2.1 Introduction

In this section we probe the rheology of granular media in split-bottom geometries. The flow profiles in the rate independent regime have been studied extensively as we discussed in chapter 1, but here we extend those studies to the rate dependent regime. Moreover, we perform the first rheological measurements in this geometry, relating the driving torque $T$ and driving rate $\Omega$. Even in the rate independent regime this is unexplored territory, and given the rich flow structure in these systems, the dependence of $T$ on filling height $H$ is non-trivial.

We start by introducing our flow geometry in section 2.2. The main differences are that the outer container now is a square box (this does not influence the flow), and that the driving disk is slightly elevated with respect to the bottom plate, without allowing particles under the disk -- this makes the rheology well defined by controlling the sharp gradients near the split.

In section 2.3 we present our flow profiles and stress measurements. First of all, in the rate independent regime we find good agreement between the measured $T(H)$ and that predicted by the torque minimization argument of Unger et al. [41] -- this allows us to measure the effective friction coefficient from rheology. Second, we probe the onset of rate dependent behavior by quantifying the changes in the flow profiles (widening and inward shift of the
shear bands for small $H/R$, and increase in axial slip for larger $H/R$ and by the increase in torque. In addition, the normal force exerted on the disk also varies with driving rate, decreasing for small $H/R$, and increasing for large $H/R$. Third, for rotation rates larger than $1 \text{ s}^{-1}$, the surface develops a depression in the center.

In section 2.5 we discuss the effect of the narrow slit for the rheology. This discontinuity of the driving can, in principle, lead to problems, although we will show that for the flow geometry discussed in sections 2.2 and 2.3 these problems do not occur. The first potential problem is that grains might get stuck, either in the gap between disk and bottom, or might be caught between their rough boundaries -- we find evidence that the latter problem plays a role for split-bottom geometries, where the roughness extends all the way to the edge of the disk (the typical case of grains glued to disk and bottom plate) and where the disk is flush with the bottom plate. Neither problem occurs in the flow geometry discussed in sections 2.2 and 2.3. The second problem can be thought of as hydrodynamic: the strain rates diverge near the split. In practice this singularity is cut off by the grain scale. We show experimentally that in a different flow geometry, where a disk is freely rotating in a bucket of grains and where this singularity does not occur, very similar flow profiles and torques are measured. Hence, this singularity does not dominate the behavior.

### 2.2 Elevated Disk Split-Bottom Setup

In this section we will describe rheological measurements on split-bottom granular flows. We have developed a flow geometry that allows for precise measurements of the driving torques necessary to sustain a certain flow rate, as well as to perform direct imaging of the surface flow profiles. Flow and torque measurements for deep layers ($H/R$) of order one, and for rate dependent flows ($\Omega$ of order one rps) are possible, as well as measurements of the pressure exerted on the driving disk. The flow cell is water tight, which allows to do similar measurements in suspension flows, as described in subsequent chapter 5. In all experiments described in this chapter, we used dry polydisperse black soda-lime glass beads (Sigmund Lindner 450X-007-L), with diameters between 2 and 2.5 mm (unless otherwise stated), and a bulk density of $2.58 \times 10^3 \text{ kg/m}^3$.

**Flow visualization setup** -- Surface velocity profiles of can be measured in the setup depicted in Fig. 2.1a. The granular medium is contained in a square transparent box with an inner width of 150 mm and height of 120 mm. In the center of the box there is a recession in the bottom plate; the radius of the
recession is 0.1 mm larger than 45 mm, and about 10 mm deep. In the recession the driving disk with radius $R_s = 45$ mm is mounted. Both disk and bottom are rough to ensure a no-slip boundary condition (Fig. 2.1d). The roughness is
created by drilling conical indentations of 5 mm in diameter and maximal depth of 6 mm. These are arranged on a honeycomb lattice with a maximum distance of 5 mm; for particle size below \( \sim 5-6 \) mm, the spacing of the lattice, this does not induce significant crystallization.

The disk in this setup is driven from below with a stepper motor (Lin Engineering 5718L-01P) and a microstepping driver (CDR-4MPS); the number of steps per revolution is set to 51200 for smooth rotation of the disk. We image the granular (free) surface with a Basler A622f B/W FireWire camera (12 bit, 1280x1024 pixels), connected to a PC with frame grabbing software (DAS Digital Image Archiver System\(^1\)). We use a fiber light source (Thorlabs OSL1-EC) to illuminate the surface; a typical image is shown in Fig. 2.1e. A sequence of these images can be used to extract surface velocity information with the particle image velocimetry technique discussed in appendix 8.

Rheological setup -- The flow geometry for the rheological experiments is identical in dimensions and roughness to that of the flow visualization setup. We drive the disk and perform the rheological measurements with a rheometer (Anton Paar MCR 501) in which the custom-built split-bottom cell is mounted. The rheometer is shown in Fig. 2.1c; number 1 in that figure is the rheometer head. We can raise the disk slightly above the surface of the static bottom to create an elevated disk split-bottom geometry -- typical disk elevations are 1 mm (Fig. 2.1c,d)

### 2.3 Flow: Profiles and Structure

In this section we investigate the onset of rate dependence in split-bottom flows. In section 2.3.1 we present data on the dependence of the surface flows on \( \Omega \), and find that rate dependence becomes detectable for \( \Omega \approx 0.1 \) s\(^{-1}\), although the changes in surface flows are relatively mild. We turn our attention to rheological measurements in sections 2.4.1 and 2.4.2. In 2.4.1 we measure the driving torque as function of filling height \( H \) in the rate-independent regime, to test a prediction from the theoretical work of Unger et al. [41]. In section 2.4.1 we turn our attention to the dependence of the driving torque and normal forces on flow rate. In section 2.4.2 we report on correlations between the fluctuations of the driving torque and normal forces.

\(^{1}\)From DVC Machinevision, Breda, The Netherlands; www.machinevision.nl
2.3.1 Surface Flow Profiles

We determine the surface flows with the Particle Image Velocimetry (PIV) technique discussed in appendix 8.1. We measure the flow profiles over more than four decades in flow rate $\Omega$, from $1.6 \times 10^{-4}$ to 1.25 rps, for a range of filling heights: $H/R_s = 0.11, 0.22, 0.33, 0.44, 0.56, 0.67, 0.89, 1.02$. We find that for low filling heights, the shear zone tends to broaden and migrate inwards with increasing rotation rate, as illustrated for $H/R_s = 0.33$ in Fig. 2.2a. Once precession (or axial slip) becomes appreciable, the shear zones develop an asymmetry with driving rate. Moreover, their center, defined as their inflection point, first moves inwards, then outwards, as shown in Fig. 2.2b. The amount of axial slip also increases markedly with $\Omega$.

To further characterize the rate dependence of the flow profiles, we fit these to error functions as in Eq. 1.3, and extract the axial slip $\omega(r = 0, z = H)$, the width of the profile $W$, and the center of the shear band $R_c$. We show the results of this in Fig. 2.3.

Our data for the axial slip at the surface as a function of filling height for slow flows (Fig. 2.3a) are consistent with earlier data [34]: $\omega(r = 0, H) \sim \exp(-H/R_s)$. We see no strong rate dependence in the axial slip -- Fig. 2.3b.

The width of the shear bands increases with driving rate as shown in (Fig. 2.3c). Jop modeled split-bottom flows with the inertial number based rheology for granular flows (see section 1.5.3), and found that the width of the shearband approximately scales as $W \sim \Omega^{0.38}$ [55]. Clearly, such models miss the

\footnote{We use frame rates between 33 mHz and 33 Hz. We obtain surface flow profiles as in Fig. 2.11. The interframe steps are 1 for all experiments but the largest filling heights, since at larger filling heights the surface flow becomes prohibitively slow.}
2.3. FLOW: PROFILES AND STRUCTURE

Figure 2.3: (a) The axial slip for slow flows in the rate independent regime $\Omega = 5 \times 10^{-4}$ rps. (b) The rate dependence of axial slip for a range of filling heights and rotation rates. (c) The width of the velocity profiles. The dotted line shows a power law $W \sim \Omega^{0.38}$ to allow comparison with predictions from Jop [55]. (d) The location of the center of the shear band. The different symbols indicate different filling heights.

Finite and rate-independent width of the shear bands at low driving rates, but the model may capture the observed broadening of the shear bands at higher driving rate. However, the range over which we can check the validity of this scaling is very limited.

The shift of the shear bands already visible in Fig. 2.2 is shown in Fig. 2.3d. Partly because the onset of rate dependence also leads to asymmetries in the flow profiles, in particular for large filling heights, the error function fits are not very good. As a result the apparent positions $R_c$ show considerable scatter, although the overall trends are clearly visible.
Figure 2.4: The setup depicted measures the evolution of the surface structure. Steady state surface structure snapshots for different $\Omega$ are shown in (b–e): (b): 0.7 rps; (c): 1.3 rps; (d): 2 rps; (e): 2.7 rps. In (e) the rotating disk is visible in the middle. The filling height $H/R_s$ here is 0.3.

### 2.3.2 Surface Recession

For driving rates above 2 rps, where one would expect the rate dependent effects reported above to be more significant, the surface starts to develop a dip in the middle. This renders an accurate determination of the surface flow profiles difficult. This effect is reminiscent of Newtons bucket, where the transverse pressure gradient in a rotating Newtonian liquid creates a so-called paraboloid of revolution [56].

Corwin [57] explored this surface deformation effect in a variant of the split-bottom geometry where the rotating disk constitutes the whole bottom. Amongst other things, he observed that for low filling heights, the center of the system is evacuated completely at higher driving rates. For larger filling heights, this evacuation does appear.
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We have probed the onset of the surface dip using the rheology setup depicted in Fig. 2.4a where a focusable diode laser (Thorlabs CPS196) behind a cylindrical lens creates a laser sheet that illuminates a line on the surface of the granular bed. We image the illuminated strip from the side and under an angle of 45° with a high-resolution digital camera (Canon EOS 20D), with an exposure time long enough to blur the otherwise patchy and scattered line of laser light. The results for \( H/R_s = 0.3 \) (\( H = 12\text{mm} \)) are shown in Fig. 2.4b-e, showing the development of the surface dip.

The onset of this effect can be estimated by comparing when the centrifugal force \( F_c = mv^2/R_s = m(2\pi\Omega)^2R_s \) becomes larger than \( \mu mg \), the maximum radial frictional force a particle can bear under its own weight. These two terms equate for \( 1/2\pi\sqrt{\mu g/R_s} \sim 1.8 \text{ rps} \), using a typical value of 0.6 for \( \mu \). Note that for high enough \( \Omega \), the center region indeed gets completely evacuated, in accordance with the observations from Corwin [57].

2.4 Rheology

2.4.1 Rate Independent Regime

![Figure 2.5: (a,b) A comparison between the numerically obtained \( T(H) \) (line), and experiments (×). The error on the average torque values is shown with the vertical bar through the data. The dashed line in (b) corresponds to \( T \sim H^2 \), the dotted line to \( T \sim H \).]

Due to the non-trivial structure of the flow profiles in the split-bottom geometry, the driving torque depends non-trivially on the filling height, even for slow, rate-independent flows. The procedure introduced by Unger (see section
1.5.3) minimizes the functional

\[ T(H) = 2\pi \mu \rho \Phi g \int_0^H (H - z)r^2 \sqrt{1 + (dr/dz)^2} \, dz. \tag{2.1} \]

Here \( \rho \) is the bulk density of the particles, \( \Phi \) is the average packing fraction (\( \sim 0.59 \)) and \( \mu \) is the effective friction coefficient -- only the product of these three is relevant for the precise value of the torque \( T \). Minimizing Eq. 2.1 not only leads to a good prediction for the position \( R_s(z) \) of the center of the shear bands but also predicts the driving torque \( T \) as a function of the height \( H \).

This prediction for the torque has not been tested experimentally. We have measured the average torque necessary to sustain a rotation rate of \( \Omega = 1.7 \times 10^{-2} \) rps (clearly in the rate independent regime) over two full rotations, for filling heights \( H/R_s = 0.085 \) to 1.5. We performed the experiments in the setup described in section 2.2.

The resulting data is shown in Fig. 2.5 and compared to the theoretical predictions obtained by minimizing the functional Eq. 2.1 numerically. We have fixed \( \rho = 2.85 \times 10^3 \) kg/m\(^3\), \( \Phi = 0.59 \) and have adjusted \( \mu \) to obtain an optimal fit -- this yields \( \mu = 0.47 \pm 0.03 \) -- the error is due to the uncertainties in \( \Phi, \rho \) and experimental noise in the determination of \( T \). The simple model describes the experimental data rather well. The minimization procedure predicts quadratic behavior of \( T(H) \) curve for small \( H/R_s \). This crosses over to linear behavior for larger filling heights, where precession sets in. Indeed, above \( H/R_s = 0.7 \) we see a profound change in the curvature of \( T(H) \): above this critical filling height, \( T(H) \) is essentially linear in \( H \).

### 2.4.2 Rate Dependent Regime

We measure the average driving torque necessary to sustain a flow over more than three decades in \( \Omega \), from \( 1.6 \times 10^{-3} \) to 1.25 rps, for five different filling heights: \( H/R_s = 0.25, 0.35, 0.45, 0.55, 0.75 \). In each experiment at fixed filling height, the rotation rate is swept from low to high; a reverse sweep gives similar results as long as the averaging timescales are long enough to allow transients to die out.

The results of the rheological experiments are shown in Fig. 2.6. In Fig. 2.6a,c we plot the average normal forces acting on the disk; in Fig. 2.6b, d we show the average driving torques necessary to sustain rotation of the disk.

In the normal forces, we can observe several trends: for large filling heights, no rate dependence is observed at all, for \( \Omega \) up to 1.25 rps. For the lower filling heights, we observe a reduction of the normal forces acting on the disk for \( \Omega = 1 \) rps and higher. This is due to the fact that at these rotation rates, the centrifugal
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Figure 2.6: Averaged normal (a,c) and shear (b,d) forces measured at different $\Omega$. Different $H/R_s$ are indicated with symbols: 0.19; + 0.29; ◦ 0.43; △ 0.52; □ 0.72; × 1.12; ◊ 1.52. In panel (c,d) the data is normalized by the average of the first 10 data points in the rate independent regime.

force is large enough to move particles in the radial direction -- the reduction of the normal forces is simply due to less particles resting on the disk. This phenomenon is accompanied by a change in the surface structure of the flow which will be discussed below.

The driving torques show more pronounced rate dependence. For all filling heights, the rate dependence sets in around 0.5 rps, and leads to an increase of the driving torque -- this increase is relatively largest for small filling heights.

2.4.3 Stress Fluctuations

During shear, the stresses fluctuate, and we have measured the normal and shear stresses exerted on the disk for a constant rate in the rate independent regime ($\Omega = 1.7 \times 10^{-2}$ rps), for a range of filling heights: $H/R_s = 0.05 - 1.5$. We sample the forces at 10 Hz, over two full rotations, amounting to 120 seconds. We ignore the first 20 seconds of each data set.
Figure 2.7: (a,b) Normal and shear forces for two filling heights, measured under continuous rotation with $\Omega = 1.6 \times 10^{-1}$ rps. The dashed line indicates the normal force one would expect based on the weight of the grains present in the cell. In these plots, larger normal forces correspond to larger weights pressing on the disk. (c) Scatter plots as shown in (a,b), here shown for all filling heights.

In Fig. 2.7a,b we present scatter plots of the torque $T$ vs normal force $N$ for $H/R_s = 0.13$ and 1.5. These illustrate that for low filling heights $N$ and $T$ are uncorrelated. The normal force deviates a few percent from the weight of the grains resting on the disk, but we attribute this to inhomogeneities in the packing density. Note that the fluctuations in the instantaneous torque are substantial.

For the large filling height data shown in Fig. 2.7b, the shear band has collapsed into a dome above the disk, and $T$ and $N$ display considerable correlations. Notice that for the dome shape, fluctuations in the normal stresses across the shear band can be expected to be picked up by the normal force exerted on the disk, while this is not the case for the shear bands on the shallow case. Hence, similar correlations between normal and shear stresses may also arise for shallow filling heights, but our probe is not sensitive to this.
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In this regime the fluctuations of the normal force are substantial, and the instantaneous normal force is always substantially larger than the normal force based on the weight of the grains (dashed line). This suggests that the normal force can be seen as due to the sum of the weight of the grains and a dynamically generated normal stress, related to the dilatancy of the grains and the shear stress.

Fig. 2.7c presents a scatter plot for all filling heights, illustrating the occurrence of a linear correlation between the normal and the shear force for all $H/R_s \gtrsim 0.6$.

$\text{(a)} \quad \text{ordinary split-bottom}$

$\text{(b)} \quad \text{elevated disk}$

Figure 2.8: (a) A schematic drawing of the standard split-bottom geometry, and (b) of the disk geometry. (c) In the region emphasized by the oval, the strain rate is diverging at $z = 0$, $r = R_s$. (d) In the disk geometry, grains touch the smooth side of the disk between $z = 0$ and $z = s$, with $s$ the thickness of the disk.

### 2.5 Flow Singularity

The discontinuity in the driving in the split-bottom geometries is a potential obstacle for doing rheology of such flows. In this section we compare the flow profiles and driving torques of the flow geometry discussed in section 2.2 to two
other geometries -- a standard split-bottom geometry where the bottom and disk are made rough by gluing grains on their surfaces and the disk and bottom are flush, and a disk geometry where the disk is freely moving in the granular medium (see Fig. 2.8).

Comparing the flow profiles and rheology of these systems allows us to identify and eliminate the problems due to individual grains being stuck near the split, and due to the ‘hydrodynamic’ singularity that is also present on the continuum scale. To probe the rheology, we focus on $T(H)$ for moderate shear rates.

We will show that the flow profiles of split-bottom geometry, elevated split-bottom geometry and disk geometry become very similar for sufficiently large filling heights. The $T(H)$ curves show differences though. We find that the standard split-bottom geometry suffers from strong rate dependence and also leads to $T(H)$ not smoothly extrapolating to zero stress for zero height. The disk geometry does not show such anomalous rate dependence, but its $T(H)$ is not described by Unger’s formalism -- drag forces exerted on the side of the disk also have to be taken into account. The elevated disk geometry, used in the previous sections of this chapter has none of these problems, showing that here the flow singularity is well controlled.

Here we first present a simple argument why the grainscale effectively limits the singularity. The maximum strain rate near the split can be estimated as $2\pi \Omega R_s/d$ -- i.e. all strain is concentrated over one grain diameter. For typical filling heights (2.5 cm), $R_s$ (4 cm), this yields that the inertial number $I$, defined as $\dot{\gamma} d/\sqrt{P/\rho}$ with $d$ the particle diameter, $P$ the local pressure, and $\rho$ the bulk density of the particles, equals $0.5 \Omega$. Hence, in practice the inertial number remains quite limited, and no singular rheology is expected.

### 2.5.1 Split-Bottom and Disk Setups

We compare the flow profiles and rheology obtained in the elevated disk geometry to two other geometries, as shown in Fig. 2.9. Both geometries consist of a transparent acrylic cylinder with an inner radius of 70 mm, with a split-bottom disk of radius $R_s$ of 40 mm. Both the bottom plate and the top surface of the disk are made rough by gluing glass beads on them. The sidewall is not made rough, but through the transparent sidewalls we do not observe any particle motion near the walls. To avoid the singularity at the split in split-bottom flows, we introduce here a modified version of the split-bottom, called the disk version. Here we remove the singularity by elevating the rough disk above the outer no-slip boundary. A hollow smooth cylinder is placed directly underneath the disk, such that no particles can get underneath the disk -- we also make the
2.5. FLOW SINGULARITY

Figure 2.9: (a) The setups used to study the effect of the flow singularity. We change between the two types by replacing the bottom of the acrylic container. (b) Typical image of grain surface, with arrow indicating the flow direction.

gap between disk and the top of the cylinder smaller than a grain diameter. See Fig. 2.8b.

We use black soda-lime glass beads (Sigmund Lindner 4504-007-L), a polydisperse mixture with a diameter range from 1 to 1.3 mm. The disk is driven by a rheometer (Anton Paar DSR 301) in constant rotation mode only. The disk axis is held by a low-friction ball bearing, and the axis is connected to the rheometer via a custom built flexure with a stiffness of 4 Nm/rad. We observe surface flow in the cell with an 8 bit camera (Basler A101f) via a mirror. We extract the surface velocity profiles with particle image velocimetry (PIV) -- see appendix 8.1. The particles are illuminated with a fiber light source (Thorlabs OSL1-EC). The light source used is a, to a good approximation, a bright point source -- this aides the PIV technique in creating high contrast in the images.
2.5.2 Rheology of Split-Bottom and Disk Geometries

We measure the average torque $T(H, \Omega)$ necessary to sustain a constant rotation rate of the disk, for different filling heights and rotation rates. The driving torque is always averaged over at least 1 full rotation, for the highest rotation rates we average over 5 rotations. We observe that with an empty cell only very low average torques values, of the order of $0.01 \text{ mNm}$ are necessary to sustain rotation of the disk -- the ball bearing does not influence the measurements.

The results of the experiments are shown in Fig. 2.10. Results for the split-bottom are shown in the left two plots. Results for the disk version are shown in the right two plots. For the split-bottom geometry $T(H)$ has a substantial offset for low filling heights, which moreover is strongly rate dependent; it increases threefold when $\Omega$ is increased from $1.7 \times 10^{-2}$ to 0.5 rps, whereas the driving torques for the elevated disk version do not show any rate dependence here (Fig. 2.10c-d).
2.6. DISCUSSION AND CONCLUSIONS

We believe that these anomalies are due to the sharp flow gradients near the split, and the possibility that grains get stuck intermittently here.

The data for $T(H)$ cannot immediately be fitted accurately by the numerical expression obtained from minimizing the torque, as was done in section 2.4.1. However, we can successfully capture $T(H)$ if we take into account the drag forces exerted on the side of the disk:

$$T(H) = T_U(H) + 2g\eta\Phi\mu R_s^2[Hs + 0.5s^2],$$

where $T_U(H)$ is the torque obtained from the Unger model, and $s$ is the thickness of the disk.

The black line in Fig. 2.10 is a fit of Eq. 2.2 to the data, while the grey line in this plot corresponds to the result without side-drag correction; the linear correction and offset are substantial but sufficient. The only fitting parameter in the model, even with the linear correction term included, is the effective friction coefficient $\mu$, which we find to be $\mu = 0.57 \pm 0.03$.

2.5.3 Flow Profile Comparison

We have probed whether the flow profiles of the ordinary split-bottom cell are similar to the ones produced by the elevated-disk split-bottom cell for a rotation rate $\Omega$ of $1.7 \times 10^{-3}$ rps. The normalized angular velocity profiles $v_\theta(r)/2\pi r \Omega$ are shown in Fig. 2.11a for the ordinary split-bottom geometry and in Fig. 2.11b for the disk geometry. The only significant point to note is how to define $H$ in the disk geometry: only if we measure $H$ from the underside of the disk, so at $H + s$, the flow profiles will overlap -- see Fig. 2.11d. This height correction works well for all other filling height with $H/R_s > 0.15$.

2.6 Discussion and Conclusions

In this chapter we have studied the rheology and flow profiles of dry granular flows in the split-bottom geometry, in both the rate independent, as well as the rate dependent regime of flow rates.

Rate independent regime -- We have seen that the large strain rates above the split between the two no-slip boundaries in the original split-bottom [31] can lead to problems when measuring the rheology in this system. We devised a modified version which we refer to as the elevated disk setup, which shows all the interesting flow phenomenology of the ordinary split-bottom geometry, but also allows for rheology experiments.
We test the Unger model prediction for $T(H)$, the height dependence of the driving torque in the rate independent regime. We do this for both small and large filling heights, and find good agreement between the prediction and the experimental results.

**Rate dependent regime** -- In the rate dependent regime, we measured rheology and flow profiles in the split-bottom geometry. While both flow profiles and rheology show rate dependence, the range is rather limited because for $\Omega > 1$ the surface develops a depression in the center region due to the centrifugal forces.

Finally we wish to suggest a simple argument for determining both the shift
of the center location of the shear zones and the increase in torque. The idea is to extend Unger's model with a rate dependent friction, and to model this rate dependence like is done in the inertial number framework. Hence, we should calculate \( I = \dot{\gamma} d / \sqrt{P/\rho} \) with \( d \) the particle diameter, \( P \) the local pressure, and \( \rho \) the bulk density of the particles, and then replace \( \mu \) by \( \mu(I) \), which we approximate as \( \mu(I) = \mu_0 + \mu_1 I \). We estimate the local strain rate to be proportional to the inverse of the width of the shear zones \( W \), and we rely on earlier measurements which find that \( W \sim z^\alpha \), with \( \alpha \) around 1/3. In principle, both the shift of the center location of the shear zones and the increase in torque should be given by minimizing the integral:

\[
T(H, \Omega) = 2\pi \rho \Phi g \int_0^H (\mu(\ell) H - z)^2 \sqrt{1 + (\text{dr/dz})^2} \, dz,
\]

(2.3)

where \( I \sim z^{-\alpha/\sqrt{P}} \).

(2.4)

Clearly, such a model contains a large number of fit parameters (in particular \( \mu_1 \)), but it would be interesting to see whether such model, at least in principle, can capture some of the observed phenomenology.