CHAPTER 7

Beyond angular entanglement

7.1 Introduction

In the community working on spatial entanglement as a resource for high-dimensional quantum information processing, particular emphasis has been put on the orbital-angular-momentum content of the field [40, 44, 47]. However, there is no intrinsic argument to restrict oneself to the azimuthal content of the field and neglect the radial content [42, 111]; the choice for this restriction seems to be merely motivated by the fact that there is no obvious approach to deal with the radial degree of freedom. It may well be, however, that exploitation of the radial content of the field provides for interesting opportunities and enhanced performance.

Also in this thesis we have focused our attention on angular entanglement. However, this is an unnecessary restriction, as our ideas on the Shannon dimensionality of measured entanglement surpass this particular application. Here, we make some first steps towards the combined use of both angular and radial degrees of freedom in bipartite spatial entanglement, and explore some of the resulting richness and associated challenges.

7.2 Setting

The approach we choose to follow - obviously one of many - is to leave the essence of our experimental scheme the same, i.e., to use two spatial-mode analysers with conjugate phase plates that can be rotated relative to each other, but to enrich the structure of the phase plates. That is to say, we incorporate the radial degree of freedom in our experiments by using phase plates carrying both angular and radial phase information. The normalised detection mode function of the angular state analyser in arm $A$ for given orientation $\alpha$ can then be written as

$$A(r, \alpha) = \frac{w_o}{\sqrt{2}} e^{-r^2/w_o^2} e^{i f(r, \alpha)},$$  \hspace{1cm} (7.1)

and similarly for $B(r, \beta)$. Here, $r = (r, \theta)$ are the polar coordinates in the transverse plane, $f(r, \alpha) = f(r, \theta - \alpha)$ is the phase profile of the phase plate and $w_o$ is the Gaussian beam waist setting the radial size of the mode.
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The effective dimensionality probed by one such analyser can be obtained from its auto-correlation function $G(\alpha, \alpha') = \langle A(\mathbf{r}, \alpha) A^*(\mathbf{r}, \alpha') \rangle$, through (see Chapter three)

$$D_{\text{eff}} = \frac{2\pi}{\int_0^{2\pi} |G(\alpha - \alpha')|^2 d(\alpha - \alpha')}.$$

(7.2)

In other words, the effective dimensionality is found by averaging the overlap probability $|G(\alpha - \alpha')|^2$ over the relative analyser settings $\alpha - \alpha'$, regardless of the exact phase profile of the plate. We conclude that the framework we developed to define the effective dimensionality of a state analyser remains intact. Note, however, that the radial coordinate becomes an essential variable that needs to be taken into account explicitly. That being said, it is immediately apparent that the radial size of the detection mode, set by the Gaussian beam waist $w_o$, has become a critical parameter.

Our experiment is essentially the same as the one discussed in the previous Chapter, with our generalised phase plates installed in the state analysers. The two-photon coincidence probability that we measure can be written as (see Chapter four)

$$P(\alpha - \beta) \propto \left| \int d\mathbf{r}_1 d\mathbf{r}_2 \Psi(\mathbf{r}_1, \mathbf{r}_2) A^*(\mathbf{r}_1, \alpha) B^*(\mathbf{r}_2, \beta) \right|^2,$$

(7.3)

where $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1, \mathbf{r}_2 | \Psi \rangle$ represents the pure spatially-entangled state as given in Eq. (2.7) from the Chapter 2. We define the Shannon dimensionality of measured entanglement as (see Chapter 4)

$$D = \frac{2\pi P_{\text{max}}}{\int_0^{2\pi} P(\alpha - \beta) d(\alpha - \beta)},$$

(7.4)

where $P_{\text{max}} = P(\alpha = \beta)$ represents the maximum coincidence probability.

7.3 Escher phase plates

It may be worthwhile to study which 2-D plate profiles maximise this quantity, in a fashion similar to our work on multi-sector phase plates from Chapter 7. For the current purpose, we manufactured a binary phase structure with formidable complexity, using the work “Vissen” by the artist M.C. Escher for the phase plate design (see Fig. 7.1(a)). In our implementation of this pattern, the colours black and white are translated into regions of phase $\Delta \phi = 0$ and $\pi$ at the wavelength of interest ($\lambda = 826$ nm). The phase plates were realised as a height relief; they were manufactured in glass and measured 3.5 × 3.5 mm² in total (see Chapter 5). The detection mode size $w_o$ used in our experiments measured 800 µm, and is indicated by the circle in Fig. 7.1(a). The cross hairs indicate the rotation centre of the plate. The magnified microscope image from Fig. 7.1(b) reveals the pixelation of the pattern due to the finite resolution of the design (800 × 800 pixels, pixel size 4.4 × 4.4 µm²). Figure 7.1(c) shows an image of the Escher phase plate, which is back-illuminated by a Gaussian beam with $w_o = 800$ µm. The cross hairs indicate the centre of mass of the Gaussian beam, to which the rotation centre of the phase plate should be aligned.
Figure 7.1: **Escher phase plate.** (a) Picture of the phase plate design, with black and white colours representing regions that impart a 0 and π phase shift at the wavelength of interest, respectively. The cross hairs indicate the centre of rotation of the plate. The circle indicates the beam waist $w_0$ of the detection mode as used in the experiment. (b) Microscope image of the central area. (c) Image of the Escher phase plate, which is back-illuminated by a Gaussian beam with a beam waist $w_0$ of the same size as indicated in (a). The cross hairs indicate the centre of mass of the Gaussian beam, to which the rotation centre of the phase plate should be aligned with µm precision. Figure (a) after M.C. Escher’s “Vissen” © the M.C. Escher Company BV-Baarn-the Netherlands, www.mcescher.com, used with kind permission.
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7.4 Relevant parameters and coincidence probability

As mentioned above, two new experimental parameters come into play when allowing the phase plates to have additional radial phase content, being the choice of the rotation centre and the detection beam waist \( w_0 \). Furthermore, at variance with the experiments discussed in Chapters 4 and 7, it turns out that we have to take the spatial properties of the down-conversion source into account explicitly when dealing with our Escher phase plates. In the following, we discuss how the choice of (i) rotation centre, (ii) detection beam waist and (iii) bandwidth of the source affect the coincidence probability and the corresponding Shannon dimensionality.

(i) Unlike purely angular phase plates (like, for instance, the multi-sector phase plates discussed in the Chapter 7), the Escher plate does not have a natural centre around which to rotate. We thus have to select a centre. In Fig. 7.1(a) and (b), the location of the plate centre we opted for in our experiments is indicated by the cross hairs. A different choice of the plate centre, however, has an immediate impact on the coincidence probability. In Fig. 7.2(a), we plot the calculated coincidence-probability curves \( P(\alpha - \beta) \) (see Eq. (7.3)) for two different choices of alignment of the two plate plates. The blue solid curve corresponds to correct alignment according to our definition from Fig. 7.1. The red dashed curve is obtained when displacing the rotation centre of the plates by 20 \( \mu m \) to the top. We observe that the exact details of the coincidence curve depend on the transverse positioning of the phase plates. The dimensionality corresponding to these two coincidence curves varies from \( D = 8.3 \) for our experimental parameters (blue solid) to \( D = 9.6 \) (green dotted).

(ii) Exploiting the radial degree of freedom, the beam size \( w_0 \) of the detection modes has become a critical parameter. In Fig. 7.2(b), we plot the calculated coincidence-probability curves for various beam sizes \( w_0 \). The blue solid curve corresponds to \( w_0 = w_{\exp} = 800 \mu m \), the value we used in our experiments. The red dashed and green dotted curves correspond to \( w_0 = 0.8w_{\exp} \) and \( w_0 = 1.2w_{\exp} \), respectively. We observe that the shape, modulation depth and peak width of the curves depend sensitively on \( w_0 \). As a result, also the Shannon dimensionality varies significantly, ranging from \( D = 8.3 \) for our experimental parameters (blue solid) to \( D = 6.3 \) (red dashed) and 10.1 (green dotted).

(iii) The spatially-entangled photon pairs are experimentally created by means of parametric down conversion in a 2 mm nonlinear periodically-poled KTP crystal (see Fig. 4.1 from Chapter 4). In Chapter 2, we experimentally investigated the emission of such a down-conversion source. Our analysis led us to conclude that the detailed spatial structure of the emission could safely be ignored, since our source generates an abundant number of entangled modes compared to the modal bandwidth of our angular-phase-plate analysers (concisely expressed in the statement that the Schmidt number of the source \( K \gg D_{\text{eff}} \)). In the present discussion regarding the Escher phase plates, however, this statement is no longer justified, as we will now argue. In Fig. 7.3(a), we show a measurement of the angular emission profile of our periodically-poled KTP source, having a Schmidt number of order \( K_{\exp} \approx 450 \). The angular-acceptance profile of the detection mode is shown in Fig. 7.3(b). Although the total angular spread of the emitted down-conversion light is larger than the angular spread of the detection mode, the intensity of the emission cannot be considered uniform for those angles probed by the analyser. This means that we have to take the spatial structure of the emission, contained in \( \Psi(r_1, r_2) \), explicitly into account.

Figure 7.2(c) shows how the calculated coincidence probability depends on the number of
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Figure 7.2: Calculated normalised coincidence-probability curves for various experimental parameters. (a) Dependence on the transverse position of phase plates’ rotation axis for alignment according to the convention as shown in Fig. 7.1(b) (blue solid); and a 20 μm vertical displacement to the top (red dashed). (b) Dependence on the radial beam size \( w_0 \) of the two detection modes for the experimental value \( w_0 = w_{\text{exp}} = 800 \) μm (blue solid); \( w_0 = 0.8w_{\text{exp}} \) (red dashed); and \( w_0 = 1.2w_{\text{exp}} \) (green dotted). (c) Dependence on the modal content of the two-photon field for our experimental conditions with Schmidt number \( K = k_{\text{exp}} \sim 430 \) (blue solid); an ideal source with \( K \to \infty \) (red dashed); and \( K \to K_{\text{exp}}/2 \) (green dotted). The blue solid curves in (a)–(c) are identical and based on the parameters used in the experiment.
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Figure 7.3: Angular intensity profiles of down-conversion source and detection modes. (a) Measured angular emission profile of the PPKTP down-conversion source (crystal length 2 mm, pump beam waist 250 µm). (b) Measured angular acceptance profile of an Escher phase plate analyser.

generated entangled modes. The blue solid curve is calculated for our experimental conditions corresponding to Fig. 7.3(a), yielding $D = 8.3$. The red dashed curve represents the case of an ideal source, i.e., a source that emits an infinite number of entangled modes ($K \to \infty$). This curve is remarkably different and the dimensionality increases dramatically to $D = 25.3$. The green dotted curve, on the contrary, demonstrates that the dimensionality drops to $D = 5.7$ when the number of generated modes is reduced by a factor of 2 compared to our experimental settings ($K \to K_{\exp}/2$).

7.5 Experimental results

In the following, we present our experimental results. Figure 7.4 shows the experimental coincidence curve as a function of the relative orientation $\alpha - \beta$ between the two Escher phase plates. Around $\alpha = \beta$, we record a maximum coincidence rate of $P_{\text{max}} = 4.8 \times 10^5$ s$^{-1}$, compared to an (orientation-independent) single-count rate of $S_e = 1.6 \times 10^5$ s$^{-1}$ at either detector. The minima of the curve around $\alpha - \beta = 90^\circ$ and $270^\circ$ are deep and measure $\sim 10$ counts s$^{-1}$ only. We corrected for an accidental coincidence rate of $S_e^2 \tau = 69$ s$^{-1}$, arising due to the finite coincidence detection window $\tau = 2.7$ ns. The solid curve represents the theoretical calculation based on Eq. (7.3), using the experimental values for the beam waist $w_0$ and the phase plate centre discussed in Fig. 7.2. The spatial structure of the entangled state $\Psi(r_1, r_2)$ was explicitly taken into account. We observe that the experimental data and theoretical model are in reasonable agreement. Deviations between the two are largest around $\alpha - \beta = 180^\circ$. This region, with reference to Fig. 7.2, is most sensitive to experimental deficiencies. The Shannon dimensionality we obtain from the experimental data set amounts to $D_{\exp} = 5.5$, whereas the theoretical model yields $D_{\text{th}} = 8.3$.

It is natural to wonder why the Shannon dimensionality $D$ has this low value. Would it not be reasonable to expect that $D$ would be much increased by using phase plates that have
7.6 Conclusions

Exploiting phase plates that carry 2-D structure, we have explored the possibility to harness spatial entanglement of two photons beyond the angular regime. As such, our experimental results constitute a first step towards the combined use of both radial and azimuthal degrees of freedom in bipartite transverse-mode entanglement.
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