when averaged across the beam.

Fried parameter after its introducer [189, 194], manifests the optical consequences of the induced phase distortions: it plays the role of a correlation length, and the constant 6.88 is chosen such that $r_0$ is that distance over which the wavefront is distorted by approximately 1 radian of root-mean-square phase difference. That is, in the absence of turbulence $r_0 \to \infty$, but when turbulence becomes stronger, the spatial coherence is reduced and hence $r_0$ shortens.

It is evident that the Fried parameter plays an important role in the field of astronomical observations. While telescopes have become ever bigger to increase their light gathering ability, the image resolution they obtain has not kept pace. The resolution is limited by atmospheric wavefront degradation and it is the Fried parameter that determines the largest aperture size that ensures a diffraction-limited imaging. For astronomical observations at visible wavelengths, typical values of the Fried parameter range between 3 cm at sea level to 30 cm at high altitude [183, 195].

Finally, we note that the model presented here constitutes the canonical Kolmogorov model of turbulence, which behaves similarly at all length scales (i.e., inner scale $\ell_0 \to 0$ and outer scale $L_0 \to \infty$). Several refinements of the model have been made to incorporate asymptotic behaviour for finite inner and outer scales [196, 197], but in many cases Eq. (8.9) suffices. In Section 8.4.2 we will discuss the validity of the model for the experimental configurations used in Chapter 9.

8.3.2 Beam propagation

We now turn to the problem of beam propagation through a turbulent atmosphere. Consider a Gaussian beam with a beam waist $w_0$. The refractive index variations along its trajectory give rise to anomalous refraction, i.e., locally producing wavefront deflections and lensing effects [198]. These random distortions vary over time and, as a consequence, the time-average wave-vector spread of the beam widens. This results in a net beam broadening (see Fig. 8.3).

From the beam broadening we can determine the relation between the Fried parameter and the beam waist,

$$\frac{w_0}{r_0} = \sqrt{(w_{le}/w_{dl})^2 - 1} / 3.0,$$

with $w_{dl}$ and $w_{le}$ the 1/e far-field radius of the diffraction-limited beam and long-exposure.
broadened beam, respectively [199]. See Fig. 9.2 for an experimental example of turbulence-induced broadening.

For a horizontal line of sight parallel to the earth's surface, the atmosphere is relatively uniform. For beam propagation in the horizontal direction the Fried parameter can be expressed as [200]

\[ r_0 = 3.02 \left( k^2 L C_n^2 \right)^{-3/5}. \]

It thus depends on the wavelength of the light via \( k = 2\pi/\lambda \), the propagation length \( L \) and the turbulence conditions. The latter are contained in the refractive-index structure constant \( C_n^2 \), which is a measure of the strength of the refractive-index inhomogeneities. Typical values of \( C_n^2 \) range between \( 10^{-12} \) and \( 10^{-15} \) m\(^{-2/3}\) for very mild and quite severe conditions, respectively. \( C_n^2 \) is largest within a few meters above the ground and drops rapidly with increasing altitude. As a numerical example, let us assume moderate inhomogeneities, \( C_n^2 = 10^{-14} - 10^{-15} \) m\(^{-2/3}\), and a visible wavelength of 550 nm. We find a Fried parameter of \( 1 - 4 \) cm for a propagation distance of 10 km. This number is larger than the size of our pupils, which explains why, under normal conditions, we do not notice turbulence effects with our bare eyes. Furthermore, Eq. (8.11) shows that the structure constant and the propagation length can be exchanged to yield the same value of the Fried parameter. We will use this property in the next Chapter to scale up our experimental conditions to real-life proportions, and estimate the outdoors propagation length feasible for quantum communication.

For a vertical line of sight from ground level through the entire column of the atmosphere, the structure constant obviously depends on the altitude: \( C_n^2 = C_n^2(z) \). For this case, Eq. (8.11) can be generalised to

\[ r_0 = \left[ 0.423 k^2 z \int_0^1 d\xi \xi^{5/3} C_n^2(\xi z) \right]^{-3/5}. \]

Typically, reported values of the Fried parameter for vertical propagation through the atmosphere are of the order of 3-30 cm, depending on weather conditions and the elevation of the observation station [183, 195].

At any given instant in time, the wavefront at the output plane of a turbulent trajectory may well be perturbed, but it remains well-defined. The time-averaged effect of turbulence, in contrast, cannot be described by a pure phase imprint only, as can be seen from the coherence function for the phase (see Eqs. (8.7) and (8.9)). This suggests that the spatial coherence of the beam is (partially) reduced.

One can quantify the loss of coherence in terms of the variance \( \Delta \phi \) of the phase fluctuations in the beam. When naively calculating the phase variance along a light path, however, one will find that it tends to blow up. Nonetheless, this has little physical consequences; what matters is the variance of the phase difference between two parallel paths (see Eq. (8.8)). This is the case because the common phase component across the beam does not degrade the coherence and can thus be omitted. The variance of the phase difference is finite and well-defined [201].

The phase variance can be obtained from the beam broadening discussed above, or more precisely, from the Strehl ratio,

\[ S = \frac{I_T}{I_0}, \]

which is experimentally accessible in a straightforward fashion; \( I_T \) and \( I_0 \) are simply the peak
8.4 Laboratory implementations of turbulence

When studying the effect of atmospheric turbulence on beam propagation, one would be tempted to simply launch a light beam through the atmosphere. There are several reasons, however, to work with artificial turbulence, the primary one being the ability to control and tune the turbulence characteristics [203]. For instance, one may want to simulate various turbulence strengths, propagation lengths or altitudes. There exist two methods to produce tailored turbulence: by means of 2-D random phase screens and by means of 3-D turbulence cells.

\[ S = e^{-\left(\Delta \phi\right)^2}. \]  

Typical values of the Strehl ratios for the experiments described in Chapter 9 lie in the range \(0.3 < S < 1\), the lower value being reached at the maximum turbulence strength at our disposal. This corresponds to a phase variance of \(0 < \Delta \phi < 2 \pi / 7\).  

It is instructive to consider the influence of this “phase smearing” on the orbital-angular-momentum detection states used in the next Chapter. In Fig. 8.4, we plot the azimuthal phase profile of the detection state for an analyser equipped with a quadrant phase plate. The black line represents the unperturbed phase profile, being a binary function between \(\phi = 0\) and \(\phi = \pi\). Due to the turbulence, the time-averaged wavefront gets spread out; the phase is no longer well-defined, but rather has a certain phase spread corresponding to the variance \(\Delta \phi\). This fuzziness is tantamount to a loss of modal resolution of the analyser.
8. ATMOSPHERIC TURBULENCE AND OPTICAL PROPAGATION

Figure 8.5: Turbulence-induced phase distortion. Example of a random phase screen profile, the phase distribution of which follows Kolmogorov statistics. Black and white shades represent various levels of phase imprint between $-30\pi$ and $50\pi$. The bar shown in the bottom left gives an estimate of the Fried parameter. This pattern was designed with software from the Max-Planck-Institut für Astronomie, Heidelberg [213].

8.4.1 Random phase screens

The first method makes use of a phase mask that carries a random phase profile, as would be imparted by the atmosphere. Advanced computational techniques exist to design realistic Kolmogorov phase patterns (see Fig. 8.5) [204]. These patterns can then be fabricated as a static diffractive element [205-207], or be printed on phase modulating devices such as spatial light modulators [208] or deformable mirrors [209]. To mimic the dynamics of the atmosphere, the phase pattern is continuously refreshed (dynamic devices), or rotated through the beam at the typical wind speed (static screens). The validity of the latter modus operandi is established by the hypothesis of frozen turbulence proposed by G. I. Taylor [210]. This hypothesis relates the turbulent fluctuations in time to those in space, and assumes their interchangeability. Poor Kolmogorov statistics due to a repetitive reoccurrence of the rotating pattern can be mitigated by using two static phase screens that rotate at slightly different frequencies.

With current technology, it is possible to micro-machine these random phase screens at small dimensions, which has enabled them to become a customary tool to test miniaturised adaptive optics eventually meant for large ground-based telescopes [211, 212].

8.4.2 Turbulence cells

In the second method, real, physical turbulence is created in a mixing chamber, by intermixing hot and cold air to bring about random refractive-index variations [203]. Alternatively, the air can be replaced by a liquid. This is particularly helpful for the creation of strong turbulence, because the refractive-index contrast in fluids can be much larger [214]. A major advantage of turbulence cells is that the generated turbulence naturally follows a Kolmogorov distribution. Full control of the desired inner and outer scales, however, is usually quite challenging.
8.4. LABORATORY IMPLEMENTATIONS OF TURBULENCE

The results presented in the Chapter 9 were performed with a home-built turbulence cell. The cell is made of a 7 cm long, 26 mm inner diameter glass tube, containing 8 resistors that produce up to 60 W of heat (see Fig. 8.6). A gentle flow of room temperature air is driven straight through the tube. We can tune the turbulence strength by varying the heating power and air flow through the tube.

The airflow is generated as follows: our detection apparatus is enclosed in a box that is kept at overpressure. A small aperture in one side of the box allows the optical beam to enter and the air to leave it. The use of this box as a source of cold air has two advantages. First, no heat can leak into the box and cause the optical equipment to lose alignment due to thermal drift. Second, the optical beam and cold air jet propagate along the same axis, avoiding directional inhomogeneities and asymmetric convection currents. Residual beam deflections in the vertical direction, which easily emerge due to upward movement of hot air, can be nullified by setting a lower limit to the air flow and optimising the beam pointing through the cell.

For reasons of experimental convenience, the beam diameter in the experiments in Chapter 9 is set to 2 mm. As this size is comparable to the inner scale of turbulence (as mentioned above typically of order 1 mm), we cannot expect to realise a turbulence strength $w_0/r_0$ much bigger than 1. In the experiments, we restricted ourselves to controllable and reproducible turbulence strengths between $0 < w_0/r_0 < 0.65$.

Both the inner and outer scale of turbulence are finite in our turbulence cell. When modeling our experiments, we should therefore question the validity of the most basic phase structure function described by Eq. (8.9), which was derived for $\ell_0 \rightarrow 0$ and $L_0 \rightarrow \infty$. From the literature, it is known that the effect of a finite inner scale $\ell_0$ on the phase structure function is usually modest [197]. A finite outer scale $L_0$, on the contrary, may have considerable influence. We therefore estimate the outer scale of our system. The size of the largest eddies that can form is constrained by the size of the turbulence cell and a conservative estimate would therefore be that $L_0 = 26$ mm, being the tube diameter. (In fact, the air is blown out of the tube and remains turbulent at some distance from it, suggesting that the outer scale is probably larger). The derivation of the theoretical curves in Fig. 9.4 from Chapter 9 is based on Eq. (8.9). We see that for large $w_0/r_0$ the canonical Kolmogorov theory indeed slightly overes-

Figure 8.6: Turbulence cell used in our experiments. The turbulence cell is composed of a glass tube (length 70 mm, diameter 26 mm) inside which several resistors are mounted, producing up to 60 W of heat. A jet of cold air is directed straight into the turbulence cell. Hence, hot and cold air are intermixed inside the tube, bringing about dynamic turbulence. The cold air emanates from a closed box that is maintained at overpressure and contains the detection apparatus. Light transits the turbulence cell and enters the box through the same aperture that serves as the outlet of cold air.
timates the turbulence in our system. Using advanced Kolmogorov modeling for finite outer scale [196, 197], however, we find that the value $L_0 = 26$ mm underestimates the turbulence by a factor of 2. The agreement between experimental and theoretical curves is sufficiently good to warrant our use of the canonical Kolmogorov theory.

8.5 Conclusions

We have outlined the Kolmogorov model of turbulence and introduced the phase structure function, which describes the loss of coherence an optical beam experiences when propagating through a turbulence medium. The relevant parameter in the model is the Fried parameter, which plays the role of a coherence length. Furthermore, we have discussed the home-built turbulence generator we will use in the next Chapter and examined its properties. It produces Kolmogorov-like turbulence up to considerable strengths and thus forms an attractive system to study orbital-angular-momentum transport through the atmosphere.