High-dimensional entanglement with orbital-angular-momentum states of light

Using a measurement-based approach, we extract and engineer high-dimensional orbital-angular-momentum entanglement of photon pairs that emerge from a parametric-downconversion source. By means of two angular-state analysers, in essence composed of a rotatable multi-sector phase plate and a single-mode fiber, we perform selective projective measurements that maximise the Shannon dimensionality $D$ of measured entanglement. The multi-sector phase plates have a binary phase profile along the azimuthal coordinate, and the arc sector sizes are optimised so as to maximise $D$. We find that the maximum dimensionality increases linearly with the number of sectors $N$. The potential of our method is illustrated with an experiment for $N = 4$, yielding $D = 16.5$.

6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

6.1 Introduction

High-dimensional bipartite entanglement has been predicted to yield a serious speed-up of many quantum-computational protocols as compared to two-qubit entanglement [17, 19, 20, 60]. As a potential candidate for such high-dimensional bipartite entanglement, orbital-angular-momentum (OAM) states of light have received a lot of attention [28]. So far, quantum cryptography [62] and quantum coin tossing [61] have been experimentally realised using OAM-entangled qutrits.

For any application of these methodologies, the generation and management of OAM-entangled states is obviously of paramount importance [173]. The most common way to generate OAM-entangled photons is through the process of spontaneous parametric down-conversion. In a typical setting, the generated spectrum of the OAM correlations is broad with long tails towards high OAM values, as determined by phase matching in the nonlinear crystal. The exact shape of the spectrum can be tailored by controlling the focusing or wavefront of the laser beam pumping the nonlinear crystal, or controlling the phase-matching characteristics of the crystal by, for instance, temperature tuning [35]. Alternatively, the problem can be addressed by designing periodically-poled nonlinear crystals with a tailored modulation of the nonlinearity. However, the fabrication of such artificial crystals is challenging [177, 178].

An alternative, measurement-based approach to tailoring entangled superposition states is based on entanglement extraction [179]. This method makes use of selective projective measurements that filter those components of a pure entangled state that are of interest, and discard others. The entanglement can be enhanced (entanglement distillation) or reduced (entanglement dilution). This idea has been used to create “triggered” OAM-entangled qutrits [180, 181].

In Chapter 4, we have used essentially these techniques to extract high-dimensional OAM entanglement from a parametric-down-conversion source of entangled photon pairs. We addressed the question how to quantify the entanglement, and we introduced the Shannon dimensionality $D$ of measured entanglement. This quantity, which may be viewed as a “filtered” Schmidt number, gives the effective number of entangled modes that are detected by the measurement apparatus [30, 87]. We demonstrated the principles of our method, and experimentally achieved $D = 3$ and $D = 6$. In the present Chapter, we expand on this idea and demonstrate its potential to extract genuinely high-dimensional OAM entanglement.

Our method is based on the use of two identical state projectors that selectively filter specific OAM components of the entangled state emitted by the down-conversion source. Such an analyser is in essence composed of a single-mode fiber and a rotatable angular phase plate, which is a diffractive optical element carrying a varying optical thickness along the azimuthal coordinate. Specifically, we use plates that carry a binary azimuthal phase profile, having $N$ elevated arc sectors that produce a $\pi$ phase shift with respect to the recessed areas between them (see Fig. 6.1). Depending on the exact distribution of the sectors, the projection state is generally an extended superposition of OAM eigenmodes, which varies with the orientation of the plate; by rotating the phase plate around its central pivot, the analyser probes a Hilbert space of large dimensionality (see Chapter 3). In this Chapter, we explore and demonstrate the potential of such multi-sector phase plates to extract high-dimensional OAM entanglement. We optimise the multi-sector profiles for given $N$ so as to maximise the Shannon dimensionality of measured entanglement, and illustrate our results with an experiment.
6.2 Setting

In this Section, we briefly review the basic principles behind our experiments. Using a down-conversion source (see Section 6.5), we create photon pairs that are entangled in their OAM degree of freedom in a fashion\(^1\) [33]

\[
|\Psi\rangle = \sum_{m=-\infty}^{\infty} c_m |m\rangle_A \otimes | - m\rangle_B,
\]

where \(|m\rangle_A \ (|m\rangle_B)\) indicates the Schmidt mode containing one photon in arm \(A(B)\) with quantised orbital angular momentum \(m\hbar\), where \(\langle \theta | m \rangle = \exp(im\theta)/\sqrt{2\pi}\) and \(\theta\) is the azimuthal angle. The complex expansion coefficients \(\{c_m\}\) obey the normalisation condition \(\sum_m |c_m|^2 = 1\). The entanglement is analysed by means of two angular-phase-plate analysers that are equipped with multi-sector phase plates. We measure the coincidence rate as a function of the orientations \(\alpha\) and \(\beta\) of the two phase plates. The coincidence probability is given by

\[
P(\alpha, \beta) = |\langle A(\alpha) \otimes B(\beta) | |\Psi\rangle |^2,
\]

where the projection state associated with the analyser in arm \(A\) is described by

\[
|A(\alpha)\rangle = \sum_m \lambda_m |m\rangle e^{im\alpha},
\]

and similarly for \(|B(\beta)\rangle\). Here, the complex coefficients \(\lambda_m = \langle m | A(a = 0)\rangle\) are fully determined by the physical profile of the phase plate (and not by its orientation); see Eq. (6.8). The absolute squares of these coefficients, \(y_m = |\lambda_m|^2\), yield the coupling strengths of an OAM mode \(|m\rangle\) to the analyser, where normalisation prescribes that \(\sum_m y_m = 1\) (i.e., neglecting photon losses).

Equation (6.1) enunciates the fact that the two entangled photons possess complementary OAM. We therefore use two complementary phase plates on the two phase-plate analysers, obeying the symmetry \(\lambda_m^A = (\lambda_{-m}^B)^*\). Here, \(\{\lambda_m^A\}\) and \(\{\lambda_m^B\}\) indicate the expansion coefficients for the projection states in arm \(A\) and \(B\), respectively. Given this symmetry, the coincidence probability described by Eq. (6.2) can be shown to take the form

\[
P(\alpha - \beta) = \sum_{m,m'} y_m y_{m'} c_m c_{m'}^* e^{i(m-m')(\alpha - \beta)}.
\]

Not surprisingly, this expression shows that the coincidence probability depends both on the source properties (through the product \(c_m c_{m'}^*\)) and on the detectors’ properties (through the

\(^1\)In fact, the photon pairs are entangled in both transverse spatial degrees of freedom. However, we will restrict ourselves to a specific single radial mode and consider the azimuthal content of the quantum correlations only.
6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

product $y_m y_{m'}$). We will use this expression to compare with our experimental results presented in Section 6.5.

We will work in the regime that the number of generated entangled modes is much larger than the number of modes our analysers have access to. We may then safely assume that the products $c_m c_{m'}^*$ vary negligibly over the range where the products $y_m y_{m'}$ are nonzero. As shown in Chapter 4, the coincidence probability $P(\alpha - \beta)$ then directly yields the Shannon dimensionality $D$ of the measured entanglement through

$$D = \frac{2\pi P_{\text{max}}}{\int_0^{2\pi} P(\alpha - \beta) d(\alpha - \beta)},$$

(6.5)

where $P_{\text{max}} = P(\alpha = \beta)$ is the maximum coincidence probability. In this regime, the Shannon dimensionality can also be expressed as

$$D = \frac{1}{\sum_m y_m^2}.$$  

That is, $D$ is fully determined by the analyser properties. The dimensionality $D$ gives the effective number of entangled modes that are detected by the measurement apparatus. Note that $D$ is maximum for a flat-topped distribution of $\{y_m\}$ (i.e., $y_m = 1/M$ for some set of OAM modes and $y_m = 0$ for all others). This case corresponds to entanglement extraction of maximally-entangled qudits, yielding $D = M$. In general, however, for a non-uniform distribution of $\{y_m\}$ we find $D < M$.

The aim of the current Chapter is to present methods to maximise the Shannon dimensionality using multi-sector phase plates. In this fashion, we maximise the amount of extracted entanglement, using phase elements only. We do so by optimising the optical profile of the multi-sector phase plates. In the next Section, we will consider these multi-sector phase plates in closer detail.

6.3 Binary sector phase plates

Let us consider an angular phase plate with an optical phase profile $f(\theta - \alpha)$ along the azimuthal coordinate $\theta$, where $\alpha$ indicates the orientation of the plate. When such a plate is part of an angular state analyser, the projection mode is described by

$$A(\theta, \alpha) = \langle \theta | A(\alpha) \rangle = \frac{1}{\sqrt{2\pi}} e^{if(\theta-\alpha)}.$$  

(6.7)

This representation allows us to write the OAM expansion coefficients $\{\lambda_m\}$ from Eq. (6.3) as

$$\lambda_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} A(\theta, 0) e^{-im\theta} d\theta,$$  

(6.8)

i.e., $\{\lambda_m\}$ are simply the Fourier expansion coefficients of the plate profile. The OAM coupling strengths $y_m = |\lambda_m|^2$ are thus obtained from the mode overlap between an OAM mode and the projection mode $A(\theta, 0)$. 

58
6.3. BINARY SECTOR PHASE PLATES

Figure 6.1: Binary multi-sector phase plate with \( N = 3 \) elevated angular sectors. The phase imprint around the azimuthal coordinate jumps alternatingly between 0 and \( \pi \) at the wavelength of interest. The \( 2N \) sector angles are denoted as \( \Theta_n \), with \( \Theta_0 = 0 \) by definition.

Next, let us specify the properties of our multi-sector phase plates. Figure 6.1 shows an impression of a \( N = 3 \) sector phase plate. Such a plate has \( N \) elevated arc sectors that produce a \( \pi \) phase shift at the wavelength of interest with respect to the recessed sections between them. The azimuthal mode function corresponding to an angular state projector equipped with such a multi-sector phase plate can be written as

\[
A(\theta, 0) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
(-1)^n, & 0 \leq \theta < \Theta_n, \\
1, & 1 \leq n < 2N - 1, \\
Theta_{2N+1} \leq \theta < 2\pi, 
\end{cases} \quad (6.9)
\]

where the angles \( \{\Theta_n\} \) represent the \( 2N \) phase edges between the arc sectors, obeying \( \Theta_n < \Theta_{n+1} < 2\pi \) and \( \Theta_0 = 0 \) by definition (see Fig. 6.1). By application of Eq. (6.8), the OAM expansion coefficients \( \lambda_m \) are expressed in a series of integrals with alternating signs (because of the \( \pi \) phase shift),

\[
2\pi \lambda_m = \int_{\Theta_n=0}^{\Theta_1} e^{-im\theta} d\theta - \int_{\Theta_1}^{\Theta_2} e^{-im\theta} d\theta + \cdots - \int_{\Theta_{2N-1}}^{2\pi} e^{-im\theta} d\theta. \quad (6.10)
\]

Evaluation of the integrals yields the OAM coupling strengths \( y_m \),

\[
y_m = \begin{cases} 
1 + \frac{1}{\pi} \sum_{n=0}^{2N-1} (-1)^n \Theta_n, & \text{for } m = 0, \\
\frac{1}{\pi^2 m^2} \sum_{n,k=0}^{2N-1} (-1)^{n+k} \cos[m(\Theta_n - \Theta_k)], & \text{otherwise.} 
\end{cases} \quad (6.12)
\]

*Calculation of these coupling coefficients may be computationally demanding, as the number of terms in the summation scales with \( N^2 \). However, we can rewrite the cosine of a difference angle as \( \cos[m(\Theta_n - \Theta_k)] = \cos(m\Theta_n) \cos(m\Theta_k) + \sin(m\Theta_n) \sin(m\Theta_k) \) and arrive at

\[
y_m = \left[ \sum_{n=0}^{2N-1} \frac{(-1)^n \cos(m\Theta_n)}{\pi m} \right]^2 + \left[ \sum_{n=0}^{2N-1} \frac{(-1)^n \sin(m\Theta_n)}{\pi m} \right]^2. \quad (6.11)
\]

The computational complexity of this alternative but equivalent representation scales with \( N \) only.
We observe that $\gamma_0$ is simply related to the difference between the sums of the elevated and recessed arc sections of the plate. The $1/m^2$ dependence of $\gamma_m$ for $m \neq 0$ stems from the hard-edged transitions between the 0 and $\pi$ phase shifted sections.

### 6.4 Optimisation of binary sector phase plates

It is our aim to maximise the dimensionality

$$D = \frac{1}{\sum_m y_m^2}$$

that can be attained using binary sector phase plates. As discussed before, $D$ is maximised if all $y_m$ have the value $1/M$ or 0, with $M$ the number of contributing modes. However, such a uniform eigenvalue distribution cannot be realised by shaping the phase profile of the plate only; also an amplitude variation would be required. The problem at hand is thus to maximise $D$ with a multi-sector phase plate, by optimising the distribution of the $2N$ sector angles $\{\Theta_0, \ldots, \Theta_{2N-1}\}$. The resulting binary profile gives rise to a distribution of $\{y_m\}$ that is as flat-topped as possible for a pure phase plate. Qualitatively, this situation is realised by randomising the $2N$ sector angles. Quantitatively, however, the optimisation of the sector angles is by no means trivial, as we will see below.

#### 6.4.1 Single-sector phase plates

To commence, we first explore the case $N = 1$, i.e., a phase plate with a single elevated arc sector. In Chapter 3, we already derived an explicit expression for $D$ in terms of the opening angle $\Theta$ of the arc sector,

$$D(\Theta) = \begin{cases} 
\frac{1}{1} & \theta \in [0, \pi], \\
\frac{1}{D(2\pi - \Theta)} & \theta \in [\pi, 2\pi]. 
\end{cases}$$

Figure 6.2(a) shows a plot of this result. We observe that the Shannon dimensionality $D$ ranges between 1 and 6, the maximum being reached with the quadrant phase plate we used in Chapter 4.

Let us consider this case of a quadrant sector ($\Theta = \pi/2$) in more detail. Of the effective dimensionality $D = 6$ that this plate supports, four dimensions can be ascribed to mutually orthogonal projection states, being the states corresponding to the four plate orientations $\alpha = 0, \pi/2, \pi$ and $3\pi/2$. The states corresponding to any other setting of $\alpha$, however, cannot be represented as a superposition of these four states alone. In fact, by varying the orientation of the plate, we scan past an infinite set of additional projection states that are each distinct, but mutually non-orthogonal. We therefore decompose the projection states $\{|A(\alpha)\rangle\}$ in the orthogonal and complete OAM basis $\{|m\rangle\}$, which constitutes an appropriate basis due to the cylindrical symmetry of the problem (see Eq. (6.8)). Figure 6.2(b) shows a histogram of the eigenvalues $\{y_m\}$. We find that we need an infinite number of OAM modes to fully describe $|A(\alpha)\rangle$. We note, however, that the eigenvalues have very different weights. In the definition of $D$, the eigenvalues are precisely weighted by their strengths, yielding an effective dimensionality of $D = 6$. A similar argument can be made for the half-sector phase plate ($\Theta = \pi$), which
6.4. OPTIMISATION OF BINARY SECTOR PHASE PLATES

possesses two orthogonal orientations for $\alpha = 0$ and $\alpha = \pi/2$, yet gives rise to $D = 3$ (see Fig. 6.2(c) for the eigenvalue spectrum).

Six is the maximum dimensionality that can be attained with a binary phase plate containing a single elevated arc sector. In order to achieve higher dimensionalities, we thus have to consider plate profiles with increased complexity. The natural thing to do is to add binary sectors to the plate and optimise the set of sector angles. In the following, we will present an effective routine to determine the optimised sector profiles.

6.4.2 Multi-sector phase plates

In order to maximise $D$, we seek the minimum of the sum $\sum_m y_m^2$ (see Eq. (6.6)). As all elements $y_m^2$ of the sum are non-negative quantities, we can therefore minimise the elements themselves. However, the minimisation of these elements is non-trivial, since the variables they depend upon are shared amongst all. We start out from the expression for $y_m$ as given in Eq. (6.12). The $m = 0$ component $y_0$ causes no serious difficulties. Taking the square for $m \neq 0$ yields

$$y_m^2 = \frac{1}{\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \cos[m(\Theta_n - \Theta_k)] \cos[m(\Theta_{n'} - \Theta_{k'})] \frac{m^3}{m^3},$$

(6.14)
Figure 6.3: Optimal multi-sector phase plate patterns. The optimal plate pattern for $N = 1 - 12$. Black and white zones indicate regions of $0$ and $\pi$ phase imprint, respectively.

The complexity of the minimisation arises from the fact that Eq. (6.14) contains all combinatorial permutations of the periodic difference angles $\Theta_n - \Theta_k$ and $\Theta_{n'} - \Theta_{k'}$. In spite of considerable effort, we have not found an analytic solution to the maximisation of $D$, and it is not obvious that such a solution exists.

Therefore, we have adopted a numerical approach. In Chapter 4, we presented the maximum dimensionality as a function of the number of sectors $N$, as found with a Monte Carlo search routine. These results are shown in Fig. 6.5 as triangles. We observe that with 10 sectors we would attain $D \approx 50$. However, apart from showing that the dimensionality increases with $N$, the exact dependence remained unclear.

The Monte Carlo routine based on Eq. (6.14) is computationally rather demanding. An alternative approach involves writing the inverse dimensionality as a polynomial series of order four (see Appendix),

$$\frac{1}{D} = \frac{1}{24\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \chi^2 (2\pi - \chi)^2 + \gamma_0^2,$$

where $\chi = \Theta_n - \Theta_k + \Theta_{n'} - \Theta_{k'} \mod 2\pi$. This simplification yields a huge speedup when applying a search routine. For example, it reduced the search time for the optimal sector profile for $N = 9$ from almost three days to less than three minutes.

Figure 6.3 shows the optimal sector profiles we found for $N = 1 - 12$, based on Eq. (6.15). Black and white sectors represent zones of $\pi$ phase difference. We note that these solutions are not unique; the optimal solution is in fact $2 \times 2 \times 2N$-fold degenerate. Namely, one could (i) interchange the $0$ and $\pi$ phase shifted sectors, (ii) reverse the handedness from clockwise to counterclockwise, and (iii) one could choose any of the $2N$ sector angles as the starting point $\Theta_0$. Figure 6.4 presents the OAM distributions $\{\gamma_m\}$ for the optimal sector profiles of Fig. 6.3.
6.5 Experimental results

In this Section, the potential of multi-sector phase plates for extracting genuinely high-dimensional OAM entanglement is illustrated by an experiment. The experimental setup we use is largely the same as that described in Chapter 4 (see Fig. 4.1). A periodically-poled KTP crystal of length 2 mm serves as a Type-I parametric down-conversion source of OAM-entangled photon pairs at 826 nm, prepared in a state described by Eq. (6.1). The effective number of azimuthal entangled modes emitted by the source, characterised by the azimuthal Schmidt number $K$, is of the order of $K = 30$ [50, 87].

As discussed in Section 6.2, conservation of OAM in the down-conversion process requires that the two state analysers are equipped with complementary phase plates. That is, the phase plate in arm $A$ must be the phase conjugate of the one in arm $B$. Evidently, the phase conjugate of a multi-sector phase plate is simply obtained by interchanging the $0$ and $\pi$ phase-shifted sectors. However, this implies that the original and the conjugate phase profile are identical up to an overall phase factor of $\pi$. Since our coincidence-detection scheme is insensitive to overall phase factors, this means that we can simply use identical multi-sector phase plates in both arms.

Figure 6.4: Orbital-angular-momentum distribution of optimal multi-sector plates. (a)-(d) Histograms of the eigenvalues $\{\gamma_m\}$ of the optimal plate patterns from Fig. 6.3 for $N = 3, 4, 7,$ and 11.

for $N = 3, 4, 7,$ and 11. We observe that with increasing $N$, the spectrum gradually becomes broader and flatter. In Fig. 6.5, we present the dependence of the maximum dimensionality on the number of sectors $N$ (circles), corresponding to the optimal plate profiles from Fig. 6.3. Note that the results obtained with our accelerated routine are superior to those found previously for $N > 5$ (triangles). Strikingly, we observe that the growth rate is essentially linear, at least up to $N = 14$, the largest value of $N$ for which we applied our routine (solid line). The slope of the linear dependence is $5.9 \pm 0.02$. Note that the linear behaviour is not exact: the data are scattered closely around the solid line.
6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

Figure 6.5: Dimensionality of the optimal multi-sector phase plates. Maximum dimensionality $D$ as a function of the number of sectors $N$. (Triangles) Results as presented in Chapter 4 using a Monte Carlo search routine with Eq. (6.14) as a starting point. (Circles) Results based on the use of a search routine with Eq. (6.15) as a starting point. These data closely follow a linear dependence with slope 5.9 (solid line). The corresponding plate profiles are shown in Fig. 6.3.

In Fig. 6.6, we first present the theoretical predictions for the coincidence-probability curves as expected for the optimal plate profiles from Fig. 6.3, for $N = 1$ (dotted), $N = 3$ (dashed), and $N = 11$ (solid). We observe that the coincidence probability is zero or close to zero over a large domain of orientations and is strongly peaked around $\alpha = \beta$. The width of this parabolic peak narrows with increasing number of sectors $N$. Note that the Shannon dimensionality $D$ is inversely proportional to the area underneath the curve, as described by Eq. (6.5).

We have performed the experiment with multi-sector phase plates of $N = 4$, the fabrication of which is discussed in Chapter 5. The four elevated sectors have an optical thickness $\delta = 1.02\pi$. The sector sizes do not exactly match those of the optimal profile from Fig. 6.3, because the production process was initiated before the calculations presented in Section 6.4 were completed. As a result, the calculated dimensionality of these plates amounts to $D = 23.0$, as compared to 23.5 for the optimal profile. The inset in Fig. 6.7(a) shows the design of the plates, which is remarkably different from the optimal plate shown in Fig. 6.3.

Figure 6.7(a) shows the experimental coincidence curve. We observe a curve that is strongly peaked around $\alpha = \beta$. Note that the single count rates at both detectors ($2.6 \times 10^6$ per 8 s) are independent of the plate orientations. Given these high single count rates and the finite 2.7 ns coincidence detection window, accidental coincidence counts arising from photons that do not belong to a pair are unavoidable; we corrected for these accidentals. At the peak of the coincidence curve, we record about $1.7 \times 10^5$ coincidence counts per 8 s. Across the domain $25^\circ < \alpha - \beta < 335^\circ$, the count rate does not exceed $2 \times 10^3$ for each setting of $\alpha - \beta$. We thus achieve a visibility as high as 99%. We verified that the coincidence curve depends on the relative orientation between the two phase plates $\alpha - \beta$ only, in agreement with Eq. (6.4). The solid curve represents the theoretical prediction, for which a trivial vertical scaling factor
6.5. EXPERIMENTAL RESULTS

Figure 6.6: Theoretical peak-normalised coincidence probability for multi-sector phase plates. Peak-normalised coincidence curves for $N = 1$ (dotted), $N = 3$ (dashed), and $N = 11$ (solid), as a function of the relative orientation $\alpha - \beta$ between the two phase plates.

was used as the only fit parameter. We observe that experiment and theory are in excellent agreement with each other.

Based on Eq. (6.5), the experimental value of the dimensionality we obtain from the data equals $D = 16.5$. This number should be compared to the theoretical value of 23.0 mentioned above. The difference between these two numbers can be ascribed to two reasons. First, the experiment is very sensitive to misalignment of the phase plates. The discrepancy between theory and experiment at the bottom of the curve is a direct result hereof (see Fig. 6.7(b) for a rescaled version of this region). Second, in our derivation of Eq. (6.6) we assumed the source to be ideal, in the sense that it generates an infinite number of entangled modes (and the products $c_m c_{m'}^*$ from Eq. (6.4) could hence be considered constant). With an increasing number of sectors on the phase plates, however, we approach the limit of validity of this approximation. This causes the peak around $\alpha = \beta$ to be somewhat blunt rather than sharply spiked. Given the fact that $D$ scales linearly with the maximum coincidence count rate (see Eq. (6.5)), the dimensionality that is experimentally realised is fairly sensitive to this effect. Increasing the Schmidt number of the source, for instance by using a shorter down-conversion crystal, would resolve this issue.

Finally, we note that the coincidence-probability curve and the accompanying Shannon dimensionality are remarkably robust against deviations of the phase step height from its target value $\delta = \pi$. For the 2% deviation of our $N = 4$ plates ($\delta = 1.02\pi$), the effect is negligible. Even for a 10% deviation, the calculated dimensionality drops only by 3%. This insensitivity makes our method very robust against possible flaws of the plates that may arise in the fabrication process.
6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

Figure 6.7: Experimental coincidence curve. (a) Coincidence rate per 8 s as a function of the relative phase plate orientation $\alpha - \beta$ between the two state analysers, performed with two identical $N = 4$ sector phase plates (see inset). The circles represent experimental data, the solid curve represents our theoretical model. The theoretically predicted dimensionality equals $D = 23.0$. Experimentally, we obtain $D = 16.5$. (b) Magnification of the bottom of figure (a), showing the noise level of the measurements.

6.6 Conclusions

In conclusion, we have introduced multi-sector phase plates as a powerful tool to tailor high-dimensional OAM entanglement of photon pairs. We have presented a numerical study of the dimensionality that can be extracted using angular state analysers with multi-sector phase plates. Qualitatively, the maximum dimensionality is obtained with phase profiles that carry the least amount of azimuthal symmetry, hence yielding a broad and flat OAM eigenvalue spectrum. Quantitatively, we have found that the maximum dimensionality grows linearly with the number of sectors $N$. We experimentally demonstrated our approach using phase plates with $N = 4$ sectors.

These results demonstrate the potential of our phase-plate technology for extracting high-dimensional OAM entanglement. They represent an important step towards harnessing the full power of high-dimensional OAM entanglement for quantum information processing.
6.7 Appendix

In this Appendix, we present a derivation of the dimensionality $D$ as given in Eq. (6.15). We start out from the definition of $D$ (see Eq. (6.6)), using the expression for $y_m^2$ from Eq. (6.14). First, notice that the spectrum of the OAM coupling probabilities $y_m^2$ is symmetric around $m = 0$, because the sector phase plates have no net phase ramp along the azimuthal coordinate $\theta$. We can thus write $1/D = \sum_{m=-\infty}^{\infty} y_m^2 = 2\sum_{m=1}^{\infty} y_m^2 + y_0^2$. Next, we can work out the infinite series over $m$ in Eq. (6.14) by exploiting the standard expansion

$$\sum_{m=1}^{\infty} \frac{\cos(mx)}{m^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi |x|^3}{12} - \frac{x^4}{48}$$

which is a fourth order polynomial in $x$ valid for $-2\pi \leq x \leq 2\pi$ [134]. Using the trigonometric identity $2 \cos a \cos b = \cos(a + b) + \cos(a - b)$ to rewrite Eq. (6.14), it follows that one can write

$$\frac{1}{D} = \frac{1}{\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \left( \frac{\pi^4}{45} - \frac{\pi^2 x^2}{6} + \frac{\pi |x|^3}{6} - \frac{x^4}{24} \right) + y_0^2,$$

where we defined

$$\chi = \Theta_n - \Theta_k + \Theta_{n'} + \Theta_{k'} \mod 2\pi.$$  

(6.18)

Here, the modulo $2\pi$ ensures that $\chi$ is bounded between 0 and $2\pi$, as is required for the application of Eq. (6.16). This function can further be simplified by appreciating that (i) the term $\pi^4/45$ cancels out after the remaining summations, and (ii) $4\pi^2 \chi^2 - 4\pi \chi^3 + \chi^4 = \chi^2 (2\pi - \chi)^2$. It then follows that

$$\frac{1}{D} = \frac{1}{24\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \chi^2 (2\pi - \chi)^2 + y_0^2,$$

(6.19)

with $\chi$ as in Eq. (6.18). Finally, Eq. (6.19) possesses many symmetries, which can be exploited to remove redundant terms from the series. Consider the set of indices $\{n, k, n', k'\}$ with $0 \leq n, k, n', k' \leq 2N - 1$, then, for instance,

1. the elements $\{a, a, a, a\}$ yield 0,
2. the elements $\{a, a, b, b\}$ yield 0,
3. swapping (i) the first and the third index, $\{a, b, c, d\} \rightarrow \{c, b, a, d\}$; (ii) the second and the fourth index, $\{a, b, c, d\} \rightarrow \{a, d, c, b\}$; or (iii) both, $\{a, b, c, d\} \rightarrow \{c, d, a, b\}$, all yield the same result.

By accounting for such symmetries and multiple identical occurrences, the computational effort is strongly reduced.
6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT